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Studying Solutions of a System of PDE Through Representations of D_4

Sarawut Saenkarun
Department of Mathematics, Ubonrachathani University, Thailand
e-mail : scsarasa@ubu.ac.th

Abstract. This paper is concerned with applications of representations of the Lie group of class D_4 to PDE. A realization of all irreducible finite-dimensional representations of D_4 is found and their application to a study of solutions of some systems of partial differential equations is given.

1. Introduction

Consider the system of four partial differential equations

$$\begin{aligned} \left[\frac{\partial}{\partial x_1}\right]^{p+1}\varphi &= 0, \\ \left[x_1\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}\right]^{q+1}\varphi &= 0, \\ \left[x_2\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_5} + x_7\frac{\partial}{\partial x_8} - x_9\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{11}}\right]^{s+1}\varphi &= 0, \\ \left[x_2\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5} + x_7\frac{\partial}{\partial x_9} - x_8\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}\right]^{t+1}\varphi &= 0, \end{aligned} \tag{1.1}$$

where p, q, s and t are non-negative integers. We will find all solutions of the system by examining the Lie algebra of differential operators generated by the linear differential operators

$$\begin{aligned} A &= \frac{\partial}{\partial x_1}, \\ B &= x_1\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}, \\ C &= x_2\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_5} + x_7\frac{\partial}{\partial x_8} - x_9\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{11}}, \\ D &= x_2\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5} + x_7\frac{\partial}{\partial x_9} - x_8\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}. \end{aligned}$$

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The system can be written as

$$\begin{aligned} A^{p+1}\varphi &= 0, \\ B^{q+1}\varphi &= 0. \\ C^{s+1}\varphi &= 0, \\ D^{t+1}\varphi &= 0. \end{aligned}$$

Let us consider the Lie algebra of differential operators generated by A, B, C and D. We define $E = [B, C] = BC - CB$, $F = [B, D]$, $G = [E, D]$, $H = [A, B]$, $I = [A, E]$, $J = [A, F]$, $K = [D, I]$ and $L = [B, K]$. The operators A, B, C, D, E, F, G, H, I, J, K, L form a basis of a twelve-dimensional nilpotent Lie algebra $\mathbb{Z}$ which turns out to be isomorphic to a maximal nilpotent subalgebra of the simple Lie algebra of class D_4. This property of the differential operators A, B, C and D turns out to be useful for studying the solutions of the system of PDE (1.1) through an examination of the representations of the Lie group D_4. It turns out that the operators A^{p+1}, B^{q+1}, C^{s+1} and D^{t+1} are intertwining operators for some pairs of representations of the group D_4, and so the space of solutions of the system of PDE (1.1) has the structure of an irreducible finite-dimensional representation of the group D_4.

We utilize the construction of the Lie algebra of class D_4 given in [1]. To do this we first remind the definition of the split Cayley algebra [2]. Following Zorn, the definition of the split Cayley algebra or vector matrix-algebra $\mathfrak{C}$ is as follows. Let V be the three-dimensional vector algebra over the field $\mathbb{R}$. Thus V has a basis i, j, k over $\mathbb{R}$ and has a bilinear scalar multiplication and a skew symmetric vector multiplication $\times$ satisfying the following conditions: let i, j, k be orthogonal unit vectors and set $i \times i = j \times j = k \times k = 0, i \times j = k,\ j \times k = i,\ k \times i = j$.

Let $\mathfrak{C}$ be the set of 2×2 matrices of the form $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R},\ a, b \in V$. Addition and multiplication by elements of $\mathbb{R}$ are as usual, so that $\mathfrak{C}$ is an eight-dimensional vector space. Define an algebra product in $\mathfrak{C}$ by $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma - (a, d) & \alpha c + \delta a + b \times d \\ \gamma b + \beta d + a \times c & \beta\delta - (b, c) \end{pmatrix}$. The split Cayley algebra is defined to be $\mathfrak{C}$ together with the vector space operations and multiplication defined above. $\mathfrak{C}$ is not associative but satisfies a weakening of the associative law called the *alternative law*: $x^2 y = x(xy), yx^2 = (yx)x$.

Let $\mathfrak{D}$ be the Lie algebra of all derivations of $\mathfrak{C}$. The subspace $\mathfrak{C}_0$ of all elements of trace 0 coincides with the space spanned by the commutators $[xy] = xy - yx$, $x, y \in \mathfrak{C}$. Hence $D\mathfrak{C}_0 \subseteq \mathfrak{C}_0$ for any derivation D. Thus $\mathfrak{C}_0$ is a seven-dimensional subspace of $\mathfrak{C}$, which is invariant under $\mathfrak{D}$. The representation of $\mathfrak{D}$ in $\mathfrak{C}_0$ is faithful and irreducible.

If T is a linear transformation of trace 0 in V, and T^* is its adjoint relative to the scalar multiplication, then it can be verified that $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \to \begin{pmatrix} 0 & Ta \\ -T^*b & 0 \end{pmatrix}$ is a derivation in $\mathfrak{C}$. The set of these derivations is a subalgebra $\mathfrak{D}_{\mathfrak{o}}$ of class A_2. In any alternative algebra (algebra with alternative law), any mapping of the form $D_{a,b} = [a_L, b_L] + [a_L, b_R] + [a_R, b_R]$, where a, b are in the algebra and a_L, a_R denote the left and right multiplications ($x \to ax$, $x \to xa$) determined by a, is a derivation. In $\mathfrak{C}$, any derivation has the form $D_{a_{12},e_1} + D_{b_{21},e_2} + D_0$, where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $a_{12} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $b_{21} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$, and $D_0 \in \mathfrak{D}_0$.

Let $\mathfrak{g}$ be the Lie algebra of all derivations and all left and right multiplications s_L and t_R of elements s and t of trace 0 of $\mathfrak{C}$.

2. Identification of $\mathfrak{g}$ with an Algebra of Matrices

We choose a basis of derivations in the split Cayley algebra as follows: In $\mathfrak{D}_0$, if $D_0 \in \mathfrak{D}_0$ corresponds to $S \in \mathfrak{sl}(3, \mathbb{R})$, we shall denote D_0 by S.

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, S_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, S_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly to D_0, for $D_{e_1,a_{12}}$ we choose

$$S_9 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, S_{10} = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, S_{11} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix},$$

for $D_{e_2,b_{21}}$,

$$S_{12} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, S_{13} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}, S_{14} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix},$$

for s_L,

$$S_{15} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}_L, S_{16} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}_L, S_{17} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}_L, S_{18} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_L,$$

$$S_{19} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}_L, S_{20} = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}_L, S_{21} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}_L,$$

and for t_R,

$$S_{22} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}_R, S_{23} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}_R, S_{24} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}_R, S_{25} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_R,$$

$$S_{26} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}_R, S_{27} = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}_R, S_{28} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}_R.$$

We choose a basis for $\mathfrak{C}$ as follows:

$$\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}.$$

By direct computation with respect to this basis in $\mathfrak{C}$, $S_n (n = 1, 2, \ldots, 28)$ are represented by matrices. In what follows, we shall also denote by $\mathfrak{g}$ the Lie algebra of matrices spanned by $S_1, \ldots, S_{28}$. For our purpose, we choose a basis for $\mathfrak{g}$ as follows: $h_1 = M_{1,1}$, $h_2 = M_{2,2}$, $h_3 = M_{3,3}$, $h_4 = M_{4,4}$, $e_{\alpha_1} = M_{1,2}$, $e_{\alpha_2} = M_{2,3}$, $e_{\alpha_3} = M_{3,4}$, $e_{\alpha_4} = M_{3,5}$, $e_{\alpha_2+\alpha_3} = M_{2,4}$, $e_{\alpha_2+\alpha_4} = M_{2,5}$, $e_{\alpha_2+\alpha_3+\alpha_4} = M_{2,6}$, $e_{\alpha_1+\alpha_2} = M_{1,3}$, $e_{\alpha_1+\alpha_2+\alpha_3} = M_{1,4}$, $e_{\alpha_1+\alpha_2+\alpha_4} = M_{1,5}$, $e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} = M_{1,6}$, $e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} = M_{1,7}$, $e_{-\alpha_1} = M_{2,1}$, $e_{-\alpha_2} = M_{3,2}$, $e_{-\alpha_3} = M_{4,3}$, $e_{-\alpha_4} = M_{5,3}$, $e_{-(\alpha_2+\alpha_3)} = M_{4,2}$, $e_{-(\alpha_2+\alpha_4)} = M_{5,2}$, $e_{-(\alpha_2+\alpha_3+\alpha_4)} = M_{6,2}$, $e_{-(\alpha_1+\alpha_2)} = M_{3,1}$, $e_{-(\alpha_1+\alpha_2+\alpha_3)} = M_{4,1}$, $e_{-(\alpha_1+\alpha_2+\alpha_4)} = M_{5,1}$, $e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} = M_{6,1}$, $e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} = M_{7,1}$ where $M_{i,j}$ is the 8×8 matrix with 1 in the (i, j) position, -1 in the $(9 - j, 9 - i)$ position, and zeros elsewhere.

3. The Lie Algebra Structure in $\mathfrak{g}$

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ spanned by h_1, h_2, h_3, h_4. From Table 3.1, $\mathfrak{h}$ is a maximal abelian subspace of $\mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is simultaneously diagonable, so that $\mathfrak{h}$ is a Cartan subalgebra. Let α_1, α_2, α_3, α_4, $\alpha_2 + \alpha_3$, $\alpha_2+\alpha_4$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, $-\alpha_1$, $-\alpha_2$, $-\alpha_3$, $-\alpha_4$, $-(\alpha_2+\alpha_3, \alpha_2+\alpha_4)$, $-(\alpha_2+\alpha_3+\alpha_4)$, $-(\alpha_1 + \alpha_2)$, $-(\alpha_1 + \alpha_2 + \alpha_3)$, $-(\alpha_1 + \alpha_2 + \alpha_4)$, $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, $-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4) \in \mathfrak{h}^*$ be such that $\alpha_1(h_1) = 1$, $\alpha_1(h_2) = -1$, $\alpha_1(h_3) = 0$, $\alpha_1(h_4) = 0$, $\alpha_2(h_1) = 0$, $\alpha_2(h_2) = 1$, $\alpha_2(h_3) = -1$, $\alpha_2(h_4) = 0$, $\alpha_3(h_1) = 0$, $\alpha_3(h_2) = 0$, $\alpha_3(h_3) = 1$, $\alpha_3(h_4) = -1$, $\alpha_4(h_1) = 0$, $\alpha_4(h_2) = 0$, $\alpha_4(h_3) = 1$,

$\alpha_4(h_4) = 1$, $(\alpha_2 + \alpha_3)(h_1) = 0$, $(\alpha_2 + \alpha_3)(h_2) = 1$, $(\alpha_2 + \alpha_3)(h_3) = 0$, $(\alpha_2+\alpha_3)h_4) = -1$, $(\alpha_2+\alpha_4)(h_1) = 0$, $(\alpha_2+\alpha_4)(h_2) = 1$, $(\alpha_2+\alpha_4)(h_3) = 0$, $(\alpha_2 + \alpha_4)(h_4) = 1$, $(\alpha_2 + \alpha_3 + \alpha_4)(h_1) = 0$, $(\alpha_2 + \alpha_3 + \alpha_4)(h_2) = 1$, $(\alpha_2 + \alpha_3 + \alpha_4)(h_3) = 1$, $(\alpha_2 + \alpha_3 + \alpha_4)(h_4) = 0$, $(\alpha_1 + \alpha_2)(h_1) = 1$, $(\alpha_1+\alpha_2)(h_2) = 0$, $(\alpha_1+\alpha_2)(h_3) = -1$, $(\alpha_1+\alpha_2)(h_4) = 0$, $(\alpha_1+\alpha_2+\alpha_3)(h_1) = 1$, $(\alpha_1+\alpha_2+\alpha_3)(h_2) = 0$, $(\alpha_1+\alpha_2+\alpha_3)(h_3) = 0$, $(\alpha_1+\alpha_2+\alpha_3)(h_4) = -1$, $(\alpha_1 + \alpha_2 + \alpha_4)(h_1) = 1$, $(\alpha_1 + \alpha_2 + \alpha_4)(h_2) = 0$, $(\alpha_1 + \alpha_2 + \alpha_4)(h_3) = 0$, $(\alpha_1+\alpha_2+\alpha_4)(h_4) = 1$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_1) = 1$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_2) = 0$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_3) = 1$, $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_4) = 0$, $(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)(h_1) = 1$, $(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)(h_2) = 1$, $(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)(h_3) = 0$, $(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)(h_4) = 0$, $-\alpha_1(h_1) = -1$, $-\alpha_1(h_2) = 1$, $-\alpha_1(h_3) = 0$, $-\alpha_1(h_4) = 0$, $-\alpha_2(h_1) = 0$, $-\alpha_2(h_2) = -1$, $-\alpha_2(h_3) = 1$, $-\alpha_2(h_4) = 0$, $-\alpha_3(h_1) = 0$, $-\alpha_3(h_2) = 0$, $-\alpha_3(h_3) = -1$, $-\alpha_3(h_4) = 1$, $-\alpha_4(h_1) = 0$, $-\alpha_4(h_2) = 0$, $-\alpha_4(h_3) = -1$, $-\alpha_4(h_4) = -1$, $-(\alpha_2 + \alpha_3)(h_1) = 0$, $-(\alpha_2 + \alpha_3)(h_2) = -1$, $-(\alpha_2 + \alpha_3)(h_3) = 0$, $-(\alpha_2 + \alpha_3)h_4) = 1$, $-(\alpha_2 + \alpha_4)(h_1) = 0$, $-(\alpha_2+\alpha_4)(h_2) = -1$, $-(\alpha_2+\alpha_4)(h_3) = 0$, $-(\alpha_2+\alpha_4)(h_4) = -1$, $-(\alpha_2+\alpha_3+\alpha_4)(h_1) = 0$, $-(\alpha_2+\alpha_3+\alpha_4)(h_2) = -1$, $-(\alpha_2+\alpha_3+\alpha_4)(h_3) = -1$, $-(\alpha_2 + \alpha_3 + \alpha_4)(h_4) = 0$, $-(\alpha_1 + \alpha_2)(h_1) = -1$, $-(\alpha_1 + \alpha_2)(h_2) = 0$, $-(\alpha_1 + \alpha_2)(h_3) = -1$, $-(\alpha_1 + \alpha_2)(h_4) = 0$, $-(\alpha_1 + \alpha_2 + \alpha_3)(h_1) = -1$, $-(\alpha_1+\alpha_2+\alpha_3)(h_2) = 0$, $-(\alpha_1+\alpha_2+\alpha_3)(h_3) = 0$,$-(\alpha_1+\alpha_2+\alpha_3)(h_4) = 1$, $-(\alpha_1+\alpha_2+\alpha_4)(h_1) = -1$, $-(\alpha_1+\alpha_2+\alpha_4)(h_2) = 0$, $-(\alpha_1+\alpha_2+\alpha_4)(h_3) = 0$, $-(\alpha_1+\alpha_2+\alpha_4)(h_4) = -1$, $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_1) = -1$, $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(h_2) = 0$, $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(h_3) = 1$, $-(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(h_4) = 0$, $-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)(h_1) = -1$, $-(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)(h_2) = -1$, $-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)(h_3) = 0$, $-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)(h_4) = 0$, then root spaces $\mathfrak{g}_{\alpha_1}$, $\mathfrak{g}_{\alpha_2}$, $\mathfrak{g}_{\alpha_3}$, $\mathfrak{g}_{\alpha_4}$, $\mathfrak{g}_{\alpha_2+\alpha_3}$, $\mathfrak{g}_{\alpha_2+\alpha_4}$, $\mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4}$, $\mathfrak{g}_{\alpha_1+\alpha_2}$, $\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$, $\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_4}$, $\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$, $\mathfrak{g}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$, $\mathfrak{g}_{-\alpha_1}$, $\mathfrak{g}_{-\alpha_2}$, $\mathfrak{g}_{-\alpha_3}$, $\mathfrak{g}_{-\alpha_4}$, $\mathfrak{g}_{-(\alpha_2+\alpha_3)}$, $\mathfrak{g}_{-(\alpha_2+\alpha_4)}$, $\mathfrak{g}_{-(\alpha_2+\alpha_3+\alpha_4)}$, $\mathfrak{g}_{-(\alpha_1+\alpha_2)}$, $\mathfrak{g}_{-(\alpha_1+\alpha_2+\alpha_3)}$, $\mathfrak{g}_{-(\alpha_1+\alpha_2+\alpha_4)}$,

Table 3.1 Commutator table for $\mathfrak{g}$

	h_1	h_2	h_3	h_4	e_{α_1}
h_1	0	0	0	0	e_{α_1}
h_2	0	0	0	0	$-e_{\alpha_1}$
h_3	0	0	0	0	0
h_4	0	0	0	0	0
e_{α_1}	$-e_{\alpha_1}$	e_{α_2}	0	0	0
e_{α_2}	0	$-e_{\alpha_2}$	e_{α_2}	0	$-e_{\alpha_1+\alpha_2}$
e_{α_3}	0	0	$-e_{\alpha_3}$	e_{α_3}	0
e_{α_4}	0	0	$-e_{\alpha_4}$	$-e_{\alpha_4}$	0
$e_{\alpha_2+\alpha_3}$	0	$-e_{\alpha_2+\alpha_3}$	0	$e_{\alpha_2+\alpha_3}$	$-e_{\alpha_1+\alpha_2+\alpha_3}$
$e_{\alpha_2+\alpha_4}$	0	$-e_{\alpha_2+\alpha_4}$	0	$-e_{\alpha_2+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_4}$
$e_{\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
$e_{\alpha_1+\alpha_2}$	$-e_{\alpha_1+\alpha_2}$	0	$e_{\alpha_1+\alpha_2}$	0	0
$e_{\alpha_1+\alpha_2+\alpha_3}$	$-e_{\alpha_1+\alpha_2+\alpha_3}$	0	0	$e_{\alpha_1+\alpha_2+\alpha_3}$	0
$e_{\alpha_1+\alpha_2+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_4}$	0	0	$-e_{\alpha_1+\alpha_2+\alpha_4}$	0
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0
$e_{-\alpha_1}$	$e_{-\alpha_1}$	$-e_{-\alpha_1}$	0	0	$h_2 - h_1$
$e_{-\alpha_2}$	0	$e_{-\alpha_2}$	$-e_{-\alpha_2}$	0	0
$e_{-\alpha_3}$	0	0	$e_{-\alpha_3}$	$-e_{-\alpha_3}$	0
$e_{-\alpha_4}$	0	0	$e_{-\alpha_4}$	$e_{-\alpha_4}$	0
$e_{-\alpha_2-\alpha_3}$	0	$e_{-\alpha_2-\alpha_3}$	0	$-e_{-\alpha_2-\alpha_3}$	0
$e_{-\alpha_2-\alpha_4}$	0	$e_{-\alpha_2-\alpha_4}$	0	$e_{-\alpha_2-\alpha_4}$	0
$e_{-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_2-\alpha_3-\alpha_4}$	0	0
$e_{-\alpha_1-\alpha_2}$	$e_{-\alpha_1-\alpha_2}$	0	$-e_{-\alpha_1-\alpha_2}$	0	$e_{-\alpha_2}$
$e_{-\alpha_1-\alpha_2-\alpha_3}$	$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	0	$-e_{-\alpha_1-\alpha_2-\alpha_3}$	$e_{-\alpha_2-\alpha_3}$
$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_4}$	0	0	$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_2-\alpha_4}$
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_2-\alpha_3-\alpha_4}$
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0	0

	e_{α_2}	e_{α_3}	e_{α_4}	$e_{\alpha_2+\alpha_3}$	$e_{\alpha_2+\alpha_4}$
h_1	e_{α_2}	0	0	0	0
h_2	e_{α_2}	0	0	$e_{\alpha_2+\alpha_3}$	$e_{\alpha_2+\alpha_4}$
h_3	$-e_{\alpha_2}$	e_{α_3}	e_{α_4}	0	0
h_4	0	$-e_{\alpha_3}$	e_{α_4}	$-e_{\alpha_2+\alpha_3}$	$e_{\alpha_2+\alpha_4}$
e_{α_1}	$e_{\alpha_1+\alpha_2}$	0	0	$e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$
e_{α_2}	0	$e_{\alpha_2+\alpha_3}$	$e_{\alpha_2+\alpha_4}$	0	0
e_{α_3}	$-e_{\alpha_2+\alpha_3}$	0	0	0	$e_{\alpha_2+\alpha_3+\alpha_4}$
e_{α_4}	$-e_{\alpha_2+\alpha_4}$	0	0	$e_{\alpha_2+\alpha_3+\alpha_4}$	0
$e_{\alpha_2+\alpha_3}$	0	0	$-e_{\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{\alpha_2+\alpha_4}$	0	$-e_{\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0	0
$e_{\alpha_1+\alpha_2}$	0	$e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$	0	0
$e_{\alpha_1+\alpha_2+\alpha_3}$	0	0	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$
$e_{\alpha_1+\alpha_2+\alpha_4}$	0	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0	0
$e_{-\alpha_1}$	0	0	0	0	0
$e_{-\alpha_2}$	$h_3 - h_2$	0	0	e_{α_3}	e_{α_4}
$e_{-\alpha_3}$	0	$h_4 - h_3$	0	$-e_{\alpha_2}$	0
$e_{-\alpha_4}$	0	0	$-h_4 - h_3$	0	$-e_{\alpha_2}$
$e_{-\alpha_2-\alpha_3}$	$e_{-\alpha_3}$	$e_{-\alpha_2}$	0	$h_4 - h_2$	0
$e_{-\alpha_2-\alpha_4}$	$e_{-\alpha_4}$	0	$e_{-\alpha_2}$	0	$-h_4 - h_2$
$e_{-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_2-\alpha_4}$	$e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_4}$	$-e_{-\alpha_3}$
$e_{-\alpha_1-\alpha_2}$	$-e_{-\alpha_1}$	0	0	0	0
$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	$e_{-\alpha_1-\alpha_2}$	0	$e_{-\alpha_1}$	0
$e_{-\alpha_1-\alpha_2-\alpha_4}$	0	0	$e_{-\alpha_1-\alpha_2}$	0	$-e_{-\alpha_1}$
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	0
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0	$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3}$

	$e_{\alpha_2+\alpha_3+\alpha_4}$	$e_{\alpha_1+\alpha_2}$	$e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
h_1	0	$e_{\alpha_1+\alpha_2}$	$e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
h_2	$e_{\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0
h_3	$e_{\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+\alpha_2}$	0	0	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$
h_4	0	0	$-e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$	0
e_{α_1}	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0
e_{α_2}	0	0	0	0	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$
e_{α_3}	0	$-e_{\alpha_1+\alpha_2+\alpha_3}$	0	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0
e_{α_4}	0	$-e_{\alpha_1+\alpha_2+\alpha_4}$	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{\alpha_2+\alpha_3}$	0	0	0	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0
$e_{\alpha_2+\alpha_4}$	0	0	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{\alpha_2+\alpha_3+\alpha_4}$	0	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0
$e_{\alpha_1+\alpha_2}$	$-e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0
$e_{\alpha_1+\alpha_2+\alpha_3}$	0	0	0	0	0
$e_{\alpha_1+\alpha_2+\alpha_4}$	0	0	0	0	0
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0	0
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	0	0	0
$e_{-\alpha_1}$	0	e_{α_2}	$e_{\alpha_2+\alpha_3}$	$e_{\alpha_2+\alpha_4}$	$e_{\alpha_2+\alpha_3+\alpha_4}$
$e_{-\alpha_2}$	0	$-e_{\alpha_1}$	0	0	0
$e_{-\alpha_3}$	$e_{\alpha_2+\alpha_4}$	0	$-e_{\alpha_1+\alpha_2}$	0	$e_{\alpha_1+\alpha_2+\alpha_4}$
$e_{-\alpha_4}$	$e_{\alpha_2+\alpha_3}$	0	0	$-e_{\alpha_1+\alpha_2}$	$e_{\alpha_1+\alpha_2+\alpha_3}$
$e_{-\alpha_2-\alpha_3}$	$-e_{\alpha_4}$	0	$-e_{\alpha_1}$	0	0
$e_{-\alpha_2-\alpha_4}$	$-e_{\alpha_3}$	0	0	$-e_{\alpha_1}$	0
$e_{-\alpha_2-\alpha_3-\alpha_4}$	$-h_3-h_2$	0	0	0	$-e_{\alpha_1}$
$e_{-\alpha_1-\alpha_2}$	0	h_3-h_1	e_{α_3}	e_{α_4}	0
$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	$e_{-\alpha_3}$	h_4-h_1	0	$-e_{\alpha_4}$
$e_{-\alpha_1-\alpha_2-\alpha_4}$	0	$e_{-\alpha_4}$	0	$-h_4-h_1$	$-e_{\alpha_3}$
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	$-e_{-\alpha_1}$	0	$e_{-\alpha_4}$	$-e_{-\alpha_3}$	$-h_3-h_1$
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-\alpha_2}$	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	$-e_{-\alpha_2-\alpha_4}$	$-e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_2}$

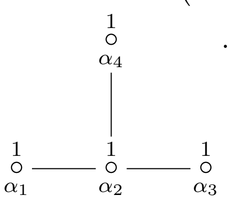
	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$e_{-\alpha_1}$	$e_{-\alpha_2}$	$e_{-\alpha_3}$	$e_{-\alpha_4}$
h_1	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$-e_{-\alpha_1}$	0	0	0
h_2	$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$e_{-\alpha_1}$	$-e_{-\alpha_2}$	0	0
h_3	0	0	$e_{-\alpha_2}$	$-e_{-\alpha_3}$	$-e_{-\alpha_4}$
h_4	0	0	0	$e_{-\alpha_3}$	$-e_{-\alpha_4}$
e_{α_1}	0	h_1-h_2	0	0	0
e_{α_2}	0	0	h_2-h_3	0	0
e_{α_3}	0	0	0	h_3-h_4	0
e_{α_4}	0	0	0	0	h_3+h_4
$e_{\alpha_2+\alpha_3}$	0	0	$-e_{\alpha_3}$	e_{α_2}	0
$e_{\alpha_2+\alpha_4}$	0	0	$-e_{\alpha_4}$	0	e_{α_2}
$e_{\alpha_2+\alpha_3+\alpha_4}$	0	0	0	$-e_{\alpha_2+\alpha_4}$	$-e_{\alpha_2+\alpha_3}$
$e_{\alpha_1+\alpha_2}$	0	$-e_{\alpha_2}$	e_{α_1}	0	0
$e_{\alpha_1+\alpha_2+\alpha_3}$	0	$-e_{\alpha_2+\alpha_3}$	0	$e_{\alpha_1+\alpha_2}$	0
$e_{\alpha_1+\alpha_2+\alpha_4}$	0	$-e_{\alpha_2+\alpha_4}$	0	0	$e_{\alpha_1+\alpha_2}$
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_2+\alpha_3+\alpha_4}$	0	$-e_{\alpha_1+\alpha_2+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_3}$
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	0	0	$-e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0
$e_{-\alpha_1}$	0	0	$-e_{-\alpha_1-\alpha_2}$	0	0
$e_{-\alpha_2}$	$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	$e_{-\alpha_1-\alpha_2}$	0	$-e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_2-\alpha_4}$
$e_{-\alpha_3}$	0	0	$e_{-\alpha_2-\alpha_3}$	0	0
$e_{-\alpha_4}$	0	0	$e_{-\alpha_2-\alpha_4}$	0	0
$e_{-\alpha_2-\alpha_3}$	$e_{\alpha_1+\alpha_2+\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	0	$e_{-\alpha_2-\alpha_3-\alpha_4}$
$e_{-\alpha_2-\alpha_4}$	$e_{\alpha_1+\alpha_2+\alpha_3}$	$e_{-\alpha_1-\alpha_2-\alpha_4}$	0	$e_{-\alpha_2-\alpha_3-\alpha_4}$	0
$e_{-\alpha_2-\alpha_3-\alpha_4}$	$e_{\alpha_1+\alpha_2}$	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0	0
$e_{-\alpha_1-\alpha_2}$	$-e_{\alpha_2+\alpha_3+\alpha_4}$	0	0	$-e_{-\alpha_1-\alpha_2-\alpha_3}$	$e_{-\alpha_1-\alpha_2-\alpha_4}$
$e_{-\alpha_1-\alpha_2-\alpha_3}$	$-e_{\alpha_2+\alpha_4}$	0	0	0	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$
$e_{-\alpha_1-\alpha_2-\alpha_4}$	$-e_{\alpha_2+\alpha_3}$	0	0	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	$-e_{\alpha_2}$	0	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	$-h_2-h_1$	0	0	0	0

	$e_{-\alpha_2-\alpha_3}$	$e_{-\alpha_2-\alpha_4}$	$e_{-\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-\alpha_2}$
h_1	0	0	0	$-e_{-\alpha_1-\alpha_2}$
h_2	$-e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_2-\alpha_4}$	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	0
h_3	0	0	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	$-e_{-\alpha_1-\alpha_2}$
h_4	$e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_2-\alpha_4}$	0	0
e_{α_1}	0	0	0	$-e_{-\alpha_2}$
e_{α_2}	$-e_{-\alpha_3}$	$-e_{-\alpha_4}$	0	$-e_{-\alpha_1}$
e_{α_3}	$e_{-\alpha_2}$	0	$-e_{-\alpha_2-\alpha_4}$	0
e_{α_4}	0	$e_{-\alpha_2}$	$-e_{-\alpha_2-\alpha_3}$	0
$e_{\alpha_2+\alpha_3}$	$h_2 - h_4$	0	$e_{-\alpha_4}$	0
$e_{\alpha_2+\alpha_4}$	0	$h_2 + h_4$	$e_{-\alpha_3}$	0
$e_{\alpha_2+\alpha_3+\alpha_4}$	e_{α_4}	e_{α_3}	$h_2 + h_3$	0
$e_{\alpha_1+\alpha_2}$	0	0	0	$-h_1 - h_3$
$e_{\alpha_1+\alpha_2+\alpha_3}$	e_{α_1}	0	0	$-e_{\alpha_3}$
$e_{\alpha_1+\alpha_2+\alpha_4}$	0	e_{α_1}	0	$-e_{\alpha_4}$
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	0	0	e_{α_1}	0
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_4}$	$-e_{\alpha_1+\alpha_2+\alpha_3}$	$-e_{\alpha_1+\alpha_2}$	$e_{\alpha_2+\alpha_3+\alpha_4}$
$e_{-\alpha_1}$	$-e_{-\alpha_1-\alpha_2-\alpha_3}$	$-e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0
$e_{-\alpha_2}$	0	0	0	0
$e_{-\alpha_3}$	0	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	0	$e_{-\alpha_1-\alpha_2-\alpha_3}$
$e_{-\alpha_4}$	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	0	0	$e_{-\alpha_1-\alpha_2-\alpha_4}$
$e_{-\alpha_2-\alpha_3}$	0	0	0	0
$e_{-\alpha_2-\alpha_4}$	0	0	0	0
$e_{-\alpha_2-\alpha_3-\alpha_4}$	0	0	0	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$
$e_{-\alpha_1-\alpha_2}$	0	0	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0
$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0
$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0	0
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0	0	0
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0	0	0

	$e_{-\alpha_1-\alpha_2-\alpha_3}$	$e_{-\alpha_1-\alpha_2-\alpha_4}$	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$
h_1	$-e_{-\alpha_1-\alpha_2-\alpha_3}$	$-e_{-\alpha_1-\alpha_2-\alpha_4}$	$-e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$
h_2	0	0	0	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$
h_3	0	0	$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0
h_4	$e_{-\alpha_1-\alpha_2-\alpha_3}$	$-e_{-\alpha_1-\alpha_2-\alpha_4}$	0	0
e_{α_1}	$-e_{-\alpha_2-\alpha_3}$	$-e_{-\alpha_2-\alpha_4}$	$-e_{-\alpha_2-\alpha_3-\alpha_4}$	0
e_{α_2}	0	0	0	$-e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$
e_{α_3}	$e_{-\alpha_1-\alpha_2}$	0	$-e_{-\alpha_1-\alpha_2-\alpha_4}$	0
e_{α_4}	0	$e_{-\alpha_1-\alpha_2}$	$-e_{-\alpha_1-\alpha_2-\alpha_3}$	0
$e_{\alpha_2+\alpha_3}$	$e_{-\alpha_1}$	0	0	$e_{-\alpha_1-\alpha_2-\alpha_4}$
$e_{\alpha_2+\alpha_4}$	0	$e_{-\alpha_1}$	0	$-e_{-\alpha_1-\alpha_2-\alpha_3}$
$e_{\alpha_2+\alpha_3+\alpha_4}$	0	0	$e_{-\alpha_1}$	$e_{-\alpha_1-\alpha_2}$
$e_{\alpha_1+\alpha_2}$	$-e_{-\alpha_3}$	$-e_{-\alpha_4}$	0	$e_{-\alpha_2-\alpha_3-\alpha_4}$
$e_{\alpha_1+\alpha_2+\alpha_3}$	$h_1 - h_4$	0	$e_{-\alpha_4}$	$e_{-\alpha_2-\alpha_4}$
$e_{\alpha_1+\alpha_2+\alpha_4}$	0	$h_1 + h_4$	$e_{-\alpha_3}$	$e_{-\alpha_2-\alpha_3}$
$e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$	e_{α_4}	e_{α_3}	$h_1 + h_3$	$e_{-\alpha_2}$
$e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$e_{\alpha_2+\alpha_4}$	$e_{\alpha_2+\alpha_3}$	e_{α_2}	$h_1 + h_4$
$e_{-\alpha_1}$	0	0	0	0
$e_{-\alpha_2}$	0	0	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0
$e_{-\alpha_3}$	0	$-e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0
$e_{-\alpha_4}$	$-e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0	0
$e_{-\alpha_2-\alpha_3}$	0	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0
$e_{-\alpha_2-\alpha_4}$	$-e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0	0
$e_{-\alpha_2-\alpha_3-\alpha_4}$	0	0	0	0
$e_{-\alpha_1-\alpha_2}$	0	0	0	0
$e_{-\alpha_1-\alpha_2-\alpha_3}$	0	0	0	0
$e_{-\alpha_1-\alpha_2-\alpha_4}$	0	0	0	0
$e_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$	0	0	0	0
$e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}$	0	0	0	0

$\mathfrak{g}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$, $\mathfrak{g}_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}$ correspond to the root vectors e_{α_1}, e_{α_2}, e_{α_3}, e_{α_4}, $e_{\alpha_2+\alpha_3}$, $e_{\alpha_2+\alpha_4}$, $e_{\alpha_2+\alpha_3+\alpha_4}$, $e_{\alpha_1+\alpha_2}$, $e_{\alpha_1+\alpha_2+\alpha_3}$, $e_{\alpha_1+\alpha_2+\alpha_4}$, $e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$, $e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$, $e_{-\alpha_1}$, $e_{-\alpha_2}$, $e_{-\alpha_3}$, $e_{-\alpha_4}$, $e_{-(\alpha_2+\alpha_3)}$, $e_{-(\alpha_2+\alpha_4)}$, $e_{-(\alpha_2+\alpha_3+\alpha_4)}$, $e_{-(\alpha_1+\alpha_2)}$, $e_{-(\alpha_1+\alpha_2+\alpha_3)}$, $e_{-(\alpha_1+\alpha_2+\alpha_4)}$, $e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$,

$e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}$ respectively. Let $\mathfrak{h}_0^*$ denote the real space spanned by the roots. We introduce an ordering in the vector space $\mathfrak{h}_0^*$. We choose for a basis of roots α_1, α_2, α_3, α_4, so that positive roots are α_1, α_2, α_3, α_4, $\alpha_2+\alpha_3$, $\alpha_2+\alpha_4$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+\alpha_3+\alpha_4$, and negative roots are $-\alpha_1$, $-\alpha_2$, $-\alpha_3$, $-\alpha_4$, $-(\alpha_2+\alpha_3$, $\alpha_2+\alpha_4)$, $-(\alpha_2+\alpha_3+\alpha_4)$, $-(\alpha_1+\alpha_2)$, $-(\alpha_1+\alpha_2+\alpha_3)$, $-(\alpha_1+\alpha_2+\alpha_4)$, $-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)$ and a simple system of roots $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Let Δ^+ be the set of all positive roots, that is, $\Delta^+ = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_2+\alpha_3,\ \alpha_2+\alpha_4,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$. The α_1 string of roots containing α_2 is α_2, $\alpha_2+\alpha_1$, the α_1 string of roots containing α_3 is α_3, the α_1 string of roots containing α_4 is α_4, the α_2 string of roots containing α_1 is α_1, $\alpha_1+\alpha_2$, the α_2 string of roots containing α_3 is α_3, $\alpha_3+\alpha_2$, the α_2 string of roots containing α_4 is α_4, $\alpha_4+\alpha_2$, the α_3 string of roots containing α_1 is α_1, the α_3 string of roots containing α_2 is α_2, $\alpha_2+\alpha_3$, the α_3 string of roots containing α_4 is α_4, the α_4 string of roots containing α_1 is α_1, the α_4 string of roots containing α_2 is α_2, $\alpha_2+\alpha_4$ and the α_4 string of roots containing α_3 is α_3, so the nondiagonal elements of Cartan matrix are $A_{12}=-1$, $A_{13}=0$, $A_{14}=0$, $A_{21}=-1$, $A_{23}=-1$, $A_{24}=-1$, $A_{31}=0$, $A_{32}=-1$, $A_{34}=0$, $A_{41}=0$, $A_{42}=-1$, and $A_{43}=0$. Thus the Cartan matrix is $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ and the corresponding Dynkin diagram is



. We have thus obtained that $\mathfrak{g}$ is a simple Lie algebra of class D_4, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma\in\Delta^+} \mathfrak{g}_\gamma \oplus \bigoplus_{\gamma\in\Delta^+} \mathfrak{g}_{-\gamma}$, where $\mathfrak{h}$ consists of diagonal matrices, $\bigoplus_{\gamma\in\Delta^+} \mathfrak{g}_\gamma$ consists of upper triangular matrices, and $\bigoplus_{\gamma\in\Delta^+} \mathfrak{g}_{-\gamma}$ consists of lower triangular matrices.

4. Constructing Representations of D_4

Let G be the Lie group with Lie algebra $\mathfrak{g}$. Then $H_1(t_{25}) = e^{t_{25}h_1}$, $H_2(t_{26}) = e^{t_{26}h_2}$, $H_3(t_{27}) = e^{t_{27}h_3}$, $H_4(t_{28}) = e^{t_{28}h_4}$, $E_{\alpha_1}(t_{13}) = e^{t_{13}e_{\alpha_1}}$, $E_{\alpha_2}(t_{19}) = e^{t_{19}e_{\alpha_2}}$, $E_{\alpha_3}(t_{23}) = e^{t_{23}e_{\alpha_3}}$, $E_{\alpha_4}(t_{24}) = e^{t_{24}e_{\alpha_4}}$, $E_{\alpha_2+\alpha_3}(t_{20}) = e^{t_{20}e_{\alpha_2+\alpha_3}}$, $E_{\alpha_2+\alpha_4}(t_{21}) = e^{t_{21}e_{\alpha_2+\alpha_4}}$, $E_{\alpha_2+\alpha_3+\alpha_4}(t_{22}) = e^{t_{22}e_{\alpha_2+\alpha_3+\alpha_4}}$, $E_{\alpha_1+\alpha_2}(t_{14}) = e^{t_{14}e_{\alpha_1+\alpha_2}}$, $E_{\alpha_1+\alpha_2+\alpha_3}(t_{15}) = e^{t_{15}e_{\alpha_1+\alpha_2+\alpha_3}}$, $E_{\alpha_1+\alpha_2+\alpha_4}(t_{16}) =$

$e^{t_{16}e_{\alpha_1+\alpha_2+\alpha_4}}$, $E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(t_{17}) = e^{t_{17}e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}}$, $E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(t_{18}) = e^{t_{18}e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}}$, $E_{-\alpha_1}(t_1) = e^{t_1e_{-\alpha_1}}$, $E_{-\alpha_2}(t_7) = e^{t_7e_{-\alpha_2}}$, $E_{-\alpha_3}(t_{11}) = e^{t_{11}e_{-\alpha_3}}$, $E_{-\alpha_4}(t_{12}) = e^{t_{12}e_{-\alpha_4}}$, $E_{-(\alpha_2+\alpha_3)}(t_8) = e^{t_8e_{-(\alpha_2+\alpha_3)}}$, $E_{-(\alpha_2+\alpha_4)}(t_9) = e^{t_9e_{-(\alpha_2+\alpha_4)}}$, $E_{-(\alpha_2+\alpha_3+\alpha_4)}(t_{10}) = e^{t_{10}e_{-(\alpha_2+\alpha_3+\alpha_4)}}$, $E_{-(\alpha_1+\alpha_2)}(t_2) = e^{t_2e_{-(\alpha_1+\alpha_2)}}$, $E_{-(\alpha_1+\alpha_2+\alpha_3)}(t_3) = e^{t_3e_{-(\alpha_1+\alpha_2+\alpha_3)}}$, $E_{-(\alpha_1+\alpha_2+\alpha_4)}(t_4) = e^{t_4e_{-(\alpha_1+\alpha_2+\alpha_4)}}$, $E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}(t_5) = e^{t_5e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}$, $E_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}(t_6) = e^{t_6e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}}$ are one parameter subgroups of Lie group G.

Using these one-parameter subgroups of group G, we shall now construct some of its subgroups that will be utilized for constructing some representations of G. The first of these subgroups is a maximal nilpotent subgroup of group G and is constructed as follows:

$Z_+ = \{ E_{\alpha_1}(t_{13})E_{\alpha_1+\alpha_2}(t_{14})E_{\alpha_1+\alpha_2+\alpha_3}(t_{15})E_{\alpha_1+\alpha_2+\alpha_4}(t_{16}) E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(t_{17})E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(t_{18})E_{\alpha_2}(t_{19})E_{\alpha_2+\alpha_3}(t_{20}) E_{\alpha_2+\alpha_4}(t_{21})E_{\alpha_2+\alpha_3+\alpha_4}(t_{22})E_{\alpha_3}(t_{23})E_{\alpha_4}(t_{24}) | t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24} \in \mathbb{R} \}$

We find it convenient to introduce the new parameters as follows: $x_1 = t_{13}, x_2 = t_{13}t_{19}+t_{14}, x_3 = (t_{13}t_{19}+t_{14})t_{23}+t_{13}t_{20}+t_{15}, x_4 = (t_{13}t_{19}+t_{14})t_{24}+t_{13}t_{21}+t_{16}, x_5 = -((t_{13}t_{19}+t_{14})t_{23}+t_{13}t_{20}+t_{15})t_{24}-(t_{13}t_{21}+t_{16})t_{23}+t_{13}t_{22}+t_{17}, x_6 = -(t_{13}t_{19}+t_{14})t_{22}-(t_{13}t_{20}+t_{15})t_{21}-t_{16}t_{20}-t_{17}t_{19}+t_{18}, x_7 = t_{19}, x_8 = t_{19}t_{23} + t_{20}, x_9 = t_{19}t_{24} + t_{21}, x_{10} = -(t_{19}t_{23} + t_{20})t_{24} - t_{21}t_{23} + t_{22}, x_{11} = t_{23}, x_{12} = t_{24}$.

Then we get

$$Z_+ = \left\{ \left(\begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & -x_1x_6 - x_2x_5 - x_3x_4 \\ 0 & 1 & x_7 & x_8 & x_9 & x_{10} & -x_7x_{10} - x_8x_9 & x_1x_8x_9 + x_1x_7x_{10} - x_6 - x_5x_7 - x_4x_8 - x_3x_9 - x_2x_{10} \\ 0 & 0 & 1 & x_{11} & x_{12} & -x_{11}x_{12} & -x_{10} - x_9x_{11} + x_7x_{11}x_{12} - x_8x_{12} & -x_5 + x_1x_{10} + x_1x_9x_{11} - x_1x_7x_{11}x_{12} + x_1x_8x_{12} - x_4x_{11} + x_2x_{11}x_{12} - x_3x_{12} \\ 0 & 0 & 0 & 1 & 0 & -x_{12} & -x_9 + x_7x_{12} & -x_4 + x_1x_9 - x_1x_7x_{12} + x_2x_{12} \\ 0 & 0 & 0 & 0 & 1 & -x_{11} & -x_8 + x_7x_{11} & -x_3 + x_1x_8 - x_1x_7x_{11} + x_2x_{11} \\ 0 & 0 & 0 & 0 & 0 & 1 & -x_7 & -x_2 + x_1x_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| \begin{array}{c} x_1, x_2, \\ x_3, x_4, \\ x_5, x_6, \\ x_7, x_8, \\ x_9, x_{10}, \\ x_{11}, x_{12} \end{array} \in \mathbb{R} \right\}$$

We also construct another maximal nilpotent subgroup as follows:

$Z_- = \{E_{-\alpha_1}(t_1)E_{-(\alpha_1+\alpha_2)}(t_2)E_{-(\alpha_1+\alpha_2+\alpha_3)}(t_3)E_{-(\alpha_1+\alpha_2+\alpha_4)}(t_4) E_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}(t_5)E_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}(t_6)E_{-\alpha_2}(t_7)E_{-(\alpha_2+\alpha_3)}(t_8) E_{-(\alpha_2+\alpha_4)}(t_9)E_{-(\alpha_2+\alpha_3+\alpha_4)}(t_{10})E_{-\alpha_3}(t_{11})E_{-\alpha_4}(t_{12}) | t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12} \in \mathbb{R}\}$. The subgroup denoted by H, which is a maximal abelian subgroup of G, is defined as follows:

$H = \{H_1(t_{25})H_2(t_{26})H_3(t_{27})H_4(t_{28})|t_{25}, t_{26}, t_{27}, t_{28} \in \mathbb{R}\}$, and the subgroup denoted by B_-, which is a maximal solvable subgroup of G, is defined as $B_- = Z_-H = \{z_-h|z_- \in Z_-, h \in H\}$.

For $(a, b, c, d) \in \mathbb{C}^4$, we define a mapping $\alpha_{a,b,c,d} : H \to \mathbb{C}$ by $H_1(t_{25})H_2(t_{26})H_3(t_{27})H_4(t_{28}) \mapsto e^{at_{25}}e^{bt_{26}}e^{ct_{27}}e^{dt_{28}}$.

By the basic property of the exponential function, $\alpha_{a,b,c,d}$ is a character of group H.

We extend $\alpha_{a,b,c,d}$ from H to B_- by the rule: for $b_- = z_-h \in B_-$, $z_- \in Z_-$, $h \in H$, $\alpha_{a,b,c,d}(b_-) = \alpha_{a,b,c,d}(z_-h) = \alpha_{a,b,c,d}(h)$.

For $(a, b, c, d) \in \mathbb{C}^4$, we define an induced representation $S^{\alpha_{a,b,c,d}} = \mathrm{ind}_{B_-}^G \alpha_{a,b,c,d}$. It operates in the space $F_{\alpha_{a,b,c,d}}(G) = \{f \in C^\infty(G) -$ the set of all complex-valued smooth functions on $G | f(gb_-) = \alpha^{-1}_{a,b,c,d}(b_-)f(g)$, $b_- \in B_-, g \in G\}$ by $S_g^{\alpha_{a,b,c,d}} f(h) = f(g^{-1}h), h, g \in G$. Because the subset B_-Z_+ is dense in G [4], the functions from the space $F_{\alpha_{a,b,c,d}}(G)$ are completely determined by their restrictions to the subgroup Z_+. This allows to realize the representations $S^{\alpha_{a,b,c,d}}$ of G in the space $C^\infty(Z_+)$. The respective representation of the Lie algebra $\mathfrak{g}$ in the space $C^\infty(Z_+)$ is realized via the differential operators as follows: It will be convenient to introduce new parameters $p = a - b$, $q = b - c$, $s = c - d$, $t = c + d$, then we get

$$\begin{aligned}
H_1 &= -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6} + p + q + \frac{1}{2}(s+t)\\
H_2 &= x_1\frac{\partial}{\partial x_1} - x_6\frac{\partial}{\partial x_6} - x_7\frac{\partial}{\partial x_7} - x_8\frac{\partial}{\partial x_8} - x_9\frac{\partial}{\partial x_9} - x_{10}\frac{\partial}{\partial x_{10}} + q + \frac{1}{2}(s+t)\\
H_3 &= x_2\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial x_5} + x_7\frac{\partial}{\partial x_7} - x_{10}\frac{\partial}{\partial x_{10}} - x_{11}\frac{\partial}{\partial x_{11}} - x_{12}\frac{\partial}{\partial x_{12}} + \frac{1}{2}(s+t)\\
H_4 &= x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} + x_8\frac{\partial}{\partial x_8} - x_9\frac{\partial}{\partial x_9} + x_{11}\frac{\partial}{\partial x_{11}} - x_{12}\frac{\partial}{\partial x_{12}} + \frac{1}{2}(t-s)\\
S_{\alpha_1} &= -\frac{\partial}{\partial x_1} - x_7\frac{\partial}{\partial x_2} - x_8\frac{\partial}{\partial x_3} - x_9\frac{\partial}{\partial x_4} - x_{10}\frac{\partial}{\partial x_5} + (x_7x_{10} + x_8x_9)\frac{\partial}{\partial x_6}\\
S_{\alpha_2} &= -\frac{\partial}{\partial x_7} - x_{11}\frac{\partial}{\partial x_8} - x_{12}\frac{\partial}{\partial x_9} + x_{11}x_{12}\frac{\partial}{\partial x_{10}}\\
S_{\alpha_3} &= -\frac{\partial}{\partial x_{11}}\\
S_{\alpha_4} &= -\frac{\partial}{\partial x_{12}}\\
S_{-\alpha_1} &= x_1^2\frac{\partial}{\partial x_1} + (x_2x_5 + x_3x_4)\frac{\partial}{\partial x_6} - x_2\frac{\partial}{\partial x_7} - x_3\frac{\partial}{\partial x_8} - x_4\frac{\partial}{\partial x_9} - x_5\frac{\partial}{\partial x_{10}} - px_1\\
S_{-\alpha_2} &= (x_2 - x_1x_7)\frac{\partial}{\partial x_1} + x_2x_7\frac{\partial}{\partial x_2} - (x_5x_7 + x_6)\frac{\partial}{\partial x_5} + x_7x_6\frac{\partial}{\partial x_6} + x_7^2\frac{\partial}{\partial x_7}\\
&\quad + x_8x_9\frac{\partial}{\partial x_{10}} - x_8\frac{\partial}{\partial x_{11}} - x_9\frac{\partial}{\partial x_{12}} - qx_7
\end{aligned}$$

$$
\begin{aligned}
S_{-\alpha_3} &= (x_3 - x_2x_{11})\frac{\partial}{\partial x_2} + x_3x_{11}\frac{\partial}{\partial x_3} - (x_4x_{11} + x_5)\frac{\partial}{\partial x_4} + x_5x_{11}\frac{\partial}{\partial x_5} \\
&\quad +(x_8 - x_7x_{11})\frac{\partial}{\partial x_7} + x_{11}x_8\frac{\partial}{\partial x_8} - (x_{10} + x_9x_{11})\frac{\partial}{\partial x_9} + x_{11}x_{10}\frac{\partial}{\partial x_{10}} \\
&\quad +x_{11}^2\frac{\partial}{\partial x_{11}} - sx_{11} \\
S_{-\alpha_4} &= (x_4 - x_2x_{12})\frac{\partial}{\partial x_2} - (x_3x_{12} + x_5)\frac{\partial}{\partial x_3} + x_4x_{12}\frac{\partial}{\partial x_4} + x_5x_{12}\frac{\partial}{\partial x_5} \\
&\quad +(x_9 - x_7x_{12})\frac{\partial}{\partial x_7} - (x_8x_{12} + x_{10})\frac{\partial}{\partial x_8} + x_9x_{12}\frac{\partial}{\partial x_9} \\
&\quad +x_{12}x_{10}\frac{\partial}{\partial x_{10}} + x_{12}^2\frac{\partial}{\partial x_{12}} - tx_{12}
\end{aligned}
$$

Since Z_+ is a group with twelve-dimensional nilpotent Lie algebra over $\mathbb{R}$, Z_+ is diffeomorphic to $\mathbb{R}^{12}$. Thus we can consider the space of all complex-valued smooth functions on $\mathbb{R}^{12}$ for the representation space of this representation of $\mathfrak{g}$. Furthermore, we can consider the space of all complex-valued polynomial functions of twelve variables for the representation space of a representation of $\mathfrak{g}$, because it is invariant under the action of all operators S.

Denote this representation by $\varphi^{p,q,s,t}$. Note that only 1 is annuled by the above positive root vectors, so 1, whose weight is $p\omega_1 + q\omega_2 + s\omega_3 + t\omega_4$, where $\omega_1, \omega_2, \omega_3, \omega_4$ are fundamental weights, is the only highest weight vector in the space of the polynomials. Thus, for $\varphi^{p,q,s,t}$, where p, q, s, t are non-negative integers, applying products of $S_{-\alpha_1}$, $S_{-\alpha_2}$, $S_{-\alpha_3}$ and $S_{-\alpha_4}$ to 1, we get an invariant subspace of the irreducible representation of $\mathfrak{g}$. For $p = 1, q = 0, s = 0, t = 0$, the subspace is spanned by 8 polynomials: 1, $P_2 = x_1$, $P_3 = -x_2+x_1x_7$, $P_4 = x_2x_{11}-x_3+x_1x_8-x_1x_7x_{11}$, $P_5 = x_2x_{12}-x_4+x_1x_9-x_1x_7x_{12}$, $P_6 = x_3x_{12} - x_{12}x_2x_{11} + x_4x_{11} + x_5 - x_{12}x_1x_8 + x_{12}x_1x_7x_{11} - x_1x_{10} - x_{11}x_1x_9$, $P_7 = x_1x_7x_{10} - x_2x_{10} - x_5x_7 - x_6 + x_9x_1x_8 - x_4x_8 - x_9x_3$, $P_8 = x_2x_5 + x_3x_4 + x_1x_6$, for $p = 0, q = 1, s = 0, t = 0$, the subspace is spanned by 28 polynomials: 1, $Q_2 = x_7$, $Q_3 = x_2$, $Q_4 = x_7x_{11} - x_8$, $Q_5 = x_7x_{12} - x_9$, $Q_6 = x_3 - x_2x_{11}$, $Q_7 = x_4 - x_2x_{12}$, $Q_8 = x_8x_{12} - x_{12}x_7x_{11}+x_{10}+x_9x_{11}$, $Q_9 = x_{12}x_2x_{11}-x_3x_{12}-x_4x_{11}-x_5$, $Q_{10} = x_8x_2-x_7x_3$, $Q_{11} = x_9x_2 - x_7x_4$, $Q_{12} = x_8x_9 + x_7x_{10}$, $Q_{13} = x_2x_{10} + x_9x_3 + x_4x_8 + x_5x_7$, $Q_{14} = 2x_5x_7+x_6-x_2x_8x_{12}+x_4x_8-x_9x_2x_{11}+x_9x_3+x_7x_{12}x_3+x_7x_{11}x_4$, $Q_{15} = x_9x_3 - 2x_9x_2x_{11} + 2x_7x_{11}x_4 + x_5x_7 - x_4x_8 - x_2x_{10}$, $Q_{16} = x_4x_8 - 2x_2x_8x_{12} + 2x_7x_{12}x_3 + x_5x_7 - x_9x_3 - x_2x_{10}$, $Q_{17} = 2x_2x_5 + 2x_3x_4$, $Q_{18} = x_9x_8x_2 - x_7^2x_5 - x_7x_6 - x_9x_7x_3 - x_8x_7x_4$, $Q_{19} = x_8x_4x_{11} + x_8x_5 - x_{10}x_3 - x_5x_{11}x_7 + x_{11}x_{10}x_2 - x_3x_{11}x_9 + x_2x_{11}^2x_9 - x_4x_{11}^2x_7$, $Q_{20} = x_9x_3x_{12} + x_9x_5 - x_4x_{12}x_8 -$

$x_5x_{12}x_7+x_{12}x_{10}x_2-x_{10}x_4+x_2x_{12}^2x_8-x_3x_{12}^2x_7$, $Q_{21}=x_2x_5x_7+x_4x_7x_3+x_2x_6$, $Q_{22}=x_7^2x_{11}x_5-x_8x_5x_7-x_8x_6+x_7x_{11}x_6-x_8x_9x_2x_{11}+x_7x_{11}x_8x_4-x_8^2x_4-x_{10}x_8x_2+x_{10}x_7x_3+x_7x_{11}x_9x_3$, $Q_{23}=x_7^2x_{12}x_5-x_9x_5x_7-x_9x_6+x_7x_{12}x_6-x_9x_2x_8x_{12}+x_7x_{12}x_9x_3-x_9^2x_3-x_{10}x_9x_2+x_7x_{12}x_8x_4+x_{10}x_7x_4$, $Q_{24}=x_2x_5x_8-x_2x_{11}x_5x_7+x_3x_4x_8-x_3x_7x_{11}x_4-x_2x_{11}x_6+x_3x_6$, $Q_{25}=x_2x_5x_9-x_2x_{12}x_5x_7+x_4x_9x_3-x_4x_7x_{12}x_3-x_2x_{12}x_6+x_4x_6$, $Q_{26}=x_8x_{12}x_2x_{10}+x_9x_{11}x_2x_{10}-x_{10}x_7x_{12}x_3-x_{10}x_7x_{11}x_4+x_5x_7x_{12}x_8+x_5x_{11}x_9x_7-x_5x_{11}x_7^2x_{12}-x_7x_{11}x_{12}x_6+x_8^2x_{12}x_4+x_9^2x_{11}x_3+x_{10}x_8x_4+x_{10}x_9x_3-x_7x_{11}x_4x_8x_{12}-x_9x_{11}x_7x_{12}x_3+x_2x_{11}x_9x_{12}x_8-x_8x_9x_5+x_8x_{12}x_6+x_9x_{11}x_6+x_{10}x_6+x_2x_{10}^2$, $Q_{27}=x_3x_4x_{12}x_7x_{11}+x_2x_5x_{12}x_7x_{11}-x_3x_{12}x_8x_4+x_2x_{11}x_{12}x_6-x_4x_{11}x_9x_3-x_5x_2x_8x_{12}-x_5x_9x_2x_{11}-x_5x_2x_{10}-x_5x_6-x_4x_{11}x_6-x_3x_{12}x_6-x_3x_4x_{10}$, $Q_{28}=x_2x_7x_5x_{10}+x_6x_2x_{10}+x_9x_3x_6+x_8x_4x_6+x_6^2+x_8x_9x_3x_4+x_8x_9x_2x_5+x_5x_7x_6+x_{10}x_3x_7x_4$, for $p=0,q=0,s=1,t=0$, the subspace is spanned by 8 polynomials: 1, $S_2=x_{11}$, $S_3=x_8$, $S_4=x_3$, $S_5=x_8x_{12}+x_{10}$, $S_6=x_3x_{12}+x_5$, $S_7=x_5x_7+x_6+x_9x_3$, $S_8=x_3x_{11}x_9+x_5x_{11}x_7-x_8x_5+x_{10}x_3+x_{11}x_6$,and for $p=0,q=0,s=0,t=1$ it is spanned by 8 polynomials: 1, $T_2=x_{12}$, $T_3=x_9$, $T_4=x_4$, $T_5=x_{10}+x_9x_{11}$, $T_6=x_4x_{11}+x_5$, $T_7=x_5x_7+x_6+x_8x_4$,$T_8=x_4x_{12}x_8+x_5x_{12}x_7-x_9x_5+x_{10}x_4+x_{12}x_6$.

5. Irreducible Representations of D_4

Let $V^{p,q,s,t}$ be the real vector space spanned by $P_2^{m_2}P_3^{m_3}P_4^{m_4}P_5^{m_5}P_6^{m_6}P_7^{m_7}P_8^{m_8}Q_2^{n_2}Q_3^{n_3}Q_4^{n_4}Q_5^{n_5}Q_6^{n_6}Q_7^{n_7}Q_8^{n_8}Q_9^{n_9}Q_{10}^{n_{10}}Q_{11}^{n_{11}}Q_{12}^{n_{12}}Q_{13}^{n_{13}}Q_{14}^{n_{14}}Q_{15}^{n_{15}}Q_{16}^{n_{16}}Q_{17}^{n_{17}}Q_{18}^{n_{18}}Q_{19}^{n_{19}}Q_{20}^{n_{20}}Q_{21}^{n_{21}}Q_{22}^{n_{22}}Q_{23}^{n_{23}}Q_{24}^{n_{24}}Q_{25}^{n_{25}}Q_{26}^{n_{26}}Q_{27}^{n_{27}}Q_{28}^{n_{28}}S_2^{k_2}S_3^{k_3}S_4^{k_4}S_5^{k_5}S_6^{k_6}S_7^{k_7}S_8^{k_8}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_5^{l_5}T_6^{l_6}T_7^{l_7}T_8^{l_8}$, where $m_2+\ldots+m_8\le p$, $n_2+\ldots+n_{28}\le q$, $k_2+\ldots+k_8\le s$, and $l_2+\ldots+l_8\le t$.

By direct computation, $V^{p,q,s,t}$ is invariant under $\varphi^{p,q,s,t}$. Thus $\varphi^{p,q,s,t}$ is a finite-dimensional representation of $\mathfrak{g}$ in $V^{p,q,s,t}$. Since 1, whose weight is $p\omega_1+q\omega_2+s\omega_3+t\omega_4$, is the only highest weight vector of $\varphi^{p,q,s,t}$ in $V^{p,q,s,t}$, and $\varphi^{p,q,s,t}$ is completely reducible, $\varphi^{p,q,s,t}$ is an irreducible representation of $\mathfrak{g}$ in $V^{p,q,s,t}$ [4].

6. Solutions of the PDE

Consider in the space $C^\infty(G)$ two representations of the group G, $S_gf(h)=f(g^{-1}h)$ and $T_gf(h)=f(hg)$ where $h,g\in G$. Observe that S_g and T_h commute in $C^\infty(G)$ for all $g,h\in G$. Let $\mathfrak{X}$ be the dual space for the space $C^\infty(G)$, that is the space of all distributions with compact support on G. Consider in $\mathfrak{X}$ two representations of the group G conjugate to S_g and T_g

that we shall denote by $\widetilde{S}_g$ and $\widetilde{T}_g$, where $\widetilde{S}_g = S^*_{g^{-1}}$ and $\widetilde{T}_g = T^*_{g^{-1}}$. Here $*$ denotes the adjoint operator, that is, if $\langle f, F\rangle$ is the canonical bilinear form for the pair $C^\infty(G)$ and $\mathfrak{X}$, then $\langle Af, F\rangle = \langle f, A^*F\rangle$ for any linear operator A in $C^\infty(G)$, $f \in C^\infty(G)$ and $F \in \mathfrak{X}$. Let 1_e be the δ-function on G with support at the identity e of the group G. Then it is easy to see that $\widetilde{S}_g 1_e = 1_g$ and $\widetilde{T}_g 1_e = 1_{g^{-1}}$, where 1_g is the δ-function with support at the point g of G and $1_{g^{-1}}$ is the same for g^{-1}. This implies that $\widetilde{S}_g 1_e = \widetilde{T}_{g^{-1}} 1_e$.

Because the representations S and T are C^∞-differentiable, we may consider their differentials, that is-the representations of the universal enveloping algebra $U(\mathfrak{g})$, which we shall also denote by S and T, and the conjugate representations will be again denoted by $\widetilde{S}$ and $\widetilde{T}$. In algebra $U(\mathfrak{g})$, we consider the principal anti-automorphism $u \longrightarrow u'$ where $u' = -u$ for $u \in \mathfrak{g}$ and $(u_1 u_2 \ldots u_k)' = u'_k u'_{k-1} \ldots u'_1$. Then we get $\widetilde{S}(u)1_e = \widetilde{T}(u')1_e$ for all $u \in U(\mathfrak{g})$.

The space $E_\sigma = F_{\alpha_{a,b,c,d}}(G) = \{f \in C^\infty(G) | f(gb_-) = \alpha^{-1}_{a,b,c,d}(b_-)f(g), b_- \in B_-, g \in G\}$ can be identified as the space of the solutions to the system of PDE

$T(x_\gamma)f = 0$, for all $\gamma \in \Pi_0$ (Π_0- the set of simple roots)

$T(x - \langle \sigma - \rho, x\rangle)f = 0$, for all $x \in \mathfrak{h}$, σ is the signature of the inducing representation and ρ the half-sum of all positive roots [4].

Denote by I_σ the cyclic submodule in $U(\mathfrak{g})$-module $\mathfrak{X}$ generated by the elements $\widetilde{S}(x_\gamma)1_e$, for all $\gamma \in \Pi_0$, $\widetilde{S}(x - \langle \sigma - \rho, x\rangle)1_e$, for all $x \in \mathfrak{h}$.

Proposition 6.1. *E_σ is the orthogonal complement for I_σ with respect to the canonical bilinear form $\langle .,. \rangle$.*

Proof. Let $f \in E_\sigma$. Then $\left\langle f, \widetilde{S}_g \widetilde{S}_{x_\beta} 1_e \right\rangle = \left\langle S_{g^{-1}} f, \widetilde{T}_{x'_\beta} 1_e \right\rangle = \langle T_{x_\beta} S_{g^{-1}} \; f, 1_e\rangle$ $= \langle S_{g^{-1}} T_{x_\beta} f, 1_e\rangle$ =0, $\beta \in \Pi_0$, and $\left\langle f, \widetilde{S}_g \widetilde{S}_{x - \langle \sigma - \rho, x\rangle} 1_e \right\rangle$
$= \left\langle S_{g^{-1}} f, \widetilde{T}_{-x - \langle \sigma - \rho, x\rangle} 1_e \right\rangle = \langle S_{g^{-1}} T_{x - \langle \sigma - \rho, x\rangle} f, 1_e\rangle = 0$, $x \in \mathfrak{h}$.
So that $E_\sigma \subset (I_\sigma)^\perp$. Reversely, let $\varphi \in (I_\sigma)^\perp$. Then $S_g \varphi \in (I_\sigma)^\perp$, $0 = \left\langle S_{g^{-1}} \varphi, \widetilde{S}_{x_\beta} 1_e \right\rangle = \langle S_{g^{-1}} T_{x_\beta} \varphi, 1_e\rangle = \left\langle T_{x_\beta} \varphi, \widetilde{S}_g 1_e \right\rangle = \langle S_{x_\beta} \varphi, 1_g\rangle$ for all $g \in G$. But this is equivalent to $T_{x_\beta} \varphi = 0$. Similarly for $x - \langle \sigma - \rho, x\rangle$, $x \in h$. Thus $E_\sigma = (I_\sigma)^\perp$. □

Following [3], let $\sigma \in \mathfrak{h}^*$ and χ be a positive root such that $\sigma(\chi) = N$, where N is a positive integer, and let M_σ be the Verma module corresponding to σ, and 1_σ is a highest weight vector of weight $\sigma - \rho$ in the module M_σ. Then $M_{\sigma - N\chi}$ is imbeddable into M_σ and so there exists $S^N_{\sigma,\chi}$ in the universal enveloping algebra of a maximal nilpotent subalgebra of $\mathfrak{g}$ spanned by all negative root vectors such that $\widetilde{1}_{\sigma-N\chi} = S^N_{\sigma,\chi}1_\sigma$, where $\widetilde{1}_{\sigma-N\chi}$ is the image of $1_{\sigma-N\chi}$ under the imbedding.

Proposition 6.2. *$T(S^N_{\sigma,\chi})$ is an intertwining operator for E_σ and $E_{\sigma'}$, $\sigma' = \sigma - N\chi$.*

Proof. It is sufficient to show that $T(S^N_{\sigma,\chi})E_\sigma \subset E_{\sigma'}$. Let $f \in E_\sigma$ and $\gamma \in \prod_0$. Then $\left\langle T_{S^N_{\sigma,\chi}} f, \widetilde{S}_g \widetilde{S}_{x_\gamma} 1_e \right\rangle = \left\langle S_{g^{-1}} f, \widetilde{S}_{x_\gamma} \widetilde{T}_{(S^N_{\sigma,\chi})'} 1_e \right\rangle = \left\langle S_{g^{-1}} f, \widetilde{S}_{x_\gamma} \widetilde{S}_{S^N_{\sigma,\chi}} 1_e \right\rangle = \left\langle S_{g^{-1}} f, \widetilde{S}_{x_\gamma S^N_{\sigma,\chi}} 1_e \right\rangle$. But $x_\gamma S^N_{\sigma,\chi} \in I_\sigma$, and so $\widetilde{S}_{x_\gamma S^N_{\sigma,\chi}} 1_e \in I_\sigma$, but since $S_{g^{-1}} f \in E_\sigma$ the last expression is equal to 0.

Let $x \in \mathfrak{h}$. Then
$\left\langle T_{S^N_{\sigma,\chi}} f, \widetilde{S}_g \widetilde{S}_{(x-\langle\sigma-N\chi-\rho,x\rangle)} 1_e \right\rangle = \left\langle S_{g^{-1}} f, \widetilde{T}_{(S^N_{\sigma,\chi})'} \ \widetilde{S}_{(x-\langle\sigma-N\chi-\rho,x\rangle)} 1_e \right\rangle$
$= \left\langle S_{g^{-1}} f, \widetilde{S}_{(x-\langle\sigma-N\chi-\rho,x\rangle) S^N_{\sigma,\chi}} 1_e \right\rangle$. But $(x - \langle \sigma - N\chi - \rho, x\rangle)\, S^N_{\sigma,\chi} \in I_\sigma$, and so $\widetilde{S}_{(x-\langle\sigma-N\alpha-\rho,x\rangle) S^N_{\sigma,\chi}} 1_e \in I_\sigma$, and therefore the last expression equals 0, because $S_{g^{-1}} f \in E_\sigma \subset I^\perp_\sigma$. This is the end of the proof. □

When γ is a simple root, then $S^N_{\sigma,\gamma} = z^N_{-\gamma}$, where for any root λ, z_λ is a root vector for λ. For the simple root α_1, $I_1 f(h) = \frac{d}{dr} T_{E^{p+1}_{\alpha_1}(r)} f(h)|_{r=0} = (\frac{\partial}{\partial x_1})^{p+1} f(h) = A^{p+1} f(h)$ is an intertwining operator for $E_{p\omega_1+q\omega_2+s\omega_3+t\omega_4}$ and $E_{-(p+2)\omega_1+(p+q+1)\omega_2+s\omega_3+t\omega_4}$. So, $V^{p,q,s,t} \subseteq Ker\ I_1$. For the simple root α_2, $I_2 f(h) = \frac{d}{dr} T_{E^{q+1}_{\alpha_2}(r)} f(h)|_{r=0} = (x_1 \frac{\partial}{\partial x_2} - x_5 \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7})^{q+1} f(h) = B^{q+1} f(h)$ is an intertwining operator for $E_{p\omega_1+q\omega_2+s\omega_3+t\omega_4}$ and $E_{(p+q+1)\omega_1-(q+2)\omega_2+(s+q+1)\omega_3+(t+q+1)\omega_4}$. So $V^{p,q,s,t} \subseteq Ker\ I_2$. For the simple root α_3, $I_3 f(h) = \frac{d}{dr} T_{E^{s+1}_{\alpha_3}(r)} f(h)|_{r=0} = (x_2 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_5} + x_7 \frac{\partial}{\partial x_8} - x_9 \frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{11}})^{s+1} f(h) = C^{s+1} f(h)$ is an intertwining operator for $E_{p\omega_1+q\omega_2+s\omega_3+t\omega_4}$ and $E_{p\omega_1+(q+s+1)\omega_2-(s+2)\omega_3+t\omega_4}$. So $V^{p,q,s,t} \subseteq Ker\ I_3$. For the simple root α_4, $I_4 f(h) = \frac{d}{dr} T_{E^{t+1}_{\alpha_4}(r)} f(h)|_{r=0} = (x_2 \frac{\partial}{\partial x_4} - x_3 \frac{\partial}{\partial x_5} + x_7 \frac{\partial}{\partial x_9} - x_8 \frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}})^{t+1} f(h) = D^{t+1} f(h)$ is an intertwining operator for $E_{p\omega_1+q\omega_2+s\omega_3+t\omega_4}$ and $E_{p\omega_1+(q+1)\omega_2+s\omega_3-(t+2)\omega_4}$. So $V^{p,q,s,t} \subseteq Ker\ I_4$.

Then $V^{p,q,s,t} = (Ker\ I_1) \cap (Ker\ I_2) \cap (Ker\ I_3) \cap (Ker\ I_4)$. That is, $P_2^{m_2} P_3^{m_3} P_4^{m_4} P_5^{m_5} P_6^{m_6} P_7^{m_7} P_8^{m_8} Q_2^{n_2} Q_3^{n_3} Q_4^{n_4} Q_5^{n_5} Q_6^{n_6} Q_7^{n_7} Q_8^{n_8} Q_9^{n_9} Q_{10}^{n_{10}} Q_{11}^{n_{11}} Q_{12}^{n_{12}} Q_{13}^{n_{13}} Q_{14}^{n_{14}} Q_{15}^{n_{15}} Q_{16}^{n_{16}} Q_{17}^{n_{17}} Q_{18}^{n_{18}} Q_{19}^{n_{19}} Q_{20}^{n_{20}} Q_{21}^{n_{21}} Q_{22}^{n_{22}} Q_{23}^{n_{23}} Q_{24}^{n_{24}} Q_{25}^{n_{25}} Q_{26}^{n_{26}} Q_{27}^{n_{27}} Q_{28}^{n_{28}} S_2^{k_2} S_3^{k_3} S_4^{k_4} S_5^{k_5} S_6^{k_6} S_7^{k_7} S_8^{k_8} T_2^{l_2} T_3^{l_3} T_4^{l_4} T_5^{l_5} T_6^{l_6} T_7^{l_7} T_8^{l_8}$, where $m_2 + \ldots + m_8 \le p$, $n_2 + \ldots + n_{28} \le q$, $k_2 + \ldots + k_8 \le s$, and $l_2 + \ldots + l_8 \le t$, are all solutions of the system (1.1), and they are the only solutions of this system of partial differential equations.

7. Conclusion

This study presents a method to find all solutions of a certain system of PDE. by considering representations of a Lie group. That is, the space of solutions of the systems of partial differential equations

$$\begin{aligned}\left[\frac{\partial}{\partial x_1}\right]^{p+1}\varphi &= 0,\\ \left[x_1\frac{\partial}{\partial x_2}-x_5\frac{\partial}{\partial x_6}+\frac{\partial}{\partial x_7}\right]^{q+1}\varphi &= 0,\\ \left[x_2\frac{\partial}{\partial x_3}-x_4\frac{\partial}{\partial x_5}+x_7\frac{\partial}{\partial x_8}-x_9\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{11}}\right]^{s+1}\varphi &= 0,\\ \left[x_2\frac{\partial}{\partial x_4}-x_3\frac{\partial}{\partial x_5}+x_7\frac{\partial}{\partial x_9}-x_8\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{12}}\right]^{t+1}\varphi &= 0,\end{aligned}$$

where p, q, s and t are non-negative integers, and the operators $\frac{\partial}{\partial x_1}$, $x_2\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_5}+x_7\frac{\partial}{\partial x_8}-x_9\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{11}}$, $x_2\frac{\partial}{\partial x_3}-x_4\frac{\partial}{\partial x_5}+x_7\frac{\partial}{\partial x_8}-x_9\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{11}}$, $x_2\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5}+x_7\frac{\partial}{\partial x_9}-x_8\frac{\partial}{\partial x_{10}}+\frac{\partial}{\partial x_{12}}$ generate a Lie algebra of differential operators that is isomorphic to a maximal nilpotent subalgebra of Lie algebra of class D_4 is the space $V^{p,q,s,t}$ of an irreducible representation spanned by the vectors $P_2^{m_2}P_3^{m_3}P_4^{m_4}P_5^{m_5}P_6^{m_6}P_7^{m_7}P_8^{m_8}Q_2^{n_2}Q_3^{n_3}Q_4^{n_4}Q_5^{n_5}Q_6^{n_6}Q_7^{n_7}Q_8^{n_8}Q_9^{n_9}Q_{10}^{n_{10}}Q_{11}^{n_{11}}Q_{12}^{n_{12}}$ $Q_{13}^{n_{13}}Q_{14}^{n_{14}}Q_{15}^{n_{15}}Q_{16}^{n_{16}}Q_{17}^{n_{17}}Q_{18}^{n_{18}}Q_{19}^{n_{19}}Q_{20}^{n_{20}}Q_{21}^{n_{21}}Q_{22}^{n_{22}}Q_{23}^{n_{23}}Q_{24}^{n_{24}}Q_{25}^{n_{25}}Q_{26}^{n_{26}}Q_{27}^{n_{27}}Q_{28}^{n_{28}}$ $S_2^{k_2}S_3^{k_3}S_4^{k_4}S_5^{k_5}S_6^{k_6}S_7^{k_7}S_8^{k_8}T_2^{l_2}T_3^{l_3}T_4^{l_4}T_5^{l_5}T_6^{l_6}T_7^{l_7}T_8^{l_8}$, where $m_2+\ldots+m_8 \le p$, $n_2+\ldots+n_{28}\le q$, $k_2+\ldots+k_8\le s$, and $l_2+\ldots+l_8\le t$, and $P_2 = x_1$, $P_3 = -x_2 + x_1x_7$, $P_4 = x_2x_{11} - x_3 + x_1x_8 - x_1x_7x_{11}$, $P_5 = x_2x_{12} - x_4 + x_1x_9 - x_1x_7x_{12}$, $P_6 = x_3x_{12} - x_{12}x_2x_{11} + x_4x_{11} + x_5 - x_{12}x_1x_8 + x_{12}x_1x_7x_{11} - x_1x_{10} - x_{11}x_1x_9$, $P_7 = x_1x_7x_{10} - x_2x_{10} - x_5x_7 - x_6 + x_9x_1x_8 - x_4x_8 - x_9x_3$, $P_8 = x_2x_5+x_3x_4+x_1x_6$, $Q_2 = x_7$, $Q_3 = x_2$, $Q_4 = x_7x_{11}-x_8$, $Q_5 = x_7x_{12}-x_9$, $Q_6 = x_3-x_2x_{11}$, $Q_7 = x_4-x_2x_{12}$, $Q_8 = x_8x_{12}-x_{12}x_7x_{11}+x_{10}+x_9x_{11}$, $Q_9 = x_{12}x_2x_{11}-x_3x_{12}-x_4x_{11}-x_5$, $Q_{10} = x_8x_2-x_7x_3$, $Q_{11} = x_9x_2-x_7x_4$, $Q_{12} = x_8x_9+x_7x_{10}$, $Q_{13} = x_2x_{10}+x_9x_3+x_4x_8+x_5x_7$, $Q_{14} = 2x_5x_7+x_6-x_2x_8x_{12}+x_4x_8-x_9x_2x_{11}+x_9x_3+x_7x_{12}x_3+x_7x_{11}x_4$, $Q_{15} = x_9x_3-2x_9x_2x_{11}+2x_7x_{11}x_4+x_5x_7-x_4x_8-x_2x_{10}$, $Q_{16} = x_4x_8-2x_2x_8x_{12}+2x_7x_{12}x_3+x_5x_7-x_9x_3-x_2x_{10}$, $Q_{17} = 2x_2x_5+2x_3x_4$, $Q_{18} = x_9x_8x_2-x_7^2x_5-x_7x_6-x_9x_7x_3-x_8x_7x_4$, $Q_{19} = x_8x_4x_{11}+x_8x_5-x_{10}x_3-x_5x_{11}x_7+x_{11}x_{10}x_2-x_3x_{11}x_9+x_2x_{11}^2x_9-x_4x_{11}^2x_7$, $Q_{20} = x_9x_3x_{12}+x_9x_5-x_4x_{12}x_8-x_5x_{12}x_7+x_{12}x_{10}x_2-x_{10}x_4+x_2x_{12}^2x_8-x_3x_{12}^2x_7$, $Q_{21} = x_2x_5x_7+x_4x_7x_3+x_2x_6$, $Q_{22} = x_7^2x_{11}x_5-x_8x_5x_7-x_8x_6+x_7x_{11}x_6-x_8x_9x_2x_{11}+x_7x_{11}x_8x_4-x_8^2x_4-x_{10}x_8x_2+x_{10}x_7x_3+x_7x_{11}x_9x_3$, $Q_{23} = x_7^2x_{12}x_5-x_9x_5x_7-x_9x_6+x_7x_{12}x_6-x_9x_2x_8x_{12}+x_7x_{12}x_9x_3-x_9^2x_3-x_{10}x_9x_2+x_7x_{12}x_8x_4+x_{10}x_7x_4$, $Q_{24} = x_2x_5x_8-x_2x_{11}x_5x_7+x_3x_4x_8-x_3x_7x_{11}x_4-x_2x_{11}x_6+x_3x_6$, $Q_{25} = x_2x_5x_9-x_2x_{12}x_5x_7+x_4x_9x_3-x_4x_7x_{12}x_3-$

$x_2x_{12}x_6+x_4x_6$, $Q_{26}=x_8x_{12}x_2x_{10}+x_9x_{11}x_2x_{10}-x_{10}x_7x_{12}x_3-x_{10}x_7x_{11}x_4+x_5x_7x_{12}x_8+x_5x_{11}x_9x_7-x_5x_{11}x_7^2x_{12}-x_7x_{11}x_{12}x_6+x_8^2x_{12}x_4+x_9^2x_{11}x_3+x_{10}x_8x_4+x_{10}x_9x_3-x_7x_{11}x_4x_8x_{12}-x_9x_{11}x_7x_{12}x_3+x_2x_{11}x_9x_{12}x_8-x_8x_9x_5+x_8x_{12}x_6+x_9x_{11}x_6+x_{10}x_6+x_2x_{10}^2$, $Q_{27}=x_3x_4x_{12}x_7x_{11}+x_2x_5x_{12}x_7x_{11}-x_3x_{12}x_8x_4+x_2x_{11}x_{12}x_6-x_4x_{11}x_9x_3-x_5x_2x_8x_{12}-x_5x_9x_2x_{11}-x_5x_2x_{10}-x_5x_6-x_4x_{11}x_6-x_3x_{12}x_6-x_3x_4x_{10}$, $Q_{28}=x_2x_7x_5x_{10}+x_6x_2x_{10}+x_9x_3x_6+x_8x_4x_6+x_6^2+x_8x_9x_3x_4+x_8x_9x_2x_5+x_5x_7x_6+x_{10}x_3x_7x_4$, $S_2=x_{11}$, $S_3=x_8$, $S_4=x_3$, $S_5=x_8x_{12}+x_{10}$, $S_6=x_3x_{12}+x_5$, $S_7=x_5x_7+x_6+x_9x_3$, $S_8=x_3x_{11}x_9+x_5x_{11}x_7-x_8x_5+x_{10}x_3+x_{11}x_6$, $T_2=x_{12}$, $T_3=x_9$, $T_4=x_4$, $T_5=x_{10}+x_9x_{11}$, $T_6=x_4x_{11}+x_5$, $T_7=x_5x_7+x_6+x_8x_4$,$T_8=x_4x_{12}x_8+x_5x_{12}x_7-x_9x_5+x_{10}x_4+x_{12}x_6$.

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