KYUNGPOOK Math. J. 55(2015), 251-258
http://dx.doi.org/10.5666/KMJ.2015.55.2.251
pISSN 1225-6951 eISSN 0454-8124

Note on the Generalized Invertibility of $a - xy^*$

Fapeng Du*
Department of Mathematics, Southeast University, Nanjing, 211100, Jiangsu Province, P.R. China
Department of Mathematics, School of mathematical & Physical Sciences, Xuzhou Institute of Technology, Xuzhou, 221008, Jiangsu Province, P.R. China
e-mail : jsdfp@163.com

Yifeng Xue
Department of Mathematics, East China Normal University, Shanghai, 200241, P.R. China
e-mail : yfxue@math.ecnu.edu.cn

Abstract. Let $\mathcal{A}$ be a unital C^*–algebra, a, x and y are elements in $\mathcal{A}$. In this paper, we present the expression of the Moore–Penrose inverse and the group inverse of $a - xy^*$under the conditions $x = aa^+x, y^* = y^*a^+a$, respectively.

1. Introduction

Let $\mathcal{A}$ be a C^*–algebra with unit 1 and $a \in \mathcal{A}$. The element $b \in \mathcal{A}$ which satisfied $aba = a$ and $bab = b$ is called the generalized inverse of a, denoted by a^+. The set of the elements which have generalized inverse in $\mathcal{A}$, denoted by $Gi(\mathcal{A})$.

The Moore–Penrose inverse of a is denoted by $a^\dagger$ and is the unique element to the following equations:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

It is well known that $a \in \mathcal{A}$ has an Moore–Penrose inverse iff a is generalized invertible in $\mathcal{A}$.

---

* Corresponding Author.
Received July 25, 2013; revised January 8, 2014; accepted January 29, 2014.
2010 Mathematics Subject Classification: 15A09, 65F20.
Key words and phrases: Generalized inverse, Moore–Penrose inverse, Group inverse.
This reseach is supported by the Foundation of Xuzhou Institute of Technology (No. XKY2014207).

The group inverse of an element $a \in \mathcal{A}$ is an unique element $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad ab = ba.$$

The Moore-Penrose inverse of $a - xy^*$ has many applications in statistics, networks, optimizations etc. (see [8],[9],[11]). Many authors have been studying the expression of $(a-xy^*)^\dagger$ and get some results similar to the famous Shermen–Morrison–Woodbury (SMW) formula (see [2],[3],[4],[5],[6],[7],[10],[12]).

In this paper, we investigate the Moore–Penrose inverse of $a - xy^*$ again. We present the explicit expressions of the Moore–Penrose inverse and the group inverse of $a - xy^*$ under the conditions $x = aa^+x; y^* = y^*a^+a$, respectively. Our results are new and cover a lots of the known results.

2. Preliminaries

Let $a \in \mathcal{A}\backslash\{0\}$. We know that aa^* is a positive element and $1 + aa^*$ is always invertible in $\mathcal{A}$. So, for an idempotent element $s \in \mathcal{A}, (1-s-s^*)^2 = 1+(s-s^*)(s-s^*)^*$ is always invertible. Thus, we have the following Lemma:

Lemma 2.1.([1],[7],[13]) *Let s be an idempotent element in $\mathcal{A}$, then $1 - s - s^*$ is invertible in $\mathcal{A}$ and $o(s) = s(s + s^* - 1)^{-1}$ is a projection (i.e. $(o(s))^2 = o(s) = (o(s))^*$) and $o(s) = ss^\dagger, o(1 - s) = 1 - s^\dagger s$.*

Lemma 2.2.([1],[7],[13]) *Let $a \in \mathcal{A}\backslash\{0\}$ with $a \in Gi(\mathcal{A})$. Then*

$$a^\dagger = [1 - o(1 - a^+a)]a^+o(aa^+) = (1 - p - p^*)^{-1}a^+(1 - q - q^*)^{-1}.$$

Here, $p = a^+a, q = aa^+$.

The Lemma 2.2 shows if a^+ exists, then $a^\dagger$ exists and

$$aa^\dagger = o(aa^+) = aa^+(aa^+ + (aa^+)^* - 1)^{-1},$$
$$a^\dagger a = 1 - o(1 - a^+a) = (a^+a + (a^+a)^* - 1)^{-1}a^+a.$$

Lemma 2.3. *Let $a, b \in \mathcal{A}$. Then $1 + ab$ is invertible iff $1 + ba$ is invertible and*

$$(1 + ab)^{-1} = 1 - a(1 + ba)^{-1}b.$$

Lemma 2.4.([13]) *Let $a \in \mathcal{A}\backslash\{0\}$ with a^+ exists. Then the following conditions are equivalent:*

(1) *$a^\#$ exists.*

(2) *$aa^+ + a^+a - 1$ is invertible in $\mathcal{A}$ for some a^+.*

(3) *$a^2a^+ + 1 - aa^+$ is invertible in $\mathcal{A}$ for some a^+.*

(4) *$a^2a^+ + 1 - aa^+$ is invertible in $\mathcal{A}$ for any a^+.*

(5) $a + 1 - aa^+$ *is invertible in* $\mathcal{A}$ *for some* a^+.

(6) $a + 1 - aa^+$ *is invertible in* $\mathcal{A}$ *for any* a^+.

Proof. Noting that $a^2a^+ + 1 - aa^+ = 1 + (a-1)aa^+$ and $1 + aa^+(a-1) = a + 1 - aa^+$. Thus, by Lemma 2.3, we have (3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6). Please see [13] for the equivalence of (1) to (4). $\square$

Lemma 2.5. *Let* $a \in \mathcal{A}$ *with* $a^\#$ *exists. Then*

$$\begin{aligned} a^\# &= (1 + a - aa^+)^{-2}a \\ &= a(1 + a - a^+a)^{-2} \\ &= (1 + a - aa^+)^{-1}a(1 + a - a^+a)^{-1}. \end{aligned}$$

Proof. Let $w = a^2a^+ + 1 - aa^+$ and $b = w^{-2}a$. Noting that $a^2a^+ = aa^+w = waa^+$, we have $ab = ba, aba = a, bab = b$. That show $b = a^\#$.

Noting that

$$(1 + a - aa^+)^{-1}a = a(1 + a - a + a)^{-1},$$

by Lemma 2.3, we have

$$\begin{aligned} a^\# &= (a^2a^+ + 1 - aa^+)^{-2}a \\ &= [1 + (a-1)aa^+]^{-2}a \\ &= [1 - (a-1)(1 + a - aa^+)^{-1}aa^+]^2a \\ &= [1 - (a-1)(1 + a - aa^+)^{-1}aa^+](2 - aa^+)(1 + a - aa^+)^{-1}a \\ &= [1 - (a-1)(1 + a - aa^+)^{-1}aa^+]a(1 + a - a^+a)^{-1} \\ &= a(1 + a - a^+a)^{-2} \\ &= (1 + a - aa^+)^{-2}a \\ &= (1 + a - aa^+)^{-1}a(1 + a - a^+a)^{-1}. \end{aligned}$$ $\square$

3. Main Results

Let $a, x, y \in \mathcal{A}$ with $a \in Gi(\mathcal{A})$. We set $e_a = 1 - aa^+, f_a = 1 - a^+a, z = 1 - y^*a^+x, u = e_ax, v = y^*f_a$ throughout this section.

Proposition 3.1. *Let* $a, x, y \in \mathcal{A}$ *with* $a \in Gi(\mathcal{A})$ *and* $u = e_ax = 0, v = y^*f_a = 0$. *If* z^+ *exists, then* $(a - xy^*)^+$ *exists and*

$$(a - xy^*)^+ = a^+ + a^+x(z^+ - f_ze_z)y^*a^+.$$

Proof. Let $\Lambda = a^+ + a^+x(z^+ - f_ze_z)y^*a^+$. Noting that $aa^+x = x, y^*a^+a = y^*$ and

$y^*a^+x = 1 - z$, we have

$$\begin{aligned}(a - xy^*)\Lambda &= (a - xy^*)(a^+ + a^+x(z^+ - f_z e_z)y^*a^+)\\ &= aa^+ - xe_z y^*a^+,\\ \Lambda(a - xy^*) &= (a^+ + a^+x(z^+ - f_z e_z)y^*a^+)(a - xy^*)\\ &= a^+a - a^+xf_z y^*\end{aligned}$$

and $(a - xy^*)\Lambda(a - xy^*) = (a - xy^*), \Lambda(a - xy^*)\Lambda = \Lambda.$ □

Corollary 3.2. *Let $x, y \in \mathcal{A}$. Then $(1 - xy^*)^+$ exists iff $(1 - y^*x)^+$ exists and*

$$(1-xy^*)^+ = 1+x\{(1-y^*x)^+ - [1-(1-y^*x)^+(1-y^*x)][1-(1-y^*x)(1-y^*x)^+]\}y^*.$$

Theorem 3.3. *Let $a, x, y \in \mathcal{A}$ with $a \in Gi(\mathcal{A})$ and $u = e_a x = 0, v = y^*f_a = 0$. If $z^\dagger$ exists, then $(a - xy^*)^\dagger$ exists and*

$$\begin{aligned}(a - xy^*)^\dagger = \{1 - (a^\dagger xf_z y^*) - (a^\dagger xf_z y^*)^*\}^{-1}\{a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger\}\\ \times \{1 - (xe_z y^*a^\dagger) - (xe_z y^*a^\dagger)^*\}^{-1}.\end{aligned}$$

In addition, if $(a^\dagger xf_z y^*)^\dagger, (xe_z y^*a^\dagger)^\dagger$ exist, then

$$\begin{aligned}(a - xy^*)^\dagger = \{1 - (a^\dagger xf_z y^*)(a^\dagger xf_z y^*)^\dagger\}\{a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger\}\\ \times \{1 - (xe_z y^*a^\dagger)^\dagger(xe_z y^*a^\dagger)\}.\end{aligned}$$

Proof. Using Proposition 3.1 and Lemma 2.2 and $(2a^\dagger a - 1)^2 = (2aa^\dagger - 1)^2 = 1$, we have

$$\begin{aligned}(a - xy^*)^\dagger = \{1 - (a^\dagger xf_z y^*) - (a^\dagger xf_z y^*)^*\}^{-1}\{a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger\}\\ \times \{1 - (xe_z y^*a^\dagger) - (xe_z y^*a^\dagger)^*\}^{-1}.\end{aligned}$$

Noting that

$$(1 - a^\dagger xf_z y^*)[a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger](1 - xe_z y^*a^\dagger) = a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger,$$

by Lemma 2.1, we get

$$\begin{aligned}(a - xy^*)^\dagger = \{1 - (a^\dagger xf_z y^*)(a^\dagger xf_z y^*)^\dagger\}\{a^\dagger + a^\dagger x(z^\dagger - f_z e_z)y^*a^\dagger\}\\ \times \{1 - (xe_z y^*a^\dagger)^\dagger(xe_z y^*a^\dagger)\}.\end{aligned}$$

□

Corollary 3.4. *Let $a, x, y \in \mathcal{A}$ with $u = e_a x = 0, v = y^*f_a = 0$ and $z = 1 - y^*a^\dagger x$. Then*

(1) *If a is invertible and $z \in Gi(\mathcal{A})$, then*

$$(a-xy^*)^\dagger = \{1-(a^{-1}xf_zy^*)-(a^{-1}xf_zy^*)^*\}^{-1}\{a^{-1}+a^{-1}x(z^\dagger-f_ze_z)y^*a^{-1}\} \times \{1-(xe_zy^*a^{-1})-(xe_zy^*a^{-1})^*\}^{-1}.$$

(2) *If $a = 1$ and $z \in Gi(\mathcal{A})$, then*

$$(1-xy^*)^\dagger = \{1-xf_zy^*-(xf_zy^*)^*\}^{-1}(1+x(z^\dagger-f_ze_z)y^*) \times \{1-xe_zy^*-(xe_zy^*)^*\}^{-1}.$$

(3) *If a and z are invertible, then*

$$(a-xy^*)^{-1} = a^{-1}+a^{-1}xz^{-1}y^*a^{-1}.$$

This is the famous SMW formula.

(4) *If a is invertible and $z = 0$, then*

$$(a-xy^*)^\dagger = \{1-a^{-1}xy^*-(a^{-1}xy^*)^*\}^{-1}(a^{-1}-a^{-1}xy^*a^{-1}) \times \{1-xy^*a^{-1}-(xy^*a^{-1})^*\}^{-1}.$$

This is better then the result of Theorem 3.5 in [3], since there is only true inverse in this formula.

(5) *If $u = v = 0$ and $z = 0$, then*

$$\begin{aligned}(a-xy^*)^\dagger &= \{1-a^\dagger xy^*-(a^\dagger xy^*)^*\}^{-1}(a^\dagger-a^\dagger xy^*a^\dagger) \\ &\quad\times\{1-xy^*a^\dagger-(xy^*a^\dagger)^*\}^{-1} \\ &= (1-(a^\dagger xy^*)(a^\dagger xy^*)^\dagger)a^\dagger(1-(xy^*a^\dagger)^\dagger(xya^\dagger)) \\ &= (1-(a^\dagger x)(a^\dagger x)^\dagger)a^\dagger(1-(y^*a^\dagger)^\dagger(y^*a^\dagger)).\end{aligned}$$

This is the result of Theorem 2.2 in [7].
Especially, when $a = 1$, we get the result of Theorem 3.3 in [3], i.e.,

$$(1-xy^*)^\dagger = (1-xx^\dagger)(1-yy^\dagger).$$

(6) *If $u = v = 0$ and $e_zy^* = 0, xf_z = 0$, then*

$$(a-xy^*)^\dagger = a^\dagger+a^\dagger xz^\dagger y^*a^\dagger.$$

This is the result of Theorem 2.2 in [4].

(7) *If $u = v = 0$ and z is invertible, then*

$$(a-xy^*)^\dagger = a^\dagger+a^\dagger xz^{-1}y^*a^\dagger.$$

This result has been proved by Du and Xue in [7, Proposition2.1] and Deng in [4, Corllary2.3], respectively.

Example 3.5. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Then

$$A^\dagger = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Z^\dagger = (I - Y^*A^\dagger X)^\dagger = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$F_Z = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad E_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(A^\dagger X F_Z Y^*)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & \frac{4}{5} \end{pmatrix}, \qquad (X E_Z Y^* A^\dagger)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $U = E_A X = 0, V = Y^* F_A = 0$. So by Theorem 3.4,

$$(A - XY^*)^\dagger = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.6. *Let $a, x, y \in \mathcal{A}$ with $u = e_a x = 0, v = y^* f_a = 0$ and $z = 1 - y^* a^+ x$. If $(a - xy^*)^\#$ exists, then*

$$(a - xy^*)^\# = (1 + a - aa^+ - xy^* + xe_z y^* a^+)^{-1}(a - xy^*)(1 + a - a^+ a - xy^* + a^+ x f_z y^*)^{-1}.$$

Proof. Using Proposition 3.1 and Lemma 2.5, we have

$$\begin{aligned} (a - xy^*)^\# &= [1 + (a - xy^*) - (a - xy^*)(a - xy^*)^+]^{-1}(a - xy^*) \\ &\quad \times [1 + (a - xy^*) - (a - xy^*)^+(a - xy^*)]^{-1} \\ &= (1 + a - aa^+ - xy^* + xe_z y^* a^+)^{-1}(a - xy^*) \\ &\quad \times (1 + a - a^+ a - xy^* + a^+ x f_z y^*)^{-1}. \end{aligned}$$ □

Corollary 3.7. *Let $a, x, y \in \mathcal{A}$ with $u = e_a x = 0, v = y^* f_a = 0$ and $z = 1 - y^* a^+ x$. Assume that $(a - xy^*)^\#$ exists.*

(1) *If $z = 0$, then*

$$\begin{aligned} (a - xy^*)^\# = (1 + a - aa^+ - xy^* + xy^* a^+)^{-1}(a - xy^*) \\ \times (1 + a - a^+ a - xy^* + a^+ xy^*)^{-1}. \end{aligned}$$

(2) *If a is invertible, then*

$$(a - xy^*)^\# = (a - xy^* + xe_z y^* a^{-1})^{-1}(a - xy^*)(a - xy^* + a^{-1}xf_z y^*)^{-1}.$$

(3) *If $a = 1$, then*

$$(1 - xy^*)^\# = (1 - xy^* + xe_z y^*)^{-1}(1 - xy^*)(1 - xy^* + xf_z y^*)^{-1}.$$

In addition, if $z = 0$, then $1 - xy^*$ is an idempotent element in $\mathcal{A}$.

Example 3.8. Let $X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then

$$Z = I - Y^*X = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \qquad Z^+ = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix},$$

$$E_Z = I - ZZ^+ = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_Z = I - Z^+Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Corollary 3.8 (3), we have

$$(I - XY^*)^\# = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

References

[1] G. Chen, Y. Xue, *The expression of generalized inverse of the perturbed operators under type I perturbation in Hilbert spaces*, Linear Algebra Appl., **285**(1998), 1–6.

[2] Y. Chen, X. Hu, Q. Xu, *The Moore–Penrose inverse of $A - XY^*$*, J. Shanghai Normal Univer., **38**(2009), 15–19.

[3] C. Deng, *On the invertibility of the operator $A - XB$*, Numb. Linear Algebra Appl., **16**(2009), 817–831.

[4] C. Deng, *A generalization of the Sherman–Morrison–Woodbury formula*, Appl. Math. Lett., **24**(2011), 1561–1564.

[5] C. Deng, *On Moore–Penrose inverse of a kind of operators*, Proceedings of the Ninth International Conference on Matrix Theory and Its Applications in China, (2010), 88–91.

[6] C. Deng, Y. Wei, *Some New Results of the Sherman–Morrison–Woodbury Formula*, Proceeding of The Sixth Iternational Conference of Matrices and Operators, **2**(2011), 220–223.

[7] F. Du, Y. Xue, *The expression of the Moore–Penrose inverse of* $A - XY^*$, J. East China Normal Univ. (Nat. Sci.), **5**(2010), 33–37.

[8] H. V. Hsnderson, Searl S. R., *On deriving the inverse of a sum of matrices*, Siam Review, **23(1)**(1981), 53–60.

[9] W. W. Hager, *Updating the inverse of a matrix*, Siam Review, **31**(1989), 221–239.

[10] Shani Jose, K. C. Sivakumar, *Moore–Penrose Inverse of Perturbed Operators on Hilbert Spaces*, Combinatorial Matrix Theory and Generalized Inverses of Matrices, (2013), 119–131.

[11] S. Kurt, A. Riedel, *A Shermen-Morrison-Woodbury identity for rank augmenting matrices with application to centering*, Siam J. Math. Anal., **12(1)**(1991), 80–95.

[12] T. Steerneman, F. P. Kleij, *Properties of the matrix* $A - XY^*$, Linear Algebra Appl., **410**(2005), 70–86.

[13] Y. Xue, Stable Perturbations of Operators and Related Topics, World Scientific, (2012).