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A Fixed Point Approach to Stability of Quintic Functional Equations in Modular Spaces

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Abstract. In this paper, we present a fixed point method to prove generalized Hyers–Ulam stability of the systems of quadratic-cubic functional equations with constant coefficients in modular spaces.

1. Introduction

The *stability problem* of functional equations started with the following question concerning stability of group homomorphisms proposed by S.M. Ulam [58] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940.

Let $(G_1, .)$ be a group and $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$.

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Given $\epsilon > 0$, does there exist a $\delta > 0$ such that, if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [24] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

If E and $E^{'}$ are Banach spaces and $f : E \longrightarrow E^{'}$ is a mapping for which there is $\varepsilon > 0$ such that $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$, then there is a unique additive mapping $L : E \longrightarrow E^{'}$ such that $\|f(x) - L(x)\| \leq \varepsilon$ for all $x \in E$.

Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [51] for linear mappings by considering an unbounded Cauchy difference, respectively.

The paper of Rassias [51] has provided a lot of influence in the development of what we now call the *generalized Hyers–Ulam stability* or as *Hyers–Ulam–Rassias stability* of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [3, 5, 6, 9, 11, 19, 25, 26, 27, 28, 31, 32, 33, 36, 52] and [46]–[53]. Recently, Sadeghi [54] presented a fixed point method to prove generalized Hyers–Ulam stability of the generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(x)$ in modular spaces.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi–additive function [1, 34]. It is natural that this equation is called a *quadratic functional equation.* In particular, every solution of the quadratic equation (1.1) is called a *quadratic function.* The Hyers–Ulam stability problem for the quadratic functional equation was solved by Skof [56]. In [5], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1.1). Eshaghi Gordji and Khodaei [20] obtained the general solution and the generalized Hyers–Ulam–Rassias stability of the following quadratic functional equation: for all $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$,

$$f(ax+by) + f(ax-by) = 2a^2 f(x) + 2b^2 f(y). \tag{1.2}$$

Jun and Kim [29] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.3). Jun et al. [30] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax+by) + f(ax-by) = ab^2(f(x+y) + f(x-y)) + 2a(a^2-b^2)f(x), \tag{1.4}$$

where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. For other cubic functional equations, see [43].

Lee et. al. [39] considered the following functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.5}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of the equation (1.5) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x, x)$ for all $x \in X$. The bi-quadratic function B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y)).$$

Obviously, the function $f(x) = cx^4$ satisfies the functional equation (1.5), which is called the *quartic functional equation*. For other quartic functional equations, see [4].

Ebadian et al. [7] considered the generalized Hyers-Ulam stability of the following systems of the additive–quartic functional equations:

$$\begin{cases} f(x_1+x_2,y)=f(x_1,y)+f(x_2,y), \\ f(x,2y_1+y_2)+f(x,2y_1-y_2) \\ \qquad = 4f(x,y_1+y_2)+4f(x,y_1-y_2)+24f(x,y_1)-6f(x,y_2) \end{cases} \tag{1.6}$$

and the quadratic-cubic functional equations:

$$\begin{cases} f(x,2y_1+y_2)+f(x,2y_1-y_2) \\ \qquad = 2f(x,y_1+y_2)+2f(x,y_1-y_2)+12f(x,y_1), \\ f(x,y_1+y_2)+f(x,y_1-y_2)=2f(x,y_1)+2f(x,y_2). \end{cases} \tag{1.7}$$

For more details about the results concerning mixed type functional equations, the readers refer to [13, 16, 17] and [18].

Recently, Ghaemi et. al. [12] investigated the the stability of the following systems of quadratic-cubic functional equations:

$$\begin{cases} f(ax_1+bx_2,y)+f(ax_1-bx_2,y)=2a^2f(x_1,y)+2b^2f(x_2,y), \\ f(x,ay_1+by_2)+f(x,ay_1-by_2) \\ \qquad = ab^2(f(x,y_1+y_2)+f(x,y_1-y_2))+2a(a^2-b^2)f(x,y_1) \end{cases} \tag{1.8}$$

in PN-spaces, where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. The function $f : R \times R \to R$ given by $f(x, y) = cx^2y^3$ is a solution of the system (1.8). In particular, letting $y = x$, we get a quintic function $g : R \to R$ in one variable given by $g(x) := f(x, x) = cx^5$.

The proof of the following propositions is evident.

Proposition 1.1. *Let X and Y be real linear spaces. If a function $f : X \times X \longrightarrow Y$ satisfies the system* (1.8)*, then $f(\lambda x, \mu y) = \lambda^2\mu^3 f(x, y)$ for all $x, y \in X$ and rational* *numbers* λ, μ.

In this paper, by using some ideas of [14, 54], we investigate the generalized Hyers–Ulame stability of a quintic mappings from linear spaces into modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [44] and were intensively developed by Amemiya, Koshi, Shimogaki, Yamamuro [37, 59] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [40, 42, 57] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [45] and interpolation theory [38, 41], which in their turn have broad applications [42]. The importance for applications consists in the richness of the structure of modular function spaces, that–besides being Banach spaces (or F–spaces in more general setting)– are equipped with modular equivalent of norm or metric notions.

Definition 1.2. Let X be an arbitrary vector space.
(a) A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary $x, y \in X$,

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

(b) if (iii) is replaced by

(iii)$'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then we say that ρ is a convex modular. A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to satisfy the Δ_2–condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$.

Definition 1.3. Let $\{x_n\}$ and x be in X_ρ. Then
(i) the sequence $\{x_n\}$, with $x_n \in X_\rho$, is ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of X_ρ is called ρ–complete complete if and only if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ–convergent to x.

Remark 1.4. Note that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) of Definition 1.2 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in X$. Moreover, if ρ is a convex modular on X and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in X$.

A convex function φ defined on the interval $[0, \infty)$, nondecreasing and continuous for $\alpha \geq 0$ and such that $\varphi(0) = 0, \varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. The Orlicz function φ satisfies the Δ_2–condition if there exists $\kappa > 0$ such that $\varphi(2\alpha) \leq \varphi(\alpha)$ for all $\alpha > 0$. Let (Ω, Σ, μ) be a measure space. Let us consider the space $L^0(\mu)$ consisting of all measurable real–valued (or complex–valued) functions on Ω. Define for every $f \in L^0(\mu)$ the Orlicz modular $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu.$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\varphi(\Omega, \mu)$ or briefly L^φ . In other words,

$$L^\varphi = \{f \in L^0(\mu) \quad | \quad \rho_\varphi(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

or equivalently as

$$L^\varphi = \{f \in L^0(\mu) \quad | \quad \rho_\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the Orlicz space L^φ is ρ_φ–complete. Moreover, $(L^\varphi, \|.\|_{\rho_\varphi})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_\varphi}$ is defined as follows

$$\|f\|_{\rho_\varphi} = \inf\left\{\lambda > 0 \quad : \quad \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\}.$$

Moreover, if $\mathfrak{L}$ is the space of sequences $x = \{x_i\}_{i=1}^\infty$ with real or complex terms x_i, $\varphi = \{\varphi_i\}_{i=1}^\infty$, φ_i are Orlicz functions and $\varrho_\varphi(x) = \Sigma_{i=1}^\infty \varphi_i(|x_i|)$, we shall write ℓ^φ in place of L^φ. The space ℓ^φ is called the generalized Orlicz sequence space. The motivation for the study of modular spaces (and Orlicz spaces) and many examples are detailed in [41, 42, 44, 45].

2. Main Results

Throughout this paper, we assume that ρ is a convex modular on X with the Fatou property such that satisfies the Δ_2–condition with $0 < \kappa \leq 2$. In this section, we establish the conditional stability of quintic functional equations.

Theorem 2.1. *Let E be a real or complex linear space and let X_ρ be a ρ–complete* *modular space. Suppose* $f : E \times E \to X_\rho$ *satisfies the condition* $f(0, y) = 0$ *and an inequality of the form*

$$\begin{aligned} \rho(f(ax_1 + bx_2, y) + f(ax_1 - bx_2, y) - 2a^2 f(x_1, y) \\ -2b^2 f(x_2, y)) &\leq \phi(x_1, x_2, y), \quad (2.1) \\ \rho(f(x, ay_1 + by_2) + f(x, ay_1 - by_2) - ab^2 f(x, y_1 + y_2) \\ -ab^2 f(x, y_1 - y_2) - 2a(a^2 - b^2) f(x, y_1)) &\leq \psi(x, y_1, y_2), \quad (2.2) \end{aligned}$$

where $\phi, \psi : E \times E \times E \to [0, \infty)$ *is a given function such that*

$$\phi(ax, 0, ay) \leq a^5 L\phi(x, 0, y), \quad \psi(a^2 x, ay, 0) \leq a^5 L\psi(ax, y, 0),$$

and has the property

$$\lim_{n\to\infty} \frac{\phi(a^n x_1, a^n x_2, a^n y)}{a^{5n}} = \frac{\psi(a^n x, a^n y_1, a^n y_2)}{a^{5n}} = 0,$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$ *and a constant* $0 < L < 1$*. Then there exists a unique quintic function* $j : E \times E \to X_\rho$ *satisfying the system* (1.8) *and*

$$\rho(j(x, y) - f(x, y)) \leq \frac{1}{1 - L} \left(\frac{1}{a^2} \phi(x, 0, y) + \frac{1}{a^5} \psi(ax, y, 0) \right),$$

for all $x, y \in E$*.*

Proof. Putting $x_1 = 2x$ and $x_2 = 0$ and replacing y by $2y$ in (2.1), we get

$$\rho(2f(2ax, 2y) - 2a^2 f(2x, 2y)) \leq \phi(2x, 0, 2y) \tag{2.3}$$

for all $x, y \in E$. Putting $y_1 = 2y$ and $y_2 = 0$ and replacing x by $2ax$ in (2.2), we get

$$\rho(2f(2ax, 2ay) - 2a^3 f(2ax, 2y)) \leq \psi(2ax, 2y, 0) \tag{2.4}$$

for all $x, y \in E$. Thus by (2.3) and (2.4) we have

$$\begin{aligned} &\rho(a^{-3} f(2ax, 2ay) - a^2 f(2x, 2y)) \\ &\leq \frac{1}{2}\rho(2a^{-3} f(2ax, 2ay) - 2f(2ax, 2y)) + \frac{1}{2}\rho(2f(2ax, 2y) - 2a^2 f(2x, 2y)) \\ &\leq \frac{1}{2a^3}\rho(2f(2ax, 2ay) - 2a^3 f(2ax, 2y)) + \frac{1}{2}\rho(2f(2ax, 2y) - 2a^2 f(2x, 2y)) \\ &\leq \frac{1}{2a^3}\psi(2ax, 2y, 0) + \frac{1}{2}\phi(2x, 0, 2y), \end{aligned}$$

for all $x, y \in E$. By last inequality we get

$$\rho(a^{-5} f(2ax, 2ay) - f(2x, 2y)) \leq \frac{1}{2a^5}\psi(2ax, 2y, 0) + \frac{1}{2a^2}\phi(2x, 0, 2y). \tag{2.5}$$

Replacing x, y by $\frac{x}{2}$, $\frac{y}{2}$ in (2.5), we have

$$\rho(a^{-5}f(ax,ay)-f(x,y)) \le \frac{1}{2a^5}\psi(ax,y,0)+\frac{1}{2a^2}\phi(x,0,y), \tag{2.6}$$

for all $x, y \in E$.

We now consider the set

$$\mathcal{M}=\{h : E\times E \to X_\rho, \quad h(0,y)=0 \text{ for all } y \in E\}$$

and introduce the convex modular $\widetilde{\rho}$ on $\mathcal{M}$ as follows,

$$\widetilde{\rho}(h)=\inf\{c>0 : \rho(h(x,y)) \le c\,\Phi(x,y)\},$$

where $\Phi(x,y) := \frac{1}{a^5}\psi(ax,y,0)+\frac{1}{a^2}\phi(x,0,y)$. It is sufficient to show that $\widetilde{\rho}$ satisfies the following condition

$$\widetilde{\rho}(\alpha g+\beta h) \le \alpha\widetilde{\rho}(g)+\beta\widetilde{\rho}(h)$$

if $\alpha+\beta=1$ and $\alpha, \beta \ge 0$. Let $\varepsilon>0$ be given. Then there exist $c_1>0$ and $c_2>0$ such that

$$c_1 \le \widetilde{\rho}(g)+\varepsilon; \quad \rho(g(x,y)) \le c_1\,\Phi(x,y)$$

and

$$c_2 \le \widetilde{\rho}(h)+\varepsilon; \quad \rho(h(x,y)) \le c_2\,\Phi(x,y).$$

If $\alpha+\beta=1$ and $\alpha, \beta \ge 0$, then we get

$$\rho(\alpha g(x,y)+\beta h(x,y)) \le \alpha\rho(g(x,y))+\beta\rho(h(x,y)) \le (\alpha c_1+\beta c_2)\Phi(x,y),$$

whence

$$\widetilde{\rho}(\alpha g+\beta h) \le \alpha\widetilde{\rho}(g)+\beta\widetilde{\rho}(h)+(\alpha+\beta)\varepsilon.$$

Hence, we have

$$\widetilde{\rho}(\alpha g+\beta h) \le \alpha\widetilde{\rho}(g)+\beta\widetilde{\rho}(h).$$

Moreover, $\widetilde{\rho}$ satisfies the Δ_2–condition with $0<\kappa\le 2$. Indeed for $\varepsilon>0$ given, there exists $c>0$ such that

$$c \le \widetilde{\rho}(h)+\varepsilon; \quad \rho(h(x,y)) \le c\,\Phi(x,y).$$

Since ρ satisfies the Δ_2–condition, we have

$$\rho(2h(x,y)) \le \kappa\,\rho(h(x,y)) \le \kappa c\,\Phi(x,y),$$

therefore $\widetilde{\rho}(2h) \le \kappa c \le \kappa\,\widetilde{\rho}(h)+\kappa\,\varepsilon$. Thus $\widetilde{\rho}$ satisfies the Δ_2–condition.

Let $\{h_n\}$ be a $\widetilde{\rho}$–Cauchy sequence in $\mathcal{M}_{\widetilde{\rho}}$ and let $\varepsilon>0$ be given. There exists a positive integer $n_0 \in \mathbb{N}$ such that $\widetilde{\rho}(h_n-h_m) \le \varepsilon$ for all $n, m \ge n_0$. Now by considering the definition of the modular $\widetilde{\rho}$, we see that

$$\rho(h_n(x,y)-h_m(x,y)) \le \varepsilon\Phi(x,y) \tag{2.7}$$

for all $x, y \in E$ and $n, m \geq n_0$. If x and y are arbitrary given points of E, (2.7) implies that $\{h_n(x,y)\}$ is a ρ–Cauchy sequence in X_ρ. Since X_ρ is ρ–complete, so $\{h_n(x,y)\}$ is ρ–convergent in X_ρ, for all $x, y \in E$. Hence, we can define a function $h : E \times E \to X_\rho$ by

$$h(x,y) = \lim_{n\to\infty} h_n(x,y),$$

for any x and $y \in E$. Let m increase to infinity, then (2.7) implies that

$$\widetilde{\rho}(h_n - h) \leq \varepsilon$$

for all $n \geq n_0$, since ρ has the Fatou property. Thus $\{h_n\}$ is $\widetilde{\rho}$–convergent sequence in $\mathcal{M}_{\widetilde{\rho}}$. Therefore $\mathcal{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$–complete.

Now, we consider the function $\mathcal{T} : \mathcal{M}_{\widetilde{\rho}} \to \mathcal{M}_{\widetilde{\rho}}$ defined by

$$\mathcal{T}h(x,y) := a^{-5}h(ax,ay)$$

for all $h \in \mathcal{M}_{\widetilde{\rho}}$. Let $g, h \in \mathcal{M}_{\widetilde{\rho}}$ and let $c \in [0,\infty]$ be an arbitrary constant with $\widetilde{\rho}(g-h) \leq c$. From the definition of $\widetilde{\rho}$, we have

$$\rho(g(x,y) - h(x,y)) \leq c\,\Phi(x,y)$$

for all $x, y \in E$. By the assumption and the last inequality, we get

$$\begin{aligned}
\rho\left(\mathcal{T}g(x,y) - \mathcal{T}h(x,y)\right) &= \rho\left(a^{-5}g(ax,ay) - a^{-5}h(ax,ay)\right) \\
&\leq \frac{1}{a^5}\rho(g(ax,ay) - h(ax,ay)) \\
&\leq \frac{1}{a^5}c\Phi(ax,ay) \\
&\leq Lc\Phi(x,y),
\end{aligned}$$

for all $x \in E$. Hence, $\widetilde{\rho}(\mathcal{T}g - \mathcal{T}h) \leq L\widetilde{\rho}(g-h)$, for all $g, h \in \mathcal{M}_{\widetilde{\rho}}$ that is, $\mathcal{T}$ is a $\widetilde{\rho}$–strict contraction. We show that the $\widetilde{\rho}$–strict mapping $\mathcal{T}$ satisfies the conditions of Theorem 3.4 of [35].

By replacing x, y by ax, ay in (2.6), we get

$$\rho(a^{-5}f(a^2x,a^2y) - f(ax,ay)) \leq \frac{1}{2a^5}\psi(a^2x,ay,0) + \frac{1}{2a^2}\phi(ax,0,ay), \tag{2.8}$$

and so

$$\rho(a^{-2(5)}f(a^2x,a^2y) - a^{-5}f(ax,ay)) \leq \frac{1}{2a^{2(5)}}\psi(a^2x,ay,0) + \frac{1}{2a^{2+5}}\phi(ax,0,ay),$$

for all $x, y \in E$. Since ρ is convex modular which satisfies the Δ_2–condition, and

$\kappa/2 \leq 1$, by (2.6) and last inequality we obtain

$$\begin{aligned}
&\rho(a^{-2(5)}f(a^2x,a^2y)-f(x,y))\\
&\leq \frac{1}{2}\rho(2a^{-2(5)}f(a^2x,a^2y)-2a^{-5}f(ax,ay))+\frac{1}{2}\rho(2a^{-5}f(ax,ay)-2f(x,y))\\
&\leq \frac{\kappa}{2}\rho(a^{-2(5)}f(a^2x,a^2y)-a^{-5}f(ax,ay))+\frac{\kappa}{2}\rho(a^{-5}f(ax,ay)-f(x,y))\\
&\leq \left\{\frac{1}{2a^{2(5)}}\psi(a^2x,ay,0)+\frac{1}{2a^5}\psi(ax,y,0)\right\}+\left\{\frac{1}{2a^{2+5}}\phi(ax,0,ay)+\frac{1}{2a^2}\phi(x,0,y)\right\},
\end{aligned}$$

for all $x,y \in E$. By mathematical induction, we can easily see that

$$\begin{aligned}
&\rho\left(\frac{f(a^nx,a^ny)}{a^{5n}}-f(x,y)\right)\\
&\leq \sum_{i=1}^{n}\frac{1}{2a^{5i}}\psi(a^ix,a^{i-1}y,0)+\sum_{i=1}^{n}\frac{1}{2a^{2+5(i-1)}}\phi(a^{i-1}x,0,a^{i-1}y)\\
&\leq \psi(ax,y,0)\sum_{i=1}^{n}\frac{a^{5(i-1)}}{2a^{5i}}L^{i-1}+\phi(x,0,y)\sum_{i=1}^{n}\frac{a^{5(i-1)}}{2a^{2+5(i-1)}}L^{i-1}\\
&\leq \frac{1}{2(1-L)}\Phi(x,y) \qquad (2.9)
\end{aligned}$$

for all $x,y \in E$. Next, we assert that $\delta_{\widetilde{\rho}}(f)=\sup\{\widetilde{\rho}(\mathcal{T}^n(f)-\mathcal{T}^m(f))\,;n,m\in\mathbb{N})\}<\infty$. It follows from inequality (2.9) that

$$\begin{aligned}
&\rho\left(\frac{f(a^nx,a^ny)}{a^n}-\frac{f(a^mx,a^my)}{a^m}\right)\\
&\leq \frac{1}{2}\rho\left(2\frac{f(a^nx,a^ny)}{a^n}-2f(x,y)\right)+\frac{1}{2}\rho\left(2\frac{f(a^mx,a^my)}{a^m}-2f(x,y)\right)\\
&\leq \frac{\kappa}{2}\rho\left(\frac{f(a^nx,a^ny)}{a^n}-f(x,y)\right)+\frac{\kappa}{2}\rho\left(\frac{f(a^mx,a^my)}{a^m}-f(x,y)\right)\\
&\leq \frac{1}{(1-L)}\Phi(x,y), \qquad (2.10)
\end{aligned}$$

for every $x,y\in E$ and $n,m\in\mathbb{N}$, which implies that

$$\widetilde{\rho}(\mathcal{T}^n(f)-\mathcal{T}^m(f))\leq\frac{1}{1-L},$$

for all $n,m\in\mathbb{N}$. By the definition of $\delta_{\widetilde{\rho}}(f)$, we have $\delta_{\widetilde{\rho}}(f)<\infty$. Lemma 3.3 of [35] shows that $\{\mathcal{T}^n(f)\}$ is $\widetilde{\rho}$–converges to $j\in\mathcal{M}_{\widetilde{\rho}}$. Since ρ has the Fatou property inequality (2.9), gives $\widetilde{\rho}(\mathcal{T}j-f)<\infty$.

If we replace m by $n+1$ in inequality (2.10), then we obtain

$$\rho\left(\frac{f(a^{n+1}x,a^{n+1}y)}{a^{n+1}}-\frac{f(a^nx,a^ny)}{a^n}\right)\leq\frac{1}{(1-L)}\Phi(x,y)$$

for all $x, y \in E$. Therefore $\widetilde{\rho}(\mathcal{T}(j) - j) \leq (1/1 - L) < \infty$. It follows from [35, Theorem 3.4] that $\widetilde{\rho}$–limit of $\{\mathcal{T}^n(f)\}$ i.e., $j \in \mathcal{M}_{\widetilde{\rho}}$ is fixed point of map $\mathcal{T}$. If we replace x_1, x_2 and y by $a^n x_1$, $a^n x_2$ and $a^n y$ in inequality (2.1), respectively, then we obtain

$$\begin{aligned}
&\rho\Big(\frac{f(a^n(ax_1+bx_2), a^n y)}{a^{5n}} + \frac{f(a^n(ax_1-bx_2), a^n y)}{a^{5n}} \\
&\qquad -2a^2\frac{f(a^n x_1, a^n y)}{a^{5n}} - 2b^2\frac{f(a^n x_2, a^n y)}{a^{5n}}\Big) \\
&\leq \frac{1}{a^{5n}}\rho\big(f(a^n(ax_1+bx_2), a^n y) + f(a^n(ax_1-bx_2), a^n y) \\
&\qquad -2a^2 f(a^n x_1, a^n y) - 2b^2 f(a^n x_2, a^n y)\big) \\
&\leq \frac{1}{a^{5n}}\phi(a^n x_1, a^n x_2, a^n y),
\end{aligned}$$

and similarly by replacing x, y_1 and y_2 by $a^n x$, $a^n y_1$ and $a^n y_2$ in inequality (2.2), respectively, we get

$$\begin{aligned}
&\rho\Big(\frac{f(a^n x, a^n(ay_1+by_2))}{a^{5n}} + \frac{f(a^n x, a^n(ay_1-by_2))}{a^{5n}} - ab^2\frac{f(a^n x, a^n(y_1+y_2))}{a^{5n}} \\
&-ab^2\frac{f(a^n x, a^n(y_1-y_2))}{a^{5n}} - 2a(a^2-b^2)\frac{f(a^n x, a^n y_1)}{a^{5n}}\Big) \leq \frac{1}{a^{5n}}\psi(a^n x, a^n y_1, a^n y_2),
\end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$. Taking the limit, we deduce that j satisfying the system (1.8), that is, j is quintic. It follows from inequality (2.9) that

$$\widetilde{\rho}(j - f) \leq \frac{1}{2(1-L)}.$$

If j^* is another fixed point of $\mathcal{T}$, then

$$\begin{aligned}
\widetilde{\rho}(j - j^*) &\leq \frac{1}{2}\widetilde{\rho}(2\mathcal{T}(j) - 2f) + \frac{1}{2}\widetilde{\rho}(2\mathcal{T}(j^*) - 2f) \\
&\leq \frac{\kappa}{2}\widetilde{\rho}(\mathcal{T}(j) - f) + \frac{\kappa}{2}\widetilde{\rho}(\mathcal{T}(j^*) - f) \leq \frac{\kappa}{2(1-L)} < \infty.
\end{aligned}$$

Since $\mathcal{T}$ is $\widetilde{\rho}$–strict contraction, we get

$$\widetilde{\rho}(j - j^*) = \widetilde{\rho}(\mathcal{T}(j) - \mathcal{T}(j^*)) \leq L\widetilde{\rho}(j - j^*),$$

which implies that $\widetilde{\rho}(j - j^*) = 0$ or $j = j^*$, since $\widetilde{\rho}(j - j^*) < \infty$. This prove the uniqueness of j. □

Corollary 2.2. *Let E be a normed space and let F be a Banach space. Suppose $f : E \times E \to F$ is a mapping with $f(0, y) = 0$ and there exist constants ϵ, ε, θ, $\vartheta \geq 0$ and $p \in [0, 5)$ such that*

$$\begin{aligned}
\|f(ax_1+bx_2, y) + f(ax_1-bx_2, y) - 2a^2 f(x_1, y) \\
-2b^2 f(x_2, y)\| &\leq \epsilon + \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \\
\|f(x, ay_1+by_2) + f(x, ay_1-by_2) - ab^2 f(x, y_1+y_2) \\
-ab^2 f(x, y_1-y_2) - 2a(a^2-b^2) f(x, y_1)\| &\leq \varepsilon + \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),
\end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$*. Then there exists a unique quintic mapping* $j : E \to F$ *such that*

$$\|f(x,y) - j(x,y)\| \leq \frac{\epsilon + \theta(\|x\|^p + \|y\|^p)}{a^2 - a^{p-3}} + \frac{\varepsilon + \vartheta(\|ax\|^p + \|y\|^p)}{a^5 - a^p},$$

for all $x, y \in E$ *and* $a \in \mathbb{Z}_+ \setminus \{1\}$*.*

Proof. It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and $\kappa = 2$. Define

$$\phi(x_1, x_2, y) = \epsilon + \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \quad \psi(x, y_1, y_2) = \varepsilon + \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),$$

and apply Theorem 2.1. □

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