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# Double Domination in the Cartesian and Tensor Products of Graphs

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Abstract. A subset $S$ of $V(G)$, where $G$ is a graph without isolated vertices, is a *double dominating set* of $G$ if for each $x \in V(G)$, $|N_G[x] \cap S| \geq 2$. This paper, shows that any positive integers $a$, $b$ and $n$ with $2 \leq a < b$, $b \geq 2a$ and $n \geq b + 2a - 2$, can be realized as domination number, double domination number and order, respectively. It also characterize the double dominating sets in the Cartesian and tensor products of two graphs and determine sharp bounds for the double domination numbers of these graphs. In particular, it show that if $G$ and $H$ are any connected non-trivial graphs of orders $n$ and $m$ respectively, then $\gamma_{\times 2}(G \Box H) \leq min\{m\gamma_2(G), n\gamma_2(H)\}$, where $\gamma_2$, is the 2-domination parameter.

## 1. Introduction

Let $G = (V(G), E(G))$ be a graph. For any vertex $x \in V(G)$, the *open neighborhood* of $x$ is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and the *closed neighborhood* of $x$ in $G$ is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of $X$ in $G$ is the set $N_G[X] = N_G(X) \cup X$.

A subset $S$ of $V(G)$ is a *dominating set* in $G$ if $N_G[S] = S \cup N_G(S) = V(G)$

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where $N_G(S) = \{v \in V(G) : xv \in E(G) \text{ for some } x \in S\}$. Equivalently, a subset $S$ of $V(G)$ is a *dominating set* in $G$ if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. The minimum cardinality of a dominating set in $G$, denoted by $\gamma(G)$, is the *domination number* of $G$. Moreover, a subset $S$ of $V(G)$ is a *2-dominating set* of the graph $G$, if every vertex $v \in V(G) \setminus S$ is adjacent to at least 2 vertices in $S$. The *2-domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of $G$. A 2-dominating set of $G$ with cardinality $\gamma_2(G)$ is called a $\gamma_2$-set.

Let $G = (V(G), E(G))$ be a graph with no isolated vertices. A subset $S$ of $V(G)$ is a *double dominating set* of $G$ if for each $x \in V(G)$, $|N_G[x] \cap S| \geq 2$. The *double domination number* of $G$, denoted by $\gamma_{\times 2}(G)$, is the minimum cardinality of a double dominating set of $G$. A double dominating set of $G$ with cardinality $\gamma_{\times 2}(G)$ is called a $\gamma_{\times 2}$-set.

Double dominating set and double domination number were first defined and introduced by F. Harary and T. W. Haynes in [4] {as cited in [5]}. They also established the Nordhaus-Gaddum inequalities for double domination. Blidia, Chellali and Haynes [2] characterized the trees having equal paired and double domination number. Atapour, Khodkar and Sheikholeslami [1] established upper bounds on the double domination subdivision number (the minimum number of edges that must be subdivided in order to increase the $\gamma_{\times 2}(G)$) for arbitrary graphs in terms of vertex degree. Khelifi et al. [6] studied the concept in relation to $\gamma_{\times 2}$-critical graphs (the removal of any edge which increases the $\gamma_{\times 2}(G)$). Cuivillas and Canoy [3] characterized and determined sharp bounds for the double dominating sets in the join, corona and lexicographic product of two graphs.

## 2. Realization Problem

It is shown in [3] that $1 + \gamma(G) \leq \gamma_{\times 2}(G)$ for any graph $G$ without isolated vertices. Also, by a remark in [3], for any graph $G$ of order $n \geq 2$ without isolated vertices, $2 \leq \gamma_{\times 2}(G) \leq n$. Thus, the following remark is immediate.

**Remark 2.1.** For any non-trivial graph $G$, $\gamma(G) < \gamma_{\times 2}(G) \leq |V(G)|$.

**Theorem 2.2.** *Given positive integers $a$, $b$ and $n$ with $2 \leq a < b$, $b \geq 2a$ and $n \geq b + 2a - 2$, there exists a connected graph $G$ with $\gamma(G) = a$, $\gamma_{\times 2}(G) = b$ and $|V(G)| = n$.*

*Proof.* Consider the following cases:
Case 1. $b = 2a$ and $n \geq b + 2a - 2$

Suppose $a = 2$. Then $b = 4$. Let $H$ be a path $[x_1, y_1, v_1, x_2]$ and let $G_1$ be a graph obtained from $H$ by adding the vertices $z_i$ for $i = 1, 2$ and the edges $z_1y_1$, $z_2v_1$, and $z_ix_i$ for $i = 1, 2$. Moreover, let $G_2$ be a graph obtained from $G_1$ by adding the paths $[x_1, u_m, z_1]$ for $m = 1, 2, ..., n$-$b$-$2a$+2 (see Figure 1). Then the set $\{x_1, x_2\}$ is a minimum dominating set of $G_1$ and $G_2$. Hence $\gamma(G_1) = \gamma(G_2) = 2 = a$. Also, a minimum double dominating set of $G_1$ and $G_2$ is the set $\{x_1, x_2\} \cup \{z_1, z_2\}$. Thus $\gamma_{\times 2}(G_1) = \gamma_{\times 2}(G_2) = 2 + 2 = 2a = b$. Now if $n = b + 2a - 2$ then $n = 6$. Thus take $G = G_1$, where $|V(G_1)| = 2+2+1+1 = n$. If not, take $G = G_2$, where $|V(G_2)| = 2+2+1+1+(n-b-2a+2) = n-b+4 = n$.

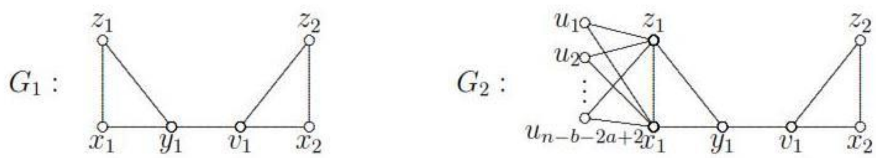


Figure 1: Graphs $G_1$ and $G_2$

Suppose now that $a > 2$. Let $P_a$ be the path $[x_1, x_2, ..., x_{a-1}, x_a]$ and let $G_3$ be the graph obtained from $P_a$ by adding the vertices $z_i$ for $i = 1, 2, ..., a$ and replacing the edges $x_i x_{i+1}$ by the paths $[x_i, y_i, v_{i+1}, x_{i+1}]$ for $i = 1, 2, ..., a-1$, then adding the edges $z_i x_i$ for $i = 1, 2, ..., a$, $z_j y_j$ for $j = 1, 2, ..., a-1$, and $z_k v_k$ for $k = 2, 3, ..., a$. Moreover, let $G_4$ be the graph obtained from $G_3$ by adding the paths $[x_1, u_m, z_1]$ for $m = 1, 2, ..., n-b-2a+2$ (see Figure 2). Then the set $\{x_1, x_2, ..., x_{a-1}, x_a\}$ is a minimum dominating set of $G_3$ and $G_4$, while a minimum double dominating set of $G_3$ and $G_4$ is the set $\{x_1, x_2, ..., x_{a-1}, x_a\} \cup \{z_1, z_2, ..., z_{a-1}, z_a\}$. Hence $\gamma(G_3) = \gamma(G_4) = a$ and $\gamma_{\times 2}(G_3) = \gamma_{\times 2}(G_4) = a + a = 2a = b$. Now if $n = b + 2a - 2$, then take $G = G_3$, where $|V(G_3)| = a+a+(a-1)+(a-1) = 4a-2 = n$. If $n > b+2a-2$, then take $G = G_4$, where $|V(G_4)| = a+a+(n-b-2a+2)+(a-1)+(a-1) = n$.

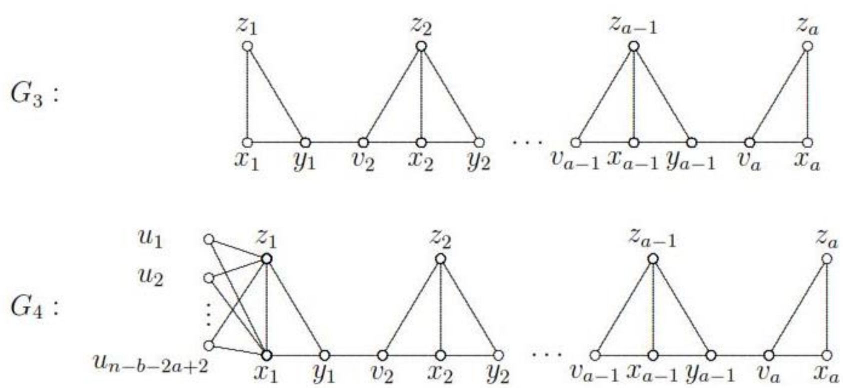


Figure 2: Graphs $G_3$ and $G_4$

Case 2. $b > 2a$ and $n \geq b + 2a - 2$

Suppose $a = 2$. Consider the graph $G_5$ obtained from $G_1$ in Figure 1 by adding the vertices $w_l$ and the edges $z_2w_l$ for $l = 1, 2, ..., b-2a$. Also, let $G_6$ be the graph obtained from $G_5$ by adding the paths $[x_1, u_m, z_1]$ for $m = 1, 2, ..., n-b-2a+2$ (see Figure 3). Then the set $\{z_1, z_2\}$ is a $\gamma$-set of $G_5$ and $G_6$. Hence, $\gamma(G_5) = \gamma(G_6) = 2 = a$. Also, a minimum double dominating set of $G_5$ and $G_6$ is the set $\{x_1, x_2\} \cup \{z_1, z_2\} \cup \{w_1, w_2, ..., w_{b-2a}\}$. Thus, $\gamma_{\times 2}(G_5) = \gamma_{\times 2}(G_6) = 2 + 2 + b - 2a = b$. Now if $n = b + 2a - 2$, then $n = b + 2$. Hence take $G = G_5$ where $|V(G_5)| = 2 + 2 + 1 + 1 + (b - 2a) = b + 2 = n$. If not, take $G = G_6$ where $|V(G_6)| = 2 + 2 + 1 + 1 + (b - 2a) + (n - b - 2a + 2) = n$.

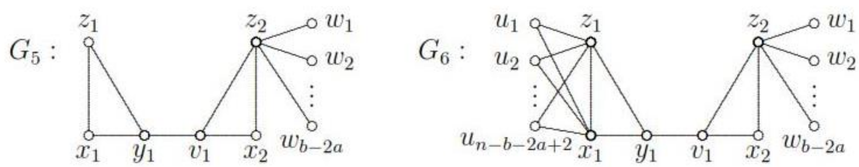


Figure 3: Graphs $G_5$ and $G_6$

Now suppose $a > 2$. Consider the graph $G_7$ obtained from $G_3$ in Figure 2 by adding the vertices $w_l$ and the edges $z_aw_l$ for $l = 1, 2, ..., b - 2a$. Moreover, let $G_8$ be the graph obtained from $G_7$ by adding the paths $[x_1, u_m, z_1]$ for $m = 1, 2, ..., n - b - 2a + 2$ (see Figure 4). Then the set $\{z_1, z_2, ..., z_{a-1}, z_a\}$ is a $\gamma$-set of $G_7$ and $G_8$ while a minimum double dominating set of $G_7$ and $G_8$ is the set $\{x_1, x_2, ..., x_{a-1}, x_a\} \cup \{z_1, z_2, ..., z_{a-1}, z_a\} \cup \{w_1, w_2, ..., w_{b-2a}\}$. Hence, $\gamma(G_7) = \gamma(G_8) = a$ and $\gamma_{\times 2}(G_7) = \gamma_{\times 2}(G_8) = a + a + (b - 2a) = b$. Now, if $n = b + 2a - 2$, then take $G = G_7$. If $n > b + 2a - 2$, then take $G = G_8$, where $|V(G_7)| = a + a + (b - 2a) + (a - 1) + (a - 1) = b + 2a - 2 = n$ and $|V(G_8)| = a + a + (n - b - 2a + 2) + (b - 2a) + (a - 1) + (a - 1) = n$.

This proves the assertion. □

## 3. Double Domination in the Cartesian Product of Graphs

The *Cartesian product* $G\Box H$ of two graphs $G$ and $H$ is the graph with vertex-set $V(G\Box H) = V(G) \times V(H)$ and edge-set $E(G\Box H)$ satisfying the following conditions: $(u, v)(u', v') \in E(G\Box H)$ if and only if either $uu' \in E(G)$ and $v = v'$ or $u = u'$ and $vv' \in V(H)$.

**Theorem 3.1.** *Let $G$ and $H$ be connected non-trivial graphs. Then $C = \bigcup_{x\in S} [\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a double dominating set of $G\Box H$ if and only if the following properties hold:*

(i) *$T_x$ is a double dominating set in $H$ for each $x \in S \setminus N(S)$;*

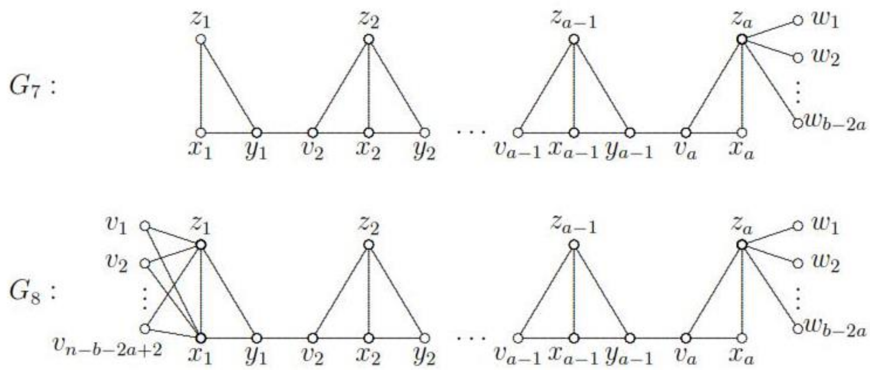


Figure 4: Graphs $G_7$ and $G_8$

(ii) *For each* $x \in S \cap N(S)$ *and for each* $a \in V(H) \setminus N_H[T_x]$, *there exist* $y, z \in N_G(x) \cap S$ *such that* $a \in T_y \cap T_z$;

(iii) *For each* $x \in S \cap N(S)$ *and for each* $a \in N_H[T_x]$, *one of the following holds:*

(a) $a \in T_x$ *and either* $|N_H[a] \cap T_x| \geq 2$ *or* $a \in T_y$ *for some* $y \in N_G(x) \cap S$;

(b) $a \notin T_x$ *and* $|N_H(a) \cap T_x| \geq 2$ *or* $a \in T_y$ *for some* $y \in N_G(x) \cap S$ *and* $|N_H(a) \cap T_x| \geq 1$ *or* $a \in T_y \cap T_z$ *for some* $y, z \in N_G(x) \cap S$ $(y \neq z)$; *and*

(iv) *For each* $x \in V(G) \setminus S$ *and for each* $a \in V(H)$, *there exist* $y, z \in N_G(x) \cap S$, *where* $y \neq z$, *such that* $a \in T_y \cap T_z$.

*Proof.* Suppose $C$ is a double dominating set of $G \Box H$. Let $x \in S \setminus N(S)$ and let $q \in V(H)$. If $q \notin T_x$, then $(x, q) \notin C$. Since $C$ is a double dominating set of $G \Box H$, there exist distinct vertices $(y, b)$ and $(z, c)$ in $C$ such that $(x, q)(y, b), (x, q)(z, c) \in E(G \Box H)$. Since $x \notin N(S)$, it follows that $x = y = z$. Thus $b, c \in N_H(q) \cap T_x$. If $q \in T_x$, then $(x, q) \in C$. Since $C$ is a total dominating set, there exist of $G \Box H$, there exists $(y, a) \in C$ such that $(x, q)(y, a) \in E(G \Box H)$. Since $x \notin N(S)$, it follows that $x = y$ and $aq \in E(H)$. Hence $a \in T_x$ and $aq \in E(H)$. In any case, $|N_H[q] \cap T_x| \geq 2$, showing that $T_x$ is a double dominating set in $H$.

Next, let $x \in S \cap N(S)$ and let $a \in V(H) \setminus N_H(T_x)$. Since $C$ is a double dominating set, there exist distinct vertices $(y, b)$ and $(z, c)$ in $C$ such that $(x, a)(y, b), (x, a)(z, c) \in E(G \Box H)$. Since $a \notin N_H(T_x)$, $a = b = c$ and $xy, xz \in E(G)$. This implies that $y, z \in N_G(x) \cap S$ and $a \in T_y \cap T_z$.

Now, let $x \in S \cap N(S)$ and let $a \in N_H[T_x]$. If $a \in T_x$, then $(x, a) \in C$. Since $C$ is a total dominating set, there exists $(y, b) \in C$ such that $(x, a)(y, b) \in E(G \Box H)$. Hence $x = y$ and $ab \in E(H)$ or $xy \in E(G)$ and $a = b$. Thus $|N_H[a] \cap T_x| \geq 2$ or

$a \in T_y$ for some $y \in N_G(x) \cap S$. Suppose $a \notin T_x$. Then $a \in N(T_x)$ and $(x,a) \notin C$. Since $C$ is a double dominating set of $G\Box H$, there exist $(y,b),(z,c) \in C$ such that $(y,b),(z,c) \in N_{G\Box H}((x,a))$. If $x = y = z$, then $b, c \in T_x \cap N_H(a)$. If either $y$ or $z$ is not $x$, say $x \neq y$, then $xy \in E(G)$ and $a = b \in T_y$. If both $y$ and $z$ is not $x$ ($y \neq z$), then $xy, xz \in E(G)$ and $a = b = c$ with $a \in T_y \cap T_z$. Thus either $|N_H(a) \cap T_x| \geq 2$ or $a \in T_y$ for some $y \in N_G(x) \cap S$ and $|N_H(a) \cap T_x| \geq 1$ or $a \in T_y \cap T_z$ for some $y, z \in N_G(x) \cap S$ ($y \neq z$)

Finally, let $x \in V(G) \setminus S$ and let $a \in V(H)$. Since $C$ is a double dominating set, there exists distinct vertices $(y,b)$ and $(z,c)$ in $C$ such that $(x,a)(y,b),(x,a)(z,c) \in E(G\Box H)$. Since $x \notin S$, $a = b = c$. It follows that $y, z \in N_G(x) \cap S$ and $a \in T_y \cap T_z$.

For the converse, suppose that $S$ is a 2-dominating set in $G$ and that properties (i), (ii), (iii), and (iv) hold. Let $(x,a) \in V(G\Box H)$. Consider the following cases:

Case 1. Suppose that $x \notin S$. Then, by $(iv)$ there exist distinct vertices $y$ and $z$ in $N_G(x) \cap S$ such that $a \in T_y \cap T_z$. It follows that $(y,a),(z,a) \in N_{G\Box H}((x,a)) \cap C$, where $(y,a) \neq (z,a)$. Thus $|N_{G\Box H}((x,a)) \cap C| \geq 2$.

Case 2. Suppose that $x \in S \setminus N(S)$. By (i), $T_x$ is a double dominating set in $H$; hence, there exist two distinct vertices $b, c \in T_x$ such that $(x,b),(x,c) \in N_{G\Box H}[(x,a)] \cap C$. Thus, $|N_{G\Box H}[(x,a)] \cap C| \geq 2$.

Case 3. Suppose that $x \in S \cap N(S)$ and $a \in V(H) \setminus N[T_x]$. Then $|N_{G\Box H}((x,a)) \cap C| \geq 2$ by (ii).

Case 4. Suppose that $x \in S \cap N(S)$ and $a \in N[T_x]$. Suppose first that $a \notin T_x$. Then there exists $b \in T_x$ such that $ab \in E(H)$. It follows that $(x,b) \in N_{G\Box H}((x,a)) \cap C$. Also, by (iii), either there exists $c \in T_x \setminus \{b\}$ or there exists $y \in N_G(x) \cap S$ such that $(y,a) \in C$. In either case, we have $|N_{G\Box H}((x,a)) \cap C| \geq 2$. Suppose now that $a \in T_x$. Then $(x,a) \in C$. By (iii), it follows that $|N_{G\Box H}[(x,a)] \cap C| \geq 2$.

Case 5. Suppose that $x \in V(G) \setminus S$ and $a \in V(H)$. Then by (iv), $|N_{G\Box H}((x,a)) \cap C| \geq 2$.

Accordingly, $C$ is a double dominating set of $G\Box H$. □

The following result is immediate.

**Corollary 3.2.** *Let $G$ and $H$ be connected non-trivial graphs of orders $n$ and $m$, respectively. Then,*

$$\gamma_{\times 2}(G\Box H) \leq min\left\{m\left[\gamma_2(G)\right], n\left[\gamma_2(H)\right]\right\}.$$

*Proof.* Let $S$ be a $\gamma_2$-set of $G$. Put $T_x = V(H)$ for each $x \in S$. Then $C = \bigcup_{x \in S}\left[\{x\} \times T_x\right]$ is a double dominating set of $G\Box H$ by Theorem 3.1. Hence,

$$\gamma_{\times 2}(G\Box H) \leq |C| = \sum_{x \in S} |T_x| = |S|\,|V(H)| = m\gamma_2(G).$$

Similarly, $\gamma_{\times 2}(G\Box H) \leq n\gamma_2(H)$. Thus,

$$\gamma_{\times 2}(G\Box H) \leq min\left\{m\left[\gamma_2(G)\right], n\left[\gamma_2(H)\right]\right\}.$$

□

**Example 3.3.** Consider the graphs $P_5 \Box P_5$ and $C_4 \Box P_3$ in Figure 5. In Figure 5$(a)$, the set $\{x_b, x_d, y_a, y_b, y_c, y_d, y_e, u_a, u_b, u_c, u_d, u_e, v_b, v_d,\}$ is a minimum double dominating set of $P_5 \Box P_5$. Thus, $\gamma_{\times 2}(P_5 \Box P_5) = 14 < 15 = m\,[\gamma_2(P_5)]$. This shows that the strict inequality in Corollary 3.2 can be attained. In Figure 5$(b)$, the set $\{x_a, x_b, x_c, z_a, z_b, z_c\}$ is a minimum double dominating set of $C_4 \Box P_3$. Hence, $\gamma_{\times 2}(C_4 \Box P_3) = 6 = m\,[\gamma_2(C_4)]$. This implies that the bound given in Corollary 3.2 is sharp.

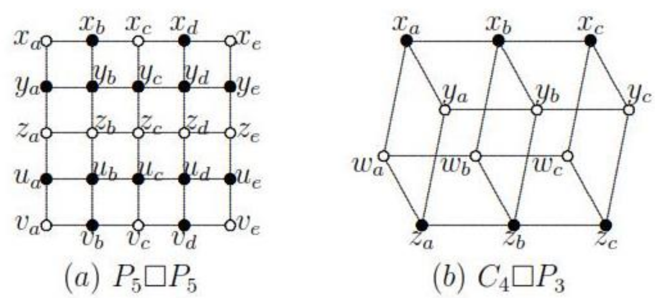


Figure 5: The Cartesian products $P_5 \Box P_5$ and $C_4 \Box P_3$

## 4. Double Domination in the Tensor Product of Graphs

The *Tensor product* $G \otimes H$ of two graphs $G$ and $H$ is the graph with vertex-set $V(G \otimes H) = V(G) \times V(H)$ and edge-set $E(G \otimes H)$ satisfying the following conditions: $(u, v)(u', v') \in E(G \otimes H)$ if and only if $uu' \in E(G)$ and $vv' \in V(H)$.

**Theorem 4.1.** *Let $G$ and $H$ be connected non-trivial graphs. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a double dominating set of $G \otimes H$ if and only if the following properties hold:*

(i) *There exists $y \in N_G(x) \cap S$ with $|N_H(p) \cap T_y| \geq 2$ or there exist $y, z \in N_G(x) \cap S$, where $y \neq z$, such that $N_H(p) \cap T_y \neq \phi$ and $N_H(p) \cap T_z \neq \phi$ whenever $x \in S$ and $p \notin T_x$ or $x \notin S$ and $p \in V(H)$; and*

(ii) *For each $x \in S$ and for each $p \in T_x$, there exists $y \in N_G(x) \cap S$ such that $|N_H(p) \cap T_y| \geq 1$.*

*Proof.* Suppose $C$ is a double dominating set of $G \otimes H$. Suppose $x \in S$ and $p \notin T_x$ (or $x \notin S$ and $p \in V(H)$). Then there exist at least two vertices $(y, q), (z, t) \in C$ such that $(x, p)(y, q), (x, p)(z, t) \in E(G \otimes H)$. Hence, $xy, xz \in E(G)$ and $pq, pt \in E(H)$. If $y = z$, then $q, t \in N_H(p) \cap T_y$, where $q \neq t$, and $y \in N_G(x) \cap S$. If $y \neq z$, then

$|N_G(x) \cap S| \geq 2$, $N_H(p) \cap T_y \neq \phi$, and $N_H(p) \cap T_z \neq \phi$. Hence, (i) holds.

Let $x \in S$ and $p \in T_x$. Since $C$ is a double dominating set, there exists $(y,q) \in C$ such that $(x,p)(y,q) \in E(G \otimes H)$. It follows that $y \in N_G(x) \cap S$ and $|N_H(p) \cap T_y| \geq 1$. This shows that (ii) holds.

For the converse, suppose that (i) and (ii) hold. Let $(x,p) \in V(G \otimes H)$ and consider the following cases:

Case 1: $(x,p) \in C$

Then $x \in S$ and $p \in T_x$. By (ii), there exists $y \in N_G(x) \cap S$ with $|N_H(p) \cap T_y| \geq 1$. Pick any $q \in N_H(p) \cap T_y$. Then $(y,q) \in C$ and $(x,p)(y,q) \in E(G \otimes H)$. It follows that $|N_{G\otimes H}[(x,p)] \cap C| \geq 2$.

Case 2: $(x,p) \notin C$

Then either $x \in S$ and $p \notin T_x$ or $x \notin S$ and $p \in V(H)$. By $(i)$, suppose there exists $y \in N_G(x) \cap S$ such that $|N_H(p) \cap T_y| \geq 2$. Pick any $t,q \in N_H(p) \cap T_y$, where $t \neq q$. Then, $(y,t),(y,q) \in C$ implying that $(x,p)(y,t),(x,p)(y,q) \in E(G \otimes H)$. Hence $|N_{G\otimes H}((x,p)) \cap C| \geq 2$. Next, suppose that there exist distinct $y,z \in N_G(x) \cap S$ with $N_H(p) \cap T_y \neq \phi$ and $N_H(p) \cap T_z \neq \phi$. Pick $q \in N_H(p) \cap T_y$ and $t \in N_H(p) \cap T_z$. Then, $(y,q)$ and $(z,t)$ are distinct elements of $N_{G\otimes H}((x,p)) \cap C$. Hence $|N_{G\otimes H}((x,p)) \cap C| \geq 2$.

Accordingly, $C$ is a double dominating set of $G \otimes H$. □

**Corollary 4.2.** *Let $G$ and $H$ be connected non-trivial graphs. If $S$ and $D$ are double dominating sets of $G$ and $H$ respectively, then $C = S \times D$ is a double dominating set of $G \otimes H$. In particular,*

$$\gamma_{\times 2}(G \otimes H) \leq \gamma_{\times 2}(G)\gamma_{\times 2}(H).$$

*Proof.* Let $T_x = D$ for each $x \in S$. Then $C = \cup_{x\in S}[\{x\} \times T_x]$. By Theorem 4.1, $C$ is a double dominating set of $G \otimes H$. Hence,

$$\begin{aligned}\gamma_{\times 2}(G \otimes H) &\leq |C| \\ &= \left|\bigcup_{x\in S}[\{x\} \times T_x]\right| \\ &= |S||D| \\ &= \gamma_{\times 2}(G)\gamma_{\times 2}(H).\end{aligned}$$

This proves the desired result. □

**Example 4.3.** Consider the graphs $P_6 \otimes P_6$ and $P_5 \otimes P_5$ in Figure 6. In Figure 6$(a)$, it can be seen that $\gamma_{\times 2}(P_6 \otimes P_6) = 20 < 25 = \gamma_{\times 2}(P_6)\gamma_{\times 2}(P_6)$. In Figure 6$(b)$, $\gamma_{\times 2}(P_5 \otimes P_5) = 16 = \gamma_{\times 2}(P_5)\gamma_{\times 2}(P_5)$. These graphs show that both the strict inequality and equality in Corollary 4.2 can be attained.

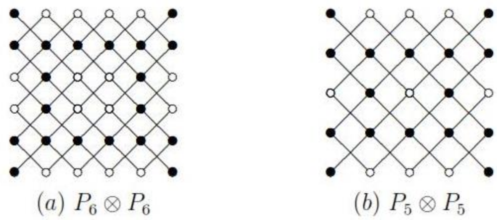


Figure 6: The tensor products of $P_6 \otimes P_6$ and $P_5 \otimes P_5$

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