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On Graded Quasi-Prime Submodules

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded quasi-prime submodules and give some basic results about graded quasi-prime submodules of graded modules. Special attention has been paid, when graded modules are graded multiplication, to find extra properties of these submodules. Furthermore, a topology related to graded quasi-prime submodules is introduced.

1. Introduction

Graded prime submodules of graded modules over graded commutative rings have been introduced and studied in [2, 5]. Here we introduce the concept of graded quasi-prime submodules and we investigate some properties of graded quasi-prime submodules of graded modules over graded commutative rings and consider some conditions under which a graded quasi-prime submodule of a graded module is graded prime. Also, the behavior of graded quasi-prime submodules under localization is studied. Furthermore, we introduce a topology on the set of graded quasi-prime submodules and some properties of this topology are given.

Before we state some results, let us introduce some notation and terminologies. Let G be a group with identity e and R be a commutative ring. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G). The elements of R_g are called homogeneous of degree g where R_g is the additive subgroup of R indexed

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by $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g \in G} R_g$. Let I be an ideal of R. Then I is called a graded ideal of (R, G) if $I = \bigoplus_{g \in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded (see Example 2.4 in [1]). Let $R = \bigoplus\limits_{g \in G} R_g$ be a G-graded ring and let I be a graded ideal of R. Then the quotient ring R/I is also a G-graded ring. Indeed, $R/I = \bigoplus\limits_{g \in G} (R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$. For the simplicity, we will denote the graded ring (R, G) by R. Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \bigoplus\limits_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup\limits_{g \in G} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \bigoplus\limits_{g \in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \bigoplus_{g \in G}(N \cap M_g)$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus\limits_{g \in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ *and* $g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus\limits_{g \in G} (S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ *and* $g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \underset{g \in G}{\cup} (S^{-1}R)_g$ and $h(S^{-1}M) = \underset{g \in G}{\cup} (S^{-1}M)_g$. Consider the graded homomorphism $\eta : M \to S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non graded case, one can prove that $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \phi$. If K is a graded submodule of an $S^{-1}R$-module $S^{-1}M$, then $K \cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. Moreover, similar to the non graded case one can prove that $S^{-1}(K \cap M) = K$. For more details, one can refer to [4].

2. Some Properties of Graded Quasi-Prime Submodules

In this section, we define the graded quasi-prime submodules and give some of their basic properties.

Definition 2.1. A proper graded submodule N of a graded R-module M is said to be a graded quasi-prime if whenever K_1 and K_2 are graded submodules of M with $K_1 \cap K_2 \subseteq N$, either $K_1 \subseteq N$ or $K_2 \subseteq N$.

The following lemma is known, but we write it here for the sake of references.

Lemma 2.2.[3, Lemma 2.1] *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I + J$ and $I \cap J$ are graded ideals.*

(ii) *If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.*

(iii) *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*

(iv) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of M.*

Recall that a proper graded submodule N of a graded R-module M is said to be graded irreducible if for each graded submodules K_1 and K_2 of M, $N = K_1 \cap K_2$ implies that either $N = K_1$ or $N = K_2$.

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded quasi-prime submodule of M, then N is a graded irreducible submodule of M.*

Proof. Assume that N is a graded quasi-prime submodule of M and K_1, K_2 are graded submodules of M such that $N = K_1 \cap K_2$. Since N is a graded quasi-prime submodule and $K_1 \cap K_2 \subseteq N$, we have either $K_1 \subseteq N$ or $K_2 \subseteq N$ and hence either $N = K_1$ or $N = K_2$. Thus N is a graded irreducible submodule. □

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and N a graded quasi-prime submodule of M. If V is a graded submodule contained in N, then N/V is a graded quasi-prime submodule of M/V.*

Proof. Let K_1 and K_2 be a graded submodules of M such that $(K_1/V) \cap (K_2/V) \subseteq N/V$. Then $K_1 \cap K_2 = (K_1+V) \cap (K_2+V) \subseteq N+V = N$. Since N is a graded quasi-prime submodule, either $K_1 \subseteq N$ or $K_2 \subseteq N$. It follows that either $K_1/V \subseteq N/V$ or $K_2/V \subseteq N/V$. Thus N/V is a graded quasi-prime submodule. □

In the following theorem, we give a characterization of graded quasi-prime submodules.

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent.*

(i) *N is a graded quasi-prime submodule of M.*

(ii) *For every pair of elements* m, $m' \in h(M)$ *such that* $mR \cap m'R \subseteq N$, *either* $m \in N$ *or* $m' \in N$.

Proof. (i) $\Rightarrow$ (ii) This follows from Lemma 2.2(2) and the definition of graded quasi-prime.

(ii) $\Rightarrow$ (i) Let K_1 and K_2 be graded submodules of M such that $K_1 \cap K_2 \subseteq N$ and $K_1 \nsubseteq N$. Then there exists an element $k_h \in (K_1 \cap h(M)) - N$. Now, let $g \in G$ and set $m = \sum_{g \in G} m_g \in K_2$. Then for all $g \in G$, $k_h R \cap m_g R \subseteq K_1 \cap K_2 \subseteq N$. By our assumption, we obtain $m_g \in N$. So $m \in N$, which indicates that $K_2 \subseteq N$. Thus N is a graded quasi-prime submodule of M. □

Recall that a graded R-module M is called graded multiplication if for each graded submodule N of M, $N = IM$ for some graded ideal I of R. One can easily show that if N is a graded submodule of a graded multiplication module M, then $N = (N :_R M)M$, (see [5, Definition 2]). Also, a proper graded submodule N of a graded R-module M is called graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, either $r \in (N :_R M)$ or $m \in N$, (see [2, Definition 2.2]). The following result provides some conditions under which a graded prime submodule is graded quasi-prime.

Theorem 2.6. *Let* R *be a* G*-graded ring,* M *a graded multiplication* R*-module and* N *a graded submodule of* M*. If* N *is a graded prime submodule of* M*, then* N *is a graded quasi-prime.*

Proof. Assume that N is a graded prime and let K_1, K_2 be graded submodules of M such that $K_1 \cap K_2 \subseteq N$ but $K_1 \nsubseteq N$ and $K_2 \nsubseteq N$. Since M is a graded multiplication, $K_1 = J_1 M$ and $K_2 = J_2 M$ for some graded ideals J_1 and J_2 of R. So there are $j_1 \in J_1 \cap h(R)$, $j_2 \in J_2 \cap h(R)$ and $m_1, m_2 \in h(M)$ such that $j_1 m_1 \notin N$ and $j_2 m_2 \notin N$. Since N is a graded prime submodule and $j_1 j_2 m_1 \in K_1 \cap K_2 \subseteq N$, we conclude that $j_2 \in (N :_R M)$, i.e., $j_2 M \subseteq N$. So $j_2 m_2 \in N$, a contradiction. Thus N is graded quasi-prime. □

Lemma 2.7. *Let* R *be a* G*-graded ring and* M *a faithful graded multiplication* R*-module. Then* $\bigcap_{\alpha \in \Delta} (I_\alpha M) = (\bigcap_{\alpha \in \Delta} I_\alpha)M$ *where* I_α *is a graded ideal of* R*.*

Proof. See [5, Theorem 8]. □

A proper graded ideal P of a graded ring R is said to be graded quasi-prime if for graded ideals J_1 and J_2 of R, the inclusion $J_1 \cap J_2 \subseteq P$ implies that either $J_1 \subseteq P$ or $J_2 \subseteq P$.

Theorem 2.8. *Let* R *be a* G*-graded ring,* M *a faithful graded multiplication* R*-module and* N *a graded submodule of* M*. Then* N *is a graded quasi-prime submodule of* M *if and only if* $(N :_R M)$ *is a graded quasi-prime ideal of* R*.*

Proof. ($\Rightarrow$) Assume that N is a graded quasi-prime submodule. By Lemma 2.2(iii), $(N :_R M)$ is a graded ideal. Let J_1 and J_2 be graded ideals of R such that $J_1 \cap J_2 \subseteq$

$(N :_R M)$, i.e., $(J_1 \cap J_2)M \subseteq N$. By Lemma 2.7, we have $(J_1 \cap J_2)M = (J_1M) \cap (J_2M) \subseteq N$. Since N is a graded quasi-prime submodule of M, either $J_1M \subseteq N$ or $J_2M \subseteq N$ and so either $J_1 \subseteq (N :_R M)$ or $J_2 \subseteq (N :_R M)$. Thus $(N :_R M)$ is a graded quasi-prime ideal of R.
($\Leftarrow$) Assume that $(N :_R M)$ is a graded quasi-prime ideal of R and let K_1, K_2 be graded submodules of M such that $K_1 \cap K_2 \subseteq N$. Then $(K_1 \cap K_2 :_R M) \subseteq (N :_R M)$ and hence $(K_1 :_R M) \cap (K_2 :_R M) \subseteq (N :_R M)$. Since $(N :_R M)$ is a graded quasi-prime ideal of R, either $(K_1 :_R M) \subseteq (N :_R M)$ or $(K_2 :_R M) \subseteq (N :_R M)$. Since M is a graded multiplication, we conclude that either $K_1 = (K_1 :_R M)M \subseteq (N :_R M)M = N$ or $K_2 = (K_2 :_R M)M \subseteq (N :_R M)M = N$. Thus N is a graded quasi-prime submodule of M. □

The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that if r is a homogeneous element of R, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$, (see [7, Definition 2.1]). Recall that a proper graded ideal P of R is said to be a *graded prime ideal* if whenever $r, s \in h(R)$ with $rs \in P$, then either $r \in P$ or $s \in P$, (see [7].) The following theorem shows the relationship between graded prime submodules and graded quasi-prime submodules.

Theorem 2.9. *Let R be a G-graded ring, M a faithful graded multiplication R-module and N a graded submodule of M such that $Gr((N :_R M) = (N :_R M)$. Then N is a graded quasi-prime submodule if and only if it is graded prime.*

Proof. ($\Rightarrow$) Assume that N is a graded quasi-prime submodule. By Theorem 2.8, $(N :_R M)$ is a graded quasi-prime ideal of R. First, we show that $(N :_R M)$ is a graded prime ideal. Let I_1, I_2 be graded ideals of R such that $I_1I_2 \subseteq (N :_R M)$. Hence by [7, Proposition 2,4], we conclude that $I_1 \cap I_2 \subseteq Gr(I_1 \cap I_2) \subseteq Gr(I_1I_2) \subseteq Gr((N :_R M)) = (N :_R M)$. Since $(N :_R M)$ is a graded quasi-prime ideal, either $I_1 \subseteq (N :_R M)$ or $I_2 \subseteq (N :_R M)$. So $(N :_R M)$ is a graded prime ideal by [7, Proposition 1.2]. It follows that N is a graded prime submodule of M by [5, Corollary 3].
($\Leftarrow$) Theorem 2.6. □

The following results study the behavior of graded quasi-prime submodules under localization.

Theorem 2.10. *Let N be a graded submodule of a graded R-module M and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If $S^{-1}N$ is a graded quasi-prime submodule of $S^{-1}M$, then $S^{-1}N \cap M$ is a graded quasi-prime submodule of M.*

Proof. Assume that $S^{-1}N$ is a graded quasi-prime submodule and let K_1, K_2 be graded submodules of M such that $K_1 \cap K_2 \subseteq S^{-1}N \cap M$. It is easy to see that $S^{-1}K_1 \cap S^{-1}K_2 \subseteq S^{-1}N$. Since $S^{-1}N$ is a graded quasi-prime, either $S^{-1}K_1 \subseteq S^{-1}N$ or $S^{-1}K_2 \subseteq S^{-1}N$ and hence either $K_1 \subseteq S^{-1}N \cap M$ or $K_2 \subseteq S^{-1}N \cap M$. Thus $S^{-1}N \cap M$ is a graded quasi-prime submodule. □

Recall that a proper graded submodule N of a graded R-module M is said to be a graded primary submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$ (see [5, Definition 6]).

Lemma 2.11. *Let N be a graded submodule of a graded R-module M and $S \subseteq h(R)$ be a multiplicatively closed subset of R such that $Gr((N :_R M)) \cap S = \phi$. If N is a graded primary submodule of M, then $S^{-1}N \cap M = N$.*

Proof. Let $x = \sum_{g \in G} x_g \in S^{-1}N \cap M$. Then for all $g \in G$, there are elements $n_h \in N \cap h(M)$ and $s \in S$ such that $\frac{x_g}{1} = \frac{n_h}{s}$. Hence there exists $t \in S$ such that $stx_g = tn_h \in N$. Since N is a graded primary submodule and $Gr((N :_R M)) \cap S = \phi$, $x_g \in N$. So $x \in N$, which shows that $S^{-1}N \cap M \subseteq N$. The opposite inclusion is obvious. Thus $S^{-1}N \cap M = N$. □

Theorem 2.12. *Let N be a graded primary submodule of a graded R-module M and $S \subseteq h(R)$ be a multiplicatively closed subset of R such that $Gr((N :_R M)) \cap S = \phi$. If N is a graded quasi-prime submodule of M, then $S^{-1}N$ is a graded quasi-prime submodule of $S^{-1}M$.*

Proof. Assume that N is a graded quasi-prime submodule of M and let K_1, K_2 be graded submodules of $S^{-1}M$ such that $K_1 \cap K_2 \subseteq S^{-1}N$. Then $(K_1 \cap M) \cap (K_2 \cap M) \subseteq S^{-1}N \cap M$. By Lemma 2.11, $S^{-1}N \cap M = N$. Since N is a graded quasi-prime submodule, either $K_1 \cap M \subseteq N$ or $K_2 \cap M \subseteq N$. So either $K_1 = S^{-1}(K_1 \cap M) \subseteq S^{-1}N$ or $K_2 = S^{-1}(K_2 \cap M) \subseteq S^{-1}N$. Thus $S^{-1}N$ is graded quasi-prime. □

3. Topology on the Graded Quasi-Prime Submodules

In this section, we introduce a topology on the set of graded quasi-prime submodules and some properties of this topology are given.

If R is a G-graded ring and M is a graded R-module, we consider $qSpec_g(M)$ which is the set of all graded quasi-prime submodules of M. We call $qSpec_g(M)$, the graded quasi-prime spectrum of M. For each subset $A \subseteq h(M)$, let $qV_g(A) = \{P \in qSpec_g(M) : A \subseteq P\}$.

Theorem 3.1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(i) *For each subset $A \subseteq h(M)$, $qV_g(A) = qV_g(N)$, where N is the graded submodule of M generated by A.*

(ii) *$qV_g(0) = qSpec_g(M)$ and $qV_g(M) = \phi$.*

(iii) *If $\{N_\alpha\}_{\alpha \in \Delta}$ is a family of graded submodules of M, then $\bigcap_{\alpha \in \Delta} qV_g(N_\alpha) = qV_g\left(\sum_{\alpha \in \Delta} N_\alpha\right)$.*

(iv) *For every pair N and K of graded submodules of M, $qV_g(N \cap K) = qV_g(N) \cup qV_g(K)$.*

Proof. (i) – (iii) Clear.

(iv) Let N, K be any graded submodules of M and $P \in qV_g(N \cap K)$. Then $N \cap K \subseteq P$. Since P is a graded quasi-prime submodule, either $N \subseteq P$ or $K \subseteq P$, i.e., $P \in qV_g(N)$ or $P \in qV_g(K)$. Hence $qV_g(N \cap K) \subseteq qV_g(N) \cup qV_g(K)$. Other side of the inclusion is obvious. Thus $qV_g(N \cap K) = qV_g(N) \cup qV_g(K)$. □

Let $q\zeta_g(M) = \{\ qV_g(N) : N$ is a graded submodule of $M\}$. Then $q\zeta_g(M)$ contains the empty set and $qSpec_g(M)$. Also, $q\zeta_g(M)$ is closed under arbitrary intersections and finite unions. Therefore, $q\zeta_g(M)$ satisfies the axioms for the closed sets of the unique topology $q\tau_g$ on $qSpec_g(M)$. Then the topology $q\tau_g(M)$ on $qSpec_g(M)$ is called the quasi-Zariski topology. Let $X = qSpec_g(M)$. For every subset S of $h(M)$, define $X_S = X - qV_g(S)$. In particular, if $S = \{a\}$, then we denote X_S by X_a.

Theorem 3.2. *Let M be a graded R-module. Then the set $\{X_a : a \in h(M)\}$ is a basis for the quasi-Zariski topology on X.*

Proof. Let U be a non-void open subset of X. Then $U = X - qV_g(N)$ for some graded submodule N of M. Assume that N is generated by $A \subseteq h(M)$. Then $U = X - qV_g(N) = X - qV_g(A) = X - qV_g(\bigcup_{a \in A}\{a\}) = X - \bigcap_{a \in A} qV_g(a) = \bigcup_{a \in A}(X - qV_g(a)) = \bigcup_{a \in A} X_a$. □

For each graded submodule N of a graded R-module M, we consider $qGr_M(N) = \{P : P$ is a graded quasi-prime submodule of M containing $N\}$.

Lemma 3.3. *Let N be a graded submodule of a graded R-module M. Then the following hold:*

(i) $qV_g(N) = qV_g(qGr_M(N))$.

(ii) *For each graded submodule K of M, $qV_g(K) \subseteq qV_g(N)$ if and only if $qGr_M(N) \subseteq qGr_M(K)$.*

Proof. Clear □

Recall that a topological space is said to be *Noetherian* if its closed sets satisfy the descending chain condition. Also, recall that a graded R-module M is called *graded Noetherian* if it is satisfies the ascending chain condition on graded submodules of M.

Theorem 3.4. *Let R be a G-graded ring and M a graded R-module. If M is graded Noetherian, then $qSpec_g(M)$ is a Noetherian topological space.*

Proof. Let $\cdots \subseteq qV_g(N_3) \subseteq qV_g(N_2) \subseteq qV_g(N_1)$ be a descending chain of closed subsets of $qSpec_g(M)$, where $\{N_k\}_{k=1}^{\infty}$ is a family of graded submodules of M.

By Lemma 3.3, we have $qGr_M(N_1) \subseteq qGr_M(N_2) \subseteq qGr_M(N_3) \subseteq \cdots$. Since M is graded Noetherian, there exists a positive integer k such that $qGr_M(N_k) = qGr_M(N_{k+i})$ for each $i = 1, 2, 3, ...$ By Lemma 3.3, we conclude that

$$qV_g(N_k) = qV_g(qGr_M(N_k)) = qV_g(qGr_M(N_{k+i})) = qV_g(N_{k+i})$$

for all $i = 1, 2, 3 \dots$ Thus $qSpec_g(M)$ is a Noetherian topological space. □

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