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Certain Polynomials with Weighted Sums

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Abstract

In this note, we provide some examples of polynomials $z^n - p(z)$, where $p(z) = \sum_{k=0}^{n-1} a_k z^k$, and $\sum_{k=0}^{n-1} a_k z^k = 1$, $a_k \geq 0$ for each k such that $p(z)$ has all its zeros on $|z| = c < 1$, and $z^n - p(z)$ has all its zeros on two circles $|z| = 1$ and $|z| = d < 1$.

Keywords: Polynomial, Weighted Sum

1. Introduction

Throughout this paper, is an integer ≥3, $p>1$, and we denote $C(r)$ by the circle of radius r with center the origin. It follows from Enestrm-Kakeya theorem[1] (see p. 136 of [1] for the statement and its proof) to

$$\frac{z^n - \sum_{k=0}^{n-1} a_k z^k}{z-1} \tag{1}$$

where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k that all zeros of (1) do not lie outside $C(1)$. To what extent, are there polynomials

$$p(z) = \sum_{k=0}^{n-1} a_k z^k$$

where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k with all its zeros on $|z| = c<1$ such that $z^n - p(z)$ has all its zeros on two circles $|z| = 1$ and $|z| = d < 1$? Whether or not certain polynomials have all their zeros on some circles is one of the most fundamental questions in the theory of distribution of polynomial zeros. Kim[2] studied polynomials of type (1),

$$z^n - \sum_{k=0}^{n-1} a_k z^k$$

whose all zeros except for $z = 1$ lie on $C(1/p)$. For convenience, we call these polynomials $C(1/p)$-polynomials, and $\sum_{k=0}^{n-1} a_k z^k$ their weighted sums, respectively.

Kim[2] showed that, given $p>1$, there exist $C(1/p)$-polynomials whose the degree of weighted sum is $n-1$. However, by estimating some coefficients of lacunary polynomials, he obtained sufficient conditions for nonexistence of certain lacunary $C(1/p)$-polynomials: If $p>n-1$, then there does not exist $C(1/p)$-polynomials whose the degree of weighted sums is $n-2$. Also, if

$$2p^4 - (n-1)(n-2)p^2 - 2(n-1)p - (n-1)(n-2) > 0$$

then there does not exist $C(1/p)$-polynomials whose the degree of weighted sum is $n-3$. In case of the degree of weighted sum $n-2$, he also showed that, by giving an example, his sufficient condition is best possible in the sense that, for all $n \geq 3$, there exist $C(1/p)$-polynomials with the degree of the weighted sums $n-2$.

2. Results and Discussion

In this section, we provide some examples of polynomials $z^n - p(z)$, where

$$p(z) = \sum_{k=0}^{n-1} a_k z^k$$

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and $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k such that $p(z)$ has all its zeros on $|z| = c<1$, and $z^n - p(z)$ has all its zeros on two circles $|z| = 1$ and $|z| = d<1$.

Proposition Let k be a positive integer. If $n = 3k$ or $n = 5k$, then there are weighted polynomial sums $p(z)$ with all its zeros on $|z| = c<1$ such that $z^n - p(z)$ has all its zeros on two circles $|z| = 1$ and $|z| = d<1$.

Proof For $n = 3k$ for some integer $k \geq 1$, we consider

$$u(z) := z^{3k} - \frac{1}{n^2}(1+(n-1)z^k + n(n-1)z^{2k})$$
$$= \frac{(z^k-1)((3kz^k)^2 + 3kz^k + 1)}{(3k)^2}$$

Then the weighted sum

$$\frac{1}{n^2}(1+(n-1)z + n(n-1)z^2)$$

has its zeros

$$\frac{1-n \pm i\sqrt{(-1+n)(1+3n)}}{2(-1+n)n}$$

with modulus $\sqrt{\frac{1}{(-1+n)n}}$, and so all zeros of (2) have modulus $\left(\frac{1}{(-1+3k)3k}\right)^{1/2k}$. Also $(3kz^k)^2 + 3kz^k + 1 = 0$ implies that $|3kz^k| = 1$ and so $|z| = \left(\frac{1}{3k}\right)^{1/k}$, which implies that all zeros of $u(z)$ lie on two circles $C(1)$ and $C\left(\left(\frac{1}{3k}\right)^{1/k}\right)$. For $n = 5k$ for some integer $k \geq 1$, we consider

$$v(z) := z^{5k} - \frac{1}{n^2}(1+(n-1)z^{2k} + n(n-1)z^{4k})$$
$$= \frac{(z^k-1)(25k^2z^{4k} + 5kz^{3k} + 5kz^{2k} + z^k + 1)}{25k^2}$$

Putting $y = z^k$ in the second factor of the numerator in above gives

$$25k^2y^4 + 5ky^3 + 5ky^2 + y + 1 = 0$$

whose roots can be computed by

$$\frac{1}{20k}\left(-1 \pm \sqrt{1+20k} \pm i\sqrt{-2+60k+2\sqrt{1+20k}}\right)$$

and their moduli are all equal to $\sqrt{\frac{1}{5k}}$. So all zeros of $v(z)$ lie on two circles $C(1)$ and $C\left(\left(\frac{1}{5k}\right)^{1/5}\right)$. Finally as in the case when $n = 3k$, we can compute that the weighted sum $\frac{1}{n^2}(1+(n-1)z^{2k}+n(n-1)z^{4k})$ has all its zeros $\left(\frac{1}{(-1+5k)5k}\right)^{1/(4k)}$. This completes the proof.

References

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[2] S.-H. Kim, "Polynomials with weighted sum", Publ. Math-Debrecen, Vol. 66, pp. 303-311, 2005.