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THE COMPOSITION SERIES OF IDEALS OF THE PARTIAL-ISOMETRIC CROSSED PRODUCT BY SEMIGROUP OF ENDOMORPHISMS

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Abstract. Let Γ^+ be the positive cone in a totally ordered abelian group Γ, and α an action of Γ^+ by extendible endomorphisms of a C^*-algebra A. Suppose I is an extendible α-invariant ideal of A. We prove that the partial-isometric crossed product $\mathcal{I} := I \times_\alpha^{\mathrm{piso}} \Gamma^+$ embeds naturally as an ideal of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$, such that the quotient is the partial-isometric crossed product of the quotient algebra. We claim that this ideal $\mathcal{I}$ together with the kernel of a natural homomorphism $\phi : A \times_\alpha^{\mathrm{piso}} \Gamma^+ \to A \times_\alpha^{\mathrm{iso}} \Gamma^+$ gives a composition series of ideals of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ studied by Lindiarni and Raeburn.

1. Introduction

Let (A, Γ^+, α) be a dynamical system consisting of the positive cone Γ^+ in a totally ordered abelian group Γ, and an action $\alpha : \Gamma^+ \to \operatorname{End} A$ of Γ^+ by extendible endomorphisms of a C^*-algebra A. A covariant representation of the system (A, Γ^+, α) is defined for which the semigroup of endomorphisms $\{\alpha_s : s \in \Gamma^+\}$ are implemented by partial isometries, and then the associated partial-isometric crossed product C^*-algebra $A \times_\alpha^{\mathrm{piso}} \Gamma^+$, generated by a universal covariant representation, is characterized by the property that its nondegenerate representations are in a bijective correspondence with covariant representations of the system. This generalizes the covariant isometric representation theory: the theory that uses isometries to represent the semigroup of endomorphisms in a covariant representation of the system. We denoted by $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ for the corresponding isometric crossed product.

Suppose I is an extendible α-invariant ideal of A, then $a + I \mapsto \alpha_x(a) + I$ defines an action of Γ^+ by extendible endomorphisms of the quotient algebra A/I. It is well-known that the isometric crossed product $I \times_\alpha^{\mathrm{iso}} \Gamma^+$ sits naturally

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as an ideal in $A \times_{\alpha}^{\text{iso}} \Gamma^+$ such that $(A \times_{\alpha}^{\text{iso}} \Gamma^+)/(I \times_{\alpha}^{\text{iso}} \Gamma^+) \simeq A/I \times_{\alpha}^{\text{iso}} \Gamma^+$. We show that this result is valid for the partial-isometric crossed product.

Moreover if $\phi : A \times_{\alpha}^{\text{piso}} \Gamma^+ \to A \times_{\alpha}^{\text{iso}} \Gamma^+$ is the natural homomorphism given by the canonical universal covariant isometric representation of (A, Γ^+, α) in $A \times_{\alpha}^{\text{iso}} \Gamma^+$, then $\ker \phi$ together with the ideal $I \times_{\alpha}^{\text{piso}} \Gamma^+$ give a composition series of ideals of $A \times_{\alpha}^{\text{piso}} \Gamma^+$, from which we recover the structure theorems of [7]. Let us now consider the framework of [7]. A system that consists of the C^*-subalgebra $A := B_{\Gamma^+}$ of $\ell^\infty(\Gamma^+)$ spanned by the functions 1_s satisfying

$$1_s(t) = \begin{cases} 1 & \text{if } t \geq s \\ 0 & \text{otherwise,} \end{cases}$$

and the action $\tau : \Gamma^+ \to \operatorname{End} B_{\Gamma^+}$ given by the translation on $\ell^\infty(\Gamma^+)$. We choose an extendible τ-invariant ideal I to be the subalgebra $B_{\Gamma^+,\infty}$ spanned by $\{1_x - 1_y : x < y \in \Gamma^+\}$. Then the composition series of ideals of $B_{\Gamma^+} \times_{\tau}^{\text{piso}} \Gamma^+$, that is given by the two ideals $\ker \phi$ and $B_{\Gamma^+,\infty} \times_{\tau}^{\text{piso}} \Gamma^+$, produces the large commutative diagram in [7, Theorem 5.6]. This result shows that the commutative diagram in [7, Theorem 5.6] exists for any totally ordered abelian subgroup (not only for subgroups of $\mathbb{R}$), and that we understand clearly where the diagram comes from.

Next, if we consider a specific semigroup Γ^+ such as the additive semigroup $\mathbb{N}$ in the group of integers $\mathbb{Z}$, then the large commutative diagram gives a clearer information about the ideals structure of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$. We can identify that the left-hand and top exact sequences in diagram [7, Theorem 5.6] are indeed equivalent to the extension of the algebra $\mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c}_0))$ of compact operators on the Hilbert module $\ell^2(\mathbb{N}, \mathbf{c}_0)$ by $\mathcal{K}(\ell^2(\mathbb{N}))$ provided by the algebra $\mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c}))$ of compact operators on $\ell^2(\mathbb{N}, \mathbf{c})$. Moreover it is known that $\operatorname{Prim} \mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c})) \simeq \operatorname{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathbf{c}) \simeq \operatorname{Prim} \mathbf{c}$ is homeomorphic to $\mathbb{N} \cup \infty$. Together with a knowledge about the primitive ideal space of the Toeplitz C^*-algebra generated by the unilateral shift, our theorem on the composition series of ideals of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$ provides a complete description of the topology on the primitive ideal space of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$.

We begin with a section containing background material about the partial-isometric crossed product by semigroups of extendible endomorphisms. In Section 3, we prove the existence of a short exact sequence of partial-isometric crossed products, which generalizes [2, Theorem 2.2] of the semigroup $\mathbb{N}$. Then we consider this and the other natural exact sequence described earlier in [4], to get the composition series of ideals in $A \times_{\alpha}^{\text{piso}} \Gamma^+$.

We proceed to Section 4 by applying our results in Section 3 to the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$ and the extendible τ-invariant ideal $B_{\Gamma^+,\infty}$ of B_{Γ^+}. It can be seen from our Proposition 4.1 that the large commutative diagram of [7, Theorem 5.6] remains valid for any subgroup Γ of a totally ordered abelian group. Finally in the last section we describe the topology of primitive ideal space of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$ by using this large diagram.

2. Preliminaries

A bounded operator V on a Hilbert space H is called an isometry if $\|V(h)\| = \|h\|$ for all $h \in H$, which is equivalent to $V^*V = 1$. A bounded operator V on a Hilbert space H is called a *partial isometry* if it is isometry on $(\ker V)^\perp$. This is equivalent to $VV^*V = V$. If V is a partial isometry, then so is the adjoint V^*, where as for an isometry V, the adjoint V^* may not be an isometry unless V is unitary. Associated to a partial isometry V, there are two orthogonal projections V^*V and VV^* on the initial space $(\ker V)^\perp$ and on the range VH respectively. In a C^*-algebra A, an element $v \in A$ is called an isometry if $v^*v = 1$ and a partial isometry if $vv^*v = v$.

An *isometric representation* of Γ^+ on a Hilbert space H is a map $S : \Gamma^+ \to B(H)$ which satisfies $S_x := S(x)$ is an isometry, and $S_{x+y} = S_xS_y$ for all $x, y \in \Gamma^+$. So an isometric representation of $\mathbb{N}$ is determined by a single isometry S_1. Similarly a *partial-isometric representation* of Γ^+ on a Hilbert space H is a map $V : \Gamma^+ \to B(H)$ which satisfies $V_x := V(x)$ is a partial isometry, and $V_{x+y} = V_xV_y$ for all $x, y \in \Gamma^+$. Note that the product VW of two partial isometries V and W is a partial isometry precisely when V^*V commutes with WW^* [7, Proposition 2.1]. Thus a partial isometry V is called a *power partial isometry* if V^n is a partial isometry for every $n \in \mathbb{N}$, so a partial-isometric representation of $\mathbb{N}$ is determined by a single power partial isometry V_1. If V is a partial-isometric representation of Γ^+, then every $V_xV_x^*$ commutes with $V_tV_t^*$, and so does $V_x^*V_x$ with $V_t^*V_t$.

Now we consider a dynamical system (A, Γ^+, α) consisting of a C^*-algebra A, an action α of Γ^+ by endomorphisms of A such that $\alpha_0 = \mathrm{id}$. Because we deal with non unital C^*-algebras and non unital endomorphisms, we require every endomorphism α_x to be extendible to a strictly continuous endomorphism $\overline{\alpha}_x$ on the multiplier algebra $M(A)$ of A. This happens precisely when there exists an approximate identity (a_λ) in A and a projection $p_{\alpha_x} \in M(A)$ such that $\alpha_x(a_\lambda)$ converges strictly to p_{α_x} in $M(A)$.

Definition 2.1. A *covariant isometric representation* of (A, Γ^+, α) on a Hilbert space H is a pair (π, S) of a nondegenerate representation $\pi : A \to B(H)$ and an isometric representation of $S : \Gamma^+ \to B(H)$ such that $\pi(\alpha_x(a)) = S_x\pi(a)S_x^*$ for all $a \in A$ and $x \in \Gamma^+$.

An *isometric crossed product* of (A, Γ^+, α) is a triple (B, j_A, j_{Γ^+}) consisting of a C^*-algebra B, a canonical covariant isometric representation (j_A, j_{Γ^+}) in $M(B)$ which satisfies the following:

(i) for every covariant isometric representation (π, S) of (A, Γ^+, α) on a Hilbert space H, there exists a nondegenerate representation $\pi \times S : B \to B(H)$ such that $(\pi \times S) \circ j_A = \pi$ and $(\overline{\pi \times S}) \circ j_{\Gamma^+} = S$; and

(ii) B is generated by $j_A(A) \cup j_{\Gamma^+}(\Gamma^+)$, we actually have

$$B = \overline{\operatorname{span}}\{j_{\Gamma^+}(x)^* j_A(a) j_{\Gamma^+}(y) : x, y \in \Gamma^+,\ a \in A\}.$$

Note that a given system (A, Γ^+, α) could have a covariant isometric representation (π, S) only with $\pi = 0$. In this case the isometric crossed product yields no information about the system. If a system admits a non trivial covariant representation, then the isometric crossed product does exist, and it is unique up to isomorphism: if there is such a covariant isometric representation (t_A, t_{Γ^+}) of (A, Γ^+, α) in a C^*-algebra C, then there is an isomorphism of C onto B which takes (t_A, t_{Γ^+}) into (j_A, j_{Γ^+}). Thus we write the isometric crossed product B as $A \times_\alpha^{\text{iso}} \Gamma^+$.

The partial-isometric crossed product of (A, Γ^+, α) is defined in a similar fashion involving partial-isometries instead of isometries.

Definition 2.2. A *covariant partial-isometric representation* of (A, Γ^+, α) on a Hilbert space H is a pair (π, S) of a nondegenerate representation $\pi : A \to B(H)$ and a partial-isometric representation $S : \Gamma^+ \to B(H)$ of Γ^+ such that $\pi(\alpha_x(a)) = S_x \pi(a) S_x^*$ for all $a \in A$ and $x \in \Gamma^+$. See in Remark 2.3 that this equation implies $S_x^* S_x \pi(a) = \pi(a) S_x^* S_x$ for $a \in A$ and $x \in \Gamma^+$. Moreover, [7, Lemma 4.2] shows that every (π, S) extends to a partial-isometric covariant representation $(\overline{\pi}, S)$ of $(M(A), \Gamma^+, \overline{\alpha})$, and the partial-isometric covariance is equivalent to $\pi(\alpha_x(a)) S_x = S_x \pi(a)$ and $S_x S_x^* = \overline{\pi}(\overline{\alpha}_x(1))$ for $a \in A$ and $x \in \Gamma^+$.

A *partial-isometric crossed product* of (A, Γ^+, α) is a triple (B, j_A, j_{Γ^+}) consisting of a C^*-algebra B, a canonical covariant partial-isometric representation (j_A, j_{Γ^+}) in $M(B)$ which satisfies the following:

(i) for every covariant partial-isometric representation (π, S) of (A, Γ^+, α) on a Hilbert space H, there exists a nondegenerate representation $\pi \times S : B \to B(H)$ such that $(\pi \times S) \circ j_A = \pi$ and $(\overline{\pi \times S}) \circ j_{\Gamma^+} = S$; and

(ii) B is generated by $j_A(A) \cup j_{\Gamma^+}(\Gamma^+)$, we actually have

$$B = \overline{\text{span}}\{j_{\Gamma^+}(x)^* j_A(a) j_{\Gamma^+}(y) : x, y \in \Gamma^+,\ a \in A\}.$$

Unlike the theory of isometric crossed product: every system (A, Γ^+, α) admits a non trivial covariant partial-isometric representation (π, S) with π faithful [7, Example 4.6]. In fact [7, Proposition 4.7] shows that a canonical covariant partial-isometric representation (j_A, j_{Γ^+}) of (A, Γ^+, α) exists in the Toeplitz algebra $\mathcal{T}_X$ associated to a discrete product system X of Hilbert bimodules over Γ^+, which (i) and (ii) are fulfilled, and it is universal: if there is such a covariant partial-isometric representation (t_A, t_{Γ^+}) of (A, Γ^+, α) in a C^*-algebra C that satisfies (i) and (ii), then there is an isomorphism of C onto B which takes (t_A, t_{Γ^+}) into (j_A, j_{Γ^+}). Thus we write the partial-isometric crossed product B as $A \times_\alpha^{\text{piso}} \Gamma^+$.

Remark 2.3. Our special thanks go to B. Kwaśniewski for showing us the proof arguments in this remark. Assuming (π, S) is covariant, then by C^*-norm equation we have $\|\pi(a) S_x^* - S_x^* \pi(\alpha_x(a))\| = 0$, therefore $\pi(a) S_x^* = S_x^* \pi(\alpha_x(a))$ for all $a \in A$ and $x \in \Gamma^+$, which means that $S_x \pi(a) = \pi(\alpha_x(a)) S_x$ for all

$a \in A$ and $x \in \Gamma^+$. So $S_x^* S_x \pi(a) = S_x^* \pi(\alpha_x(a)) S_x = (\pi(\alpha_x(a^*)) S_x)^* S_x = (S_x \pi(a^*))^* S_x = \pi(a) S_x^* S_x$.

More details on the proof are available in [6, Lemma 1.2].

3. The short exact sequence of partial-isometric crossed products

Theorem 3.1. *Suppose that $(A \times_\alpha^{\mathrm{piso}} \Gamma^+, i_A, V)$ is the partial-isometric crossed product of a dynamical system (A, Γ^+, α), and I is an extendible α-invariant ideal of A. Then there is a short exact sequence*

$$0 \longrightarrow I \times_\alpha^{\mathrm{piso}} \Gamma^+ \xrightarrow{\mu} A \times_\alpha^{\mathrm{piso}} \Gamma^+ \xrightarrow{\gamma} A/I \times_{\tilde{\alpha}}^{\mathrm{piso}} \Gamma^+ \longrightarrow 0, \tag{3.1}$$

where μ is an isomorphism of $I \times_\alpha^{\mathrm{piso}} \Gamma^+$ onto the ideal

$$\mathcal{D} := \overline{\operatorname{span}}\{V_x^* i_A(i) V_y : i \in I,\ x, y \in \Gamma^+\} \text{ of } A \times_\alpha^{\mathrm{piso}} \Gamma^+.$$

If $q : A \to A/I$ is the quotient map, i_I, W denote the maps $I \to I \times_\alpha^{\mathrm{piso}} \Gamma^+$, $W : \Gamma^+ \to M(I \times_\alpha^{\mathrm{piso}} \Gamma^+)$, and similarly for $i_{A/I}, U$ the maps $A/I \to A/I \times_{\tilde{\alpha}}^{\mathrm{piso}} \Gamma^+$, $\Gamma^+ \to M(A/I \times_{\tilde{\alpha}}^{\mathrm{piso}} \Gamma^+)$, then

$$\mu \circ i_I = i_A|_I, \quad \overline{\mu} \circ W = V \quad \text{and} \quad \gamma \circ i_A = i_{A/I} \circ q, \quad \overline{\gamma} \circ V = U.$$

Proof. We make some minor adjustment to the proof of [1, Theorem 3.1] for partial isometries. First, we check that $\mathcal{D}$ is indeed an ideal of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$. Let $\xi = V_x^* i_A(i) V_y \in \mathcal{D}$. Then $V_s^* \xi$ is trivially contained in $\mathcal{D}$, and computations below show that $i_A(a)\xi$ and $V_s \xi$ are all in $\mathcal{D}$ for $a \in A$ and $s \in \Gamma^+$:

$$\begin{aligned}
i_A(a)\xi &= i_A(a) V_x^* i_A(i) V_y = (V_x i_A(a^*))^* i_A(i) V_y \\
&= (i_A(\alpha_x(a^*)) V_x)^* i_A(i) V_y = V_x^* i_A(\alpha_x(a) i) V_y; \\
V_s \xi &= V_s V_x^* i_A(i) V_y = V_s (V_s^* V_s V_x^* V_x) V_x^* i_A(i) V_y \\
&= V_s V_u^* V_u V_x^* i_A(i) V_y, \quad u := \max\{s, x\} \\
&= (V_s V_s^* V_{u-s}^*)(V_{u-x} V_x V_x^*) i_A(i) V_y = V_{u-s}^* (V_u V_u^* V_u V_u^*)(V_{u-x} i_A(i)) V_y \\
&= V_{u-s}^* V_u V_u^* i_A(\alpha_{u-x}(i)) V_{u-x} V_y = V_{u-s}^* \overline{i}_A(\overline{\alpha}_u(1)) i_A(\alpha_{u-x}(i)) V_{u-x+y}.
\end{aligned}$$

This ideal $\mathcal{D}$ gives us a nondegenerate homomorphism $\psi : A \times_\alpha^{\mathrm{piso}} \Gamma^+ \to M(D)$ which satisfies $\psi(\xi)d = \xi d$ for $\xi \in A \times_\alpha^{\mathrm{piso}} \Gamma^+$ and $d \in \mathcal{D}$. Let $j_I : I \xrightarrow{i_A} A \times_\alpha^{\mathrm{piso}} \Gamma^+ \xrightarrow{\psi} M(\mathcal{D})$, and $S : \Gamma^+ \xrightarrow{V} M(A \times_\alpha^{\mathrm{piso}} \Gamma^+) \xrightarrow{\overline{\psi}} M(\mathcal{D})$. We use extendibility of ideal I to show j_I is nondegenerate. Take an approximate identity (e_λ) for I, and let $\varphi : A \to M(I)$ be the homomorphism satisfying $\varphi(a)i = ai$ for $a \in A$ and $i \in I$. Then $i_A(\alpha_s(e_\lambda)i)$ converges in norm to $i_A(\overline{\varphi}(\overline{\alpha}_s(1_{M(A)}))i)$. However

$$i_A(\overline{\varphi}(\overline{\alpha}_s(1_{M(A)}))i) = \overline{i}_A(\overline{\alpha}_s(1_{M(A)})) i_A(i) = V_s V_s^* i_A(i).$$

So $i_A(\alpha_s(e_\lambda)i)$ converges in norm to $V_s V_s^* i_A(i)$. Since $j_I(e_\lambda) V_s^* i_A(i) V_t = V_s^* i_A(\alpha_s(e_\lambda)i) V_t$ by covariance, it follows that $j_I(e_\lambda) V_s^* i_A(i) V_t$ converges in norm to $V_s^* i_A(i) V_t$. We can similarly show that $V_s^* i_A(i) V_t j_I(e_\lambda)$ converges in norm to $V_s^* i_A(i) V_t$. Thus $j_I(e_\lambda) \to 1_{M(\mathcal{D})}$ strictly, and hence j_I is nondegenerate.

We claim that the triple $(\mathcal{D}, j_I, S)$ is a partial-isometric crossed product of (I, Γ^+, α). A routine computations show the covariance of (j_I, S) for (I, Γ^+, α). Suppose now (π, T) is a covariant representation of (I, Γ^+, α) on a Hilbert space H. Let $\rho : A \xrightarrow{\varphi} M(I) \xrightarrow{\overline{\pi}} B(H)$. Then by extendibility of ideal I, that is $\overline{\alpha|_I} \circ \varphi = \varphi \circ \alpha$, the pair (ρ, T) is a covariant representation of (A, Γ^+, α). The restriction $(\rho \times T)|_{\mathcal{D}}$ to $\mathcal{D}$ of $\rho \times T$ is a nondegenerate representation of $\mathcal{D}$ which satisfies the requirement $(\rho \times T)|_{\mathcal{D}} \circ j_I = \pi$ and $\overline{(\rho \times T)|_{\mathcal{D}}} \circ S = T$. Thus the triple $(\mathcal{D}, j_I, S)$ is a partial-isometric crossed product for (I, Γ^+, α), and we have the homomorphism $\mu = i_A|_I \times V$.

Next we show the exactness. Let Φ be a nondegenerate representation of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ with kernel $\mathcal{D}$. Since $I \subset \ker \Phi \circ i_A$, we can have a representation $\tilde{\Phi}$ of A/I, which together with $\overline{\Phi} \circ V$ is a covariant partial-isometric representation of $(A/I, \Gamma^+, \tilde{\alpha})$. Then $\tilde{\Phi} \times (\overline{\Phi} \circ V)$ lifts to Φ, and therefore $\ker \gamma \subset \ker \Phi = \mathcal{D}$. $\square$

Corollary 3.2. *Let (A, Γ^+, α) be a dynamical system, and I an extendible α-invariant ideal of A. Then there is a commutative diagram*:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \phi_I & \longrightarrow & I \times_\alpha^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_I} & I \times_\alpha^{\mathrm{iso}} \Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \mu \downarrow & & \mu^{\mathrm{iso}} \downarrow & & \\
0 & \longrightarrow & \ker \phi_A & \longrightarrow & A \times_\alpha^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_A} & A \times_\alpha^{\mathrm{iso}} \Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \gamma \downarrow & & \gamma^{\mathrm{iso}} \downarrow & & \\
0 & \longrightarrow & \ker \phi_{A/I} & \longrightarrow & A/I \times_{\tilde{\alpha}}^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_{A/I}} & A/I \times_{\tilde{\alpha}}^{\mathrm{iso}} \Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. The three row exact sequences follow from [4], the middle column from Theorem 3.1 and the right column exact sequence from [1]. By inspection on the spanning elements, one can see that $\mu(\ker \phi_I)$ is an ideal of $\ker \phi_A$ and $\mu^{\mathrm{iso}} \circ \phi_I = \phi_A \circ \mu$, thus first and second rows commute. Then Snake Lemma gives the commutativity of all rows and columns. $\square$

4. The example

We consider a dynamical system $(B_{\Gamma^+}, \Gamma^+, \tau)$ consisting of a unital C^*-subalgebra B_{Γ^+} of $\ell^\infty(\Gamma^+)$ spanned by the set $\{1_s : s \in \Gamma^+\}$ of characteristic functions 1_s of $\{x \in \Gamma^+ : x \geq s\}$, the action τ of Γ^+ on B_{Γ^+} is given by $\tau_x(1_s) = 1_{s+x}$. The ideal $B_{\Gamma^+,\infty} = \overline{\mathrm{span}}\{1_i - 1_j : i < j \in \Gamma^+\}$ is an extendible τ-invariant ideal of B_{Γ^+}. Then we want to show in Proposition 4.1 that an application of Corollary 3.2 to the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ and the ideal $B_{\Gamma^+,\infty}$ gives [7, Theorem 5.6].

The crossed product $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ is a universal C^*-algebra generated by the canonical isometric representation t of Γ^+: every isometric representation w of Γ^+ gives a covariant isometric representation (π_w, w) of $(B_{\Gamma^+}, \Gamma^+, \tau)$. Suppose $\{\varepsilon_x : x \in \Gamma^+\}$ is the usual orthonormal basis in $\ell^2(\Gamma^+)$, and let $T_s(\varepsilon_x) = \varepsilon_{x+s}$ for every $s \in \Gamma^+$. Then $s \mapsto T_s$ is an isometric representation of Γ^+, and the Toeplitz algebra $\mathcal{T}(\Gamma)$ is the C^*-subalgebra of $B(\ell^2(\Gamma^+))$ generated by $\{T_s : s \in \Gamma^+\}$. So there exists a representation $\mathfrak{T} := \pi_T \times T$ of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ on $\ell^2(\Gamma^+)$ such that $\mathfrak{T}(t_x) = T_x$ and $\mathfrak{T}(1_x) = T_x T_x^*$ for all $x \in \Gamma^+$. This representation is faithful by [3, Theorem 2.4]. Thus $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ and the Toeplitz algebra $\mathcal{T}(\Gamma) = \pi_T \times T(B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+)$ are isomorphic, and the isomorphism takes the ideal $B_{\Gamma^+,\infty} \times_\tau^{\mathrm{iso}} \Gamma^+$ of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ onto the commutator ideal $\mathcal{C}_\Gamma = \overline{\mathrm{span}}\{T_x(1-TT^*)T_y^* : x, y \in \Gamma^+\}$ of $\mathcal{T}(\Gamma)$.

Similarly, the crossed product $B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+$ has a partial-isometric version of universal property by [7, Proposition 5.1]: every partial-isometric representation v of Γ^+ gives a covariant partial-isometric representation (π_v, v) of $(B_{\Gamma^+}, \Gamma^+, \tau)$ with $\pi_v(1_x) = v_x v_x^*$, and then $B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+$ is the universal C^*-algebra generated by the canonical partial-isometric representation v of Γ^+. Now since $x \mapsto T_x$ and $x \mapsto T_x^*$ are partial-isometric representations of Γ^+ in the Toeplitz algebra $\mathcal{T}(\Gamma)$, there exist (by the universality) a homomorphism φ_T and φ_{T^*} of $B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+$ onto $\mathcal{T}(\Gamma)$.

Next consider the algebra $C(\hat{\Gamma})$ generated by $\{\lambda_x : x \in \Gamma\}$ of the evaluation maps $\lambda_x(\xi) = \xi(x)$ on $\hat{\Gamma}$. Let ψ_T and ψ_{T^*} be the homomorphisms of $\mathcal{T}(\Gamma)$ onto $C(\hat{\Gamma})$ defined by $\psi_T(T_x) = \lambda_x$ and $\psi_{T^*}(T_x) = \lambda_{-x}$.

Proposition 4.1 ([7, Theorem 5.6]). *Let Γ^+ be the positive cone in a totally ordered abelian group Γ. Then the following commutative diagram exists*:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\varphi_T \cap \ker\varphi_{T^*} & \longrightarrow & \ker\varphi_{T^*} & \xrightarrow{\varphi_T|} & \mathcal{C}_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\varphi_T & \longrightarrow & B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\varphi_T} & \mathcal{T}(\Gamma) & \longrightarrow & 0 \\
 & & \downarrow & & \varphi_{T^*}\downarrow & \searrow^{\Psi} & \downarrow\psi_{T^*} & & \\
0 & \longrightarrow & \mathcal{C}_\Gamma & \longrightarrow & \mathcal{T}(\Gamma) & \xrightarrow{\psi_T} & C(\hat{\Gamma}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{4.1}
$$

where Ψ maps each generator $v_x \in B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+$ to $\delta_x^* \in C^*(\Gamma) \simeq C(\hat{\Gamma})$.

Proof. Apply Corollary 3.2 to the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ and the extendible ideal $B_{\Gamma^+,\infty}$. Let $Q^{\mathrm{piso}} := B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau}}^{\mathrm{piso}} \Gamma^+$ and $Q^{\mathrm{iso}} := B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau}}^{\mathrm{iso}} \Gamma^+$. Then we have:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \phi_{B_{\Gamma^+,\infty}} & \longrightarrow & B_{\Gamma^+,\infty} \times_{\tau}^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_{B_{\Gamma^+,\infty}}} & B_{\Gamma^+,\infty} \times_{\tau}^{\mathrm{iso}} \Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \mu\downarrow & & \downarrow \mu_{B_{\Gamma^+,\infty}} & & \\
0 & \longrightarrow & \ker \phi_{B_{\Gamma^+}} & \longrightarrow & B_{\Gamma^+} \times_{\tau}^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_{B_{\Gamma^+}}} & B_{\Gamma^+} \times_{\tau}^{\mathrm{iso}} \Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \gamma\downarrow & & \downarrow \gamma_{B_{\Gamma^+,\infty}} & & \\
0 & \longrightarrow & \ker \phi_Q & \longrightarrow & Q^{\mathrm{piso}} & \xrightarrow{\phi_Q} & Q^{\mathrm{iso}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.2}$$

We claim that exact sequences in this diagram and (4.1) are equivalent. The middle exact sequences of (4.1) and (4.2) are trivially equivalent via the isomorphism $\mathfrak{T} : B_{\Gamma^+} \times_{\tau}^{\mathrm{iso}} \Gamma^+ \to \mathcal{T}(\Gamma)$. By viewing B_{Γ^+} as the algebra of functions that have limit, the map $f \in B_{\Gamma^+} \mapsto \lim_{x\in\Gamma^+} f(x)$ induces an isomorphism $B_{\Gamma^+}/B_{\Gamma^+,\infty} \to \mathbb{C}$, which intertwines the action $\tilde{\tau}$ and the trivial action id on $\mathbb{C}$. So $(B_{\Gamma^+}/B_{\Gamma^+,\infty}, \Gamma^+, \tilde{\tau}) \simeq (\mathbb{C}, \Gamma^+, \mathrm{id})$. Moreover, $\mathfrak{T}$ combines with the isomorphism $h : B_{\Gamma^+}/B_{\Gamma^+,\infty} \times_{\tilde{\tau}}^{\mathrm{iso}} \Gamma^+ \to \mathbb{C} \times_{\mathrm{id}}^{\mathrm{iso}} \Gamma^+ \to C^*(\Gamma) \simeq C(\hat{\Gamma})$ to identify the right-hand exact sequence equivalently to $0 \to \mathcal{C}_\Gamma \to \mathcal{T}(\Gamma) \overset{\psi_{T^*}}{\to} C(\hat{\Gamma}) \to 0$.

For the bottom sequence, we consider the pair of

$$\iota_{\mathbb{C}} : z \in \mathbb{C} \mapsto z1_{\mathcal{T}(\Gamma)} \text{ and } \iota_{\Gamma^+} : x \in \Gamma^+ \mapsto T_x^* \in \mathcal{T}(\Gamma).$$

It is a partial-isometric covariant representation, such that $(\mathcal{T}(\Gamma), \iota_{\mathbb{C}}, \iota_{\Gamma^+})$ is a partial-isometric crossed product of $(\mathbb{C}, \Gamma^+, \mathrm{id})$. So we have an isomorphism

$$\Upsilon : Q^{\mathrm{piso}} \to \mathbb{C} \times_{\mathrm{id}}^{\mathrm{piso}} \Gamma^+ \overset{\iota}{\to} \mathcal{T}(\Gamma) \text{ in which } \Upsilon(i_{\Gamma^+}(x)) = T_x^* \text{ for all } x,$$

and moreover if (j_Q, u) denotes the canonical covariant partial-isometric representation of the system $(Q := B_{\Gamma^+}/B_{\Gamma^+,\infty}, \Gamma^+, \tilde{\tau})$ in Q^{piso}, then Υ satisfies the equations $\Upsilon(u_x) = T_x^*$ and $\Upsilon(j_Q(1_x + B_{\Gamma^+,\infty})) = \iota_{\mathbb{C}}(\lim_y 1_x(y)) = 1$ for all $x \in \Gamma^+$. To see $\Upsilon(\ker \phi_Q) = \mathcal{C}_\Gamma$, recall from [4, Proposition 2.3] that

$$\ker \phi_Q := \overline{\mathrm{span}}\{u_x^* j_Q(a)(1 - u_z^* u_z)u_y : a \in Q,\ x, y, z \in \Gamma^+\}.$$

Since $\Upsilon(u_x^* j_Q(a)(1 - u_z^* u_z)u_y)$ is a scalar multiplication of $T_x(1 - T_z T_z^*)T_y^*$, therefore $\Upsilon(\ker \phi_Q) = \mathcal{C}_\Gamma$. Consequently the two exact sequences are equivalent:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \phi_Q & \to & Q^{\mathrm{piso}} & \xrightarrow{\phi_Q} & Q^{\mathrm{iso}} & \longrightarrow & 0 \\
 & & \Upsilon\downarrow & & \Upsilon\downarrow & & h\downarrow & & \\
0 & \longrightarrow & \mathcal{C}_\Gamma & \longrightarrow & \mathcal{T}(\Gamma) & \xrightarrow{\psi_T} & C(\hat{\Gamma}) & \longrightarrow & 0.
\end{array}$$

For the second column exact sequence, we note that the isomorphism $\jmath : Q^{\mathrm{piso}} \simeq \mathbb{C} \times_{\mathrm{id}}^{\mathrm{piso}} \Gamma^+ \to \mathcal{T}(\Gamma)$ satisfies $\jmath \circ \gamma = \varphi_{T^*}$. This implies

$$B_{\Gamma^+,\infty} \times_\tau^{\mathrm{piso}} \Gamma^+ \simeq \ker(\jmath \circ \gamma) = \ker \varphi_{T^*},$$

and therefore the second column sequence of diagram (4.1) is equivalent to $0 \to \ker \varphi_{T^*} \to B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+ \to \mathcal{T}(\Gamma) \to 0$.

Next we are working for the first row. The homomorphism $\phi_{B_{\Gamma^+}}$ in the following diagram

$$\begin{array}{ccc} B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+ & \xrightarrow{\phi_{B_{\Gamma^+}}} & B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+ \\ & \searrow^{\varphi_T} & \downarrow \mathfrak{T} \\ & & \mathcal{T}(\Gamma), \end{array}$$

restricts to the homomorphism $\phi_{B_{\Gamma^+,\infty}}$ of the ideal $B_{\Gamma^+,\infty} \times_\tau^{\mathrm{piso}} \Gamma^+ \simeq \ker \varphi_{T^*}$ onto $B_{\Gamma^+,\infty} \times_\tau^{\mathrm{iso}} \Gamma^+ \simeq \mathcal{C}_\Gamma$. So the homomorphism $\varphi_T| : \ker \varphi_{T^*} \to \mathcal{C}_\Gamma$ has kernel $I := \ker \varphi_{T^*} \cap \ker \varphi_T$, and therefore first row exact sequence of the two diagrams are indeed equivalent.

Finally we show that such Ψ exists. Consider $C(\hat{\Gamma}) \simeq C^*(\Gamma) \simeq \mathbb{C} \times_{\mathrm{id}} \Gamma$ is the C^*-algebra generated by the unitary representation $x \in \Gamma \mapsto \delta_x \in \mathbb{C} \times_{\mathrm{id}} \Gamma$. Then we have a homomorphism $\pi_{\delta^*} \times \delta^* : B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+ \to \mathbb{C} \times_{\mathrm{id}} \Gamma$ which satisfies $\pi_{\delta^*} \times \delta^*(v_x) = \delta_x^*$ for all $x \in \Gamma^+$, and hence it is surjective. By looking at the spanning elements of $\ker \varphi_T$ and $\ker \varphi_{T^*}$ we can see that these two ideals are contained in $\ker(\pi_{\delta^*} \times \delta^*)$, therefore $\mathcal{J} := \ker \varphi_T + \ker \varphi_{T^*}$ must be also in $\ker(\pi_{\delta^*} \times \delta^*)$. For the other inclusion, let ρ be a unital representation of $B_{\Gamma^+} \times_\tau^{\mathrm{piso}} \Gamma^+$ on a Hilbert space H_ρ with $\ker \rho = \mathcal{J}$. Then for $s \in \Gamma^+$ we have $\rho((1 - v_s v_s^*) - (1 - v_s^* v_s)) = 0$ because $1 - v_s v_s^* \in \ker \varphi_{T^*}$ and $1 - v_s^* v_s \in \ker \varphi_T$ belong to $\mathcal{J}$. So $0 = \rho(v_s^* v_s - v_s v_s^*)$, which implies that $\rho(v_s^* v_s) = \rho(v_s v_s^*)$. On the other hand the equation $\rho((1 - v_s v_s^*) + (1 - v_s^* v_s)) = 0$ gives $\rho(v_s v_s^*) = I$. Therefore $\rho(v_s v_s^*) = \rho(v_s^* v_s) = I$, and this means $\rho(v_s)$ is unitary for every $s \in \Gamma^+$. Consequently a representation $\tilde{\rho} : \mathbb{C} \times_{\mathrm{id}} \Gamma \to B(H_\rho)$ exists, and it satisfies $\tilde{\rho} \circ (\pi_{\delta^*} \times \delta^*) = \rho$. Thus $\ker \pi_{\delta^*} \times \delta^* \subset \ker \rho = \mathcal{J}$, and the composition $\pi_{\delta^*} \times \delta^*$ with the Fourier transform $C^*(\Gamma) \simeq C(\hat{\Gamma})$ is the wanted homomorphism Ψ. □

5. The primitive ideals of $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$

Suppose Γ^+ is now the additive semigroup $\mathbb{N}$. The algebra $B_\mathbb{N}$ is conveniently viewed as the C^*-algebra $\mathbf{c}$ of convergent sequences, the ideal $B_{\mathbb{N},\infty}$ with $\mathbf{c}_0$, and the action τ of $\mathbb{N}$ on $\mathbf{c}$ is generated by the unilateral shift: $\tau_1(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$. The universal C^*-algebra $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ is generated by a power partial isometry $v := i_\mathbb{N}(1)$. The Toeplitz algebra $\mathcal{T}(\mathbb{Z})$ is the C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by isometries $\{T_n : n \in \mathbb{N}\}$, where $T_n(e_i) = e_{n+i}$ on the set of usual orthonormal basis $\{e_i : i \in \mathbb{N} \cup \{0\}\}$ of $\ell^2(\mathbb{N})$, and the commutator ideal of $\mathcal{T}(\mathbb{Z})$ is $\mathcal{K}(\ell^2(\mathbb{N}))$. Kernels of φ_T and φ_{T^*} are identified in [7, Lemma 6.2] by

$$\ker\varphi_T = \overline{\operatorname{span}}\{g^m_{i,j} : i,j,m \in \mathbb{N}\}; \quad \ker\varphi_{T^*} = \overline{\operatorname{span}}\{f^m_{i,j} : i,j,m \in \mathbb{N}\},$$

where

$$g^m_{i,j} = v_i^* v_m v_m^* (1 - v^* v) v_j \quad \text{and} \quad f^m_{i,j} = v_i v_m^* v_m (1 - v v^*) v_j^*.$$

Moreover $\mathcal{I} := \ker\varphi_T \cap \ker\varphi_{T^*}$ is an essential ideal in $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ [7, Lemma 6.8], given by

$$\overline{\operatorname{span}}\{f^m_{i,j} - f^{m+1}_{i,j} = g^m_{m-i,m-j} - g^{m+1}_{m-i,m-j} : m \in \mathbb{N},\ 0 \le i,j \le m\}.$$

The main point of [7, §6] is to show that there exist isomorphisms of $\ker\varphi_T$ and $\ker\varphi_{T^*}$ onto the algebra

$$\mathcal{A} := \{f : \mathbb{N} \to K(\ell^2(\mathbb{N})) : f(n) \in P_n K(\ell^2(\mathbb{N})) P_n \text{ and } \varepsilon_\infty(f) = \lim_n f(n) \text{ exists}\},$$

where $P_n := 1 - T_{n+1} T^*_{n+1}$ is the projection of $\ell^2(\mathbb{N})$ onto the subspace spanned by $\{e_i : i = 0,1,2,\ldots,n\}$, and such that they restrict to isomorphisms of $\mathcal{I}$ onto the ideal

$$\mathcal{A}_0 := \{f \in \mathcal{A} : \lim_n f(n) = 0\} \text{ of } \mathcal{A}.$$

We shall show in Proposition 5.1 that $\mathcal{A}$ and $\mathcal{A}_0$ are related to the algebras of compact operators on the Hilbert $\mathbf{c}$-module $\ell^2(\mathbb{N}, \mathbf{c})$ and on the closed sub-$\mathbf{c}$-module $\ell^2(\mathbb{N}, \mathbf{c}_0)$. We supply our readers with some basic theory of the C^*-algebra of operators on this Hilbert module, and let us begin with recalling the module structure of $\ell^2(\mathbb{N}, \mathbf{c})$ (and its closed sub-module). The vector space $\ell^2(\mathbb{N}, \mathbf{c})$, containing all $\mathbf{c}$-valued functions $\mathsf{a} : \mathbb{N} \to \mathbf{c}$ such that the series $\sum_{n\in\mathbb{N}} \mathsf{a}(n)^* \mathsf{a}(n)$ converges in the norm of $\mathbf{c}$, forms a Hilbert $\mathbf{c}$-module with the module structure defined by $(\mathsf{a} \cdot x)(n) = \mathsf{a}(n)x$ for $x \in \mathbf{c}$, and its $\mathbf{c}$-valued inner product given by $\langle \mathsf{a}, \mathsf{b} \rangle = \sum_{n\in\mathbb{N}} \mathsf{a}(n)^* \mathsf{b}(n)$. In fact the module $\ell^2(\mathbb{N}, \mathbf{c})$ is naturally isomorphic to the Hilbert module $\ell^2(\mathbb{N}) \otimes \mathbf{c}$ that arises from the completion of algebraic (vector space) tensor product $\ell^2(\mathbb{N}) \odot \mathbf{c}$ associated to the $\mathbf{c}$-valued inner product defined on simple tensor product by $\langle \xi \otimes x, \eta \otimes y \rangle = \langle \xi, \eta \rangle x^* y$ for $\xi, \eta \in \ell^2(\mathbb{N})$ and $x, y \in \mathbf{c}$. The isomorphism is implemented by the map ϕ that takes $(e_i \otimes x) \in \ell^2(\mathbb{N}) \otimes \mathbf{c}$ to the element $\phi(e_i \otimes x) \in \ell^2(\mathbb{N}, \mathbf{c})$ which is the function $[\phi(e_i \otimes x)](n) = \begin{cases} x & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$ By exactly the same arguments, we see that the two Hilbert $\mathbf{c}_0$-modules $\ell^2(\mathbb{N}, \mathbf{c}_0)$ and $\ell^2(\mathbb{N}) \otimes \mathbf{c}_0$ are isomorphic. However since $\mathbf{c}_0$ is an ideal of $\mathbf{c}$, it follows that the $\mathbf{c}_0$-module $\ell^2(\mathbb{N}, \mathbf{c}_0)$ is a closed sub-$\mathbf{c}$-module of $\ell^2(\mathbb{N}, \mathbf{c})$, and respectively $\ell^2(\mathbb{N}) \otimes \mathbf{c}_0$ is a closed sub-$\mathbf{c}$-module of $\ell^2(\mathbb{N}) \otimes \mathbf{c}$. Moreover the $\mathbf{c}$-module isomorphism ϕ restricts to $\mathbf{c}_0$-module isomorphism $\ell^2(\mathbb{N}, \mathbf{c}_0) \simeq \ell^2(\mathbb{N}) \otimes \mathbf{c}_0$.

Next, we consider the C^*-algebra $\mathcal{L}(\ell^2(\mathbb{N}, \mathbf{c}))$ of adjointable operators on $\ell^2(\mathbb{N}, \mathbf{c})$, and the ideal $\mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c}))$ of $\mathcal{L}(\ell^2(\mathbb{N}, \mathbf{c}))$ spanned by the set $\{\theta_{\mathsf{a},\mathsf{b}} : \mathsf{a}, \mathsf{b} \in \ell^2(\mathbb{N}, \mathbf{c})\}$ of compact operators on the module $\ell^2(\mathbb{N}, \mathbf{c})$. The algebra $\mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c}_0))$ is defined by the same arguments, and note that $\mathcal{K}(\ell^2(\mathbb{N}, \mathbf{c}_0))$ is an ideal of $\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))$. The isomorphism of two modules $\ell^2(\mathbb{N},\mathbf{c})$ and $\ell^2(\mathbb{N})\otimes\mathbf{c}$, implies that $\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))\simeq\mathcal{K}(\ell^2(\mathbb{N})\otimes\mathbf{c})$, which by the Hilbert module theorem, this is the C^*-algebraic tensor product $\mathcal{K}(\ell^2(\mathbb{N}))\otimes\mathbf{c}$ of $\mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathbf{c}$. We shall often use the characteristic functions $\{1_n : n\in\mathbb{N}\}$ as generator elements of $\mathbf{c}$ and the spanning set $\{\theta_{e_i\otimes 1_n, e_j\otimes 1_n} : i,j,n\in\mathbb{N}\}$ of $\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))$ in our computations.

There is another ingredient that we need to consider to state the Proposition. Suppose $S\in\mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}))$ is an operator defined by $S(\mathsf{a})(i)=\mathsf{a}(i-1)$ for $i\geq 1$ and zero otherwise. One can see that $S^*S=1$, i.e., S is an isometry. Let $p\in\mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}))$ be the projection $(p(\mathsf{a}))(n)=1_n\mathsf{a}(n)$ for $\mathsf{a}\in\ell^2(\mathbb{N},\mathbf{c})$, and similarly $q\in\mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}_0))$ be the projection $(q(\mathsf{a}))(n)=1_n\mathsf{a}(n)$ for $\mathsf{a}\in\ell^2(\mathbb{N},\mathbf{c}_0)$. Then the following two partial isometric representations of $\mathbb{N}$ in $p\mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}))p$ defined by

$$w : n\in\mathbb{N}\mapsto pS_n^*p \quad\text{ and }\quad t : n\in\mathbb{N}\mapsto pS_np,$$

induce the representations $\pi_w\times w$ and $\pi_t\times t$ of $\mathbf{c}\times_\tau^{\text{piso}}\mathbb{N}$ in $p\mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}))p$ which satisfy $\pi_w\times w(v_i)=pS_i^*p$ and $\pi_t\times t(v_i)=pS_ip$ for all $i\in\mathbb{N}$. These $\pi_w\times w$ and $\pi_t\times t$ are faithful representations [4, Example 4.3].

Proposition 5.1. *The representations* $\pi_w\times w$ *and* $\pi_t\times t$ *map* $\ker\varphi_T$ *and* $\ker\varphi_{T^*}$ *isomorphically onto the full corner* $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$. *Moreover, they restrict to isomorphisms of the ideal* $\ker\varphi_T\cap\ker\varphi_{T^*}$ *onto the full corner* $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$.

Remark 5.2. It follows from this proposition that $\operatorname{Prim}\ker\varphi_T$ and $\operatorname{Prim}\ker\varphi_{T^*}$ are both homeomorphic to $\operatorname{Prim}\mathbf{c}$. In fact, since $\ker\varphi_{T^*}\simeq\mathbf{c}_0\times_\tau^{\text{piso}}\mathbb{N}$ by [2, Corollary 3.1], we can therefore deduce that $\mathbf{c}_0\times_\tau^{\text{piso}}\mathbb{N}$ is Morita equivalent to $\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))$. This is a useful fact for our subsequential work on the partial-isometric crossed product of lattice semigroup $\mathbb{N}\times\mathbb{N}$.

Proof of Proposition 5.1. We only have to show that

$$\pi_t\times t(\ker\varphi_{T^*})=p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \quad\text{and}$$
$$\pi_t\times t(\ker\varphi_T\cap\ker\varphi_{T^*})=q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q.$$

The rest of arguments is done in [4, Example 4.3].

Note that the algebra $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$ is spanned by $\{p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p : i,j,n\in\mathbb{N}\}$. Since $\pi_t\times t(f^n_{i,j})=p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p$ for every $i,j,n\in\mathbb{N}$, therefore $\pi_t\times t(\ker\varphi_{T^*})=p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$.

Similarly we consider that $\{q\theta^{\mathbf{c}_0}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_{\{n\}}}q : i,j\leq n\in\mathbb{N}\}$ spans $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$. We use the equation $\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_0}=\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}$ for every $n\in\mathbb{N}$, in the computations below, to see that

$$\begin{aligned}\pi_t\times t(f^n_{i,j}-f^{n+1}_{i,j}) &= p(\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_0}-\theta^{\mathbf{c}}_{e_i\otimes 1_{n+1}, e_j\otimes 1_0})p\\ &= p(\theta^{\mathbf{c}}_{(e_i\otimes 1_n)-(e_i\otimes 1_{n+1}), (e_j\otimes 1_0)})p\\ &= p(\theta^{\mathbf{c}}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_0})p\end{aligned}$$

$$= p(\theta^{\mathbf{c}}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_{\{n\}}})p.$$

To convince that every $p(\theta^{\mathbf{c}}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_{\{n\}}})p$ belongs to $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$, we need the embedding $\iota^{\mathcal{K}}$ of $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ in $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$ stated in Lemma 5.3. In fact, every element $p(\theta^{\mathbf{c}}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_{\{n\}}})p$ spans $\iota^{\mathcal{K}}(q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q)$, therefore $\pi_t \times t(\ker\varphi_{T^*}\cap\ker\varphi_T) = \iota^{\mathcal{K}}(q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q)$. □

Lemma 5.3. *Let $p \in \mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}))$ and $q \in \mathcal{L}(\ell^2(\mathbb{N},\mathbf{c}_0))$ are the projections defined by $(p(\mathsf{a}))(n) = 1_n\mathsf{a}(n)$ for $\mathsf{a}\in\ell^2(\mathbb{N},\mathbf{c})$, and $(q(\mathsf{a}))(n) = 1_n\mathsf{a}(n)$ for $\mathsf{a}\in\ell^2(\mathbb{N},\mathbf{c}_0)$. Then the full corner $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ embeds naturally via $\iota^{\mathcal{K}}(q\theta^{\mathbf{c}_0}_{\mathsf{a},\mathsf{b}}q) = p\theta^{\mathbf{c}}_{\mathsf{a},\mathsf{b}}p$ as an ideal in $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$, and there exists a short exact sequence*

$$0 \longrightarrow q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q \xrightarrow{\iota^{\mathcal{K}}} p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \xrightarrow{q^{\mathcal{K}}} \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow 0,$$

where $q^{\mathcal{K}}(p\theta^{\mathbf{c}}_{\mathsf{a},\mathsf{b}}p) = \theta_{x,y}$ with $x,y\in\ell^2(\mathbb{N})$ are given by $x_i = \lim_{n\to\infty}(1_i\mathsf{a}(i))(n)$ and $y_i = \lim_{n\to\infty}(1_i\mathsf{b}(i))(n)$. In particular we have

$$\begin{aligned} q^{\mathcal{K}}(p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_m}p) &= q^{\mathcal{K}}(\theta^{\mathbf{c}}_{p(e_i\otimes 1_n), p(e_j\otimes 1_m)}) \\ &= q^{\mathcal{K}}(\theta^{\mathbf{c}}_{e_i\otimes 1_{n\vee i}, e_j\otimes 1_{m\vee j}}) \\ &= T_i(1-TT^*)T_j^* \in \mathcal{K}(\ell^2(\mathbb{N})). \end{aligned}$$

Proof. Apply [5, Lemma 2.6] for the module $X := \ell^2(\mathbb{N},\mathbf{c})$ and $I = \mathbf{c}_0$. In this case we have the submodule $XI = \ell^2(\mathbb{N},\mathbf{c}_0)$. Note that if $\mathsf{a}\in\ell^2(\mathbb{N},\mathbf{c})$, then every sequence $\mathsf{a}(i)\in\mathbf{c}$ is convergent in $\mathbb{C}$, and the map $\mathsf{q} : \mathsf{a}\mapsto(\mathsf{q}(\mathsf{a}))(i) = \lim_{n\to\infty}(\mathsf{a}(i))(n)$ gives $0\to\ell^2(\mathbb{N},\mathbf{c}_0)\to\ell^2(\mathbb{N},\mathbf{c})\xrightarrow{\mathsf{q}}\ell^2(\mathbb{N})\to 0$. Moreover [5, Lemma 2.6] proves that $\iota^{\mathcal{K}}(\theta^{XI}_{\mathsf{a},\mathsf{b}}) = \theta^X_{\mathsf{a},\mathsf{b}}$ and $q^{\mathcal{K}}(\theta^X_{\mathsf{a},\mathsf{b}}) = \theta^{X/XI}_{\mathsf{q}(\mathsf{a}),\mathsf{q}(\mathsf{b})}$ give the exactness of the sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0)) \xrightarrow{\iota^{\mathcal{K}}} \mathcal{K}(\ell^2(\mathbb{N},\mathbf{c})) \xrightarrow{q^{\mathcal{K}}} \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow 0.$$

Since $\iota^{\mathcal{K}}(q\theta^{XI}_{\mathsf{a},\mathsf{b}}q) = \theta^X_{q(\mathsf{a}),q(\mathsf{b})} = p\theta^X_{\mathsf{a},\mathsf{b}}p$ for every a and b in $\ell^2(\mathbb{N},\mathbf{c}_0)$, the corner $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ is embedded into $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$ such that $q^{\mathcal{K}}$ is defined by $q^{\mathcal{K}}(p\theta^X_{\mathsf{a},\mathsf{b}}p) = q^{\mathcal{K}}(\theta^X_{p(\mathsf{a}),p(\mathsf{b})}) = \theta_{x,y}$ where $x_i = \lim_{n\to\infty}(1_i\mathsf{a}(i))(n)$ and $y_i = \lim_{n\to\infty}(1_i\mathsf{b}(i))(n)$. Thus we obtain the required exact sequence. □

Proposition 5.4. *There are isomorphisms $\Theta : p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \to \ker\varphi_T$ and $\Theta_* : p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \to \ker\varphi_{T^*}$ defined by $\Theta(p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p) = g^n_{i,j}$ and*

$\Theta_*(p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p) = f^n_{i,j}$ *for all* $i, j, n \in \mathbb{N}$ *such that the following commutative diagram has all rows and columns exact*:

(5.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q & \xrightarrow{\iota^{\mathcal{K}}} & p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p & \xrightarrow{q^{\mathcal{K}}} & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota^{\mathcal{K}}\circ\alpha} & & \downarrow{\scriptstyle \Theta_*} & & \downarrow & & \\
0 & \longrightarrow & p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p & \xrightarrow{\Theta} & \mathbf{c}\times^{\mathrm{piso}}_{\tau}\mathbb{N} & \xrightarrow{\varphi_T} & \mathcal{T}(\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q^{\mathcal{K}}} & & \downarrow{\scriptstyle \varphi_{T^*}} & \searrow{\scriptstyle \Psi} & \downarrow{\scriptstyle \psi_{T^*}} & & \\
0 & \longrightarrow & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & \mathcal{T}(\mathbb{Z}) & \xrightarrow{\psi_T} & C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. We apply Proposition 4.1 to the system $(\mathbf{c}, \mathbb{N}, \tau)$. Let $\{v_i : i \in \mathbb{N}\}$ denote the generators of $\mathbf{c}\times^{\mathrm{piso}}_{\tau}\mathbb{N}$, and $\{\delta_i : i \in \mathbb{Z}\}$ the generator of $C^*(\mathbb{Z})$. Then the homomorphism $\Psi : \mathbf{c}\times^{\mathrm{piso}}_{\tau}\mathbb{N} \to C(\mathbb{T})$ given by Proposition 4.1 satisfies $\Psi(v_i) = \delta^*_i = (z \mapsto \overline{z}^i) \in C(\mathbb{T})$ for every $i \in \mathbb{N}$. Moreover $\Theta = (\pi_w \times w)^{-1}$ and $\Theta_* = (\pi_t \times t)^{-1}$, by Proposition 5.1, satisfy $\Theta(p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p) = g^n_{i,j}$ and $\Theta_*(p\theta^{\mathbf{c}}_{e_i\otimes 1_n, e_j\otimes 1_n}p) = f^n_{i,j}$ for all $i, j, n \in \mathbb{N}$. So the first row sequence is exact, and which is equivalent to the one of (4.1) for $\Gamma^+ = \mathbb{N}$ because

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} & \xrightarrow{\mathrm{id}} & \ker\varphi_{T^*} & \xrightarrow{\varphi_T|} & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_t\times t} & & \downarrow{\scriptstyle \pi_t\times t} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q & \xrightarrow{\iota^{\mathcal{K}}} & p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p & \xrightarrow{q^{\mathcal{K}}} & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & 0.
\end{array}$$

For the first column we use the automorphism α of $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ defined on its spanning element by $\alpha(q\theta^{\mathbf{c}_0}_{e_i\otimes 1_{\{n\}}, e_j\otimes 1_{\{n\}}}q) = q\theta^{\mathbf{c}_0}_{e_{n-i}\otimes 1_{\{n\}}, e_{n-j}\otimes 1_{\{n\}}}q$. Then by inspections on the spanning elements of the algebras involved, we can see that the diagram (5.1) commutes. □

Thus we know from the diagram that the set $\mathrm{Prim}\,\mathbf{c}\times^{\mathrm{piso}}_{\tau}\mathbb{N}$ is given by the sets $\mathrm{Prim}\,\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))$ and $\mathrm{Prim}\,\mathcal{T}(\mathbb{Z})$. Since

$$\mathrm{Prim}\,\mathcal{T}(\mathbb{Z}) = \mathrm{Prim}\,\mathcal{K}(\ell^2(\mathbb{N})) \cup \mathrm{Prim}\,C(\mathbb{T}) = \{0\} \cup \mathbb{T},$$

and $\mathrm{Prim}\,\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))$ is homeomorphic to

$$\mathrm{Prim}\,\mathbf{c} = \mathrm{Prim}\,\mathbf{c}_0 \cup \mathrm{Prim}\,\mathbb{C} \simeq \mathbb{N} \cup \{\infty\},$$

therefore $\mathrm{Prim}\,\mathbf{c}\times^{\mathrm{piso}}_{\tau}\mathbb{N}$ consists of a copy of $\{I_n\}$ of $\mathbb{N}$ embedded as an open subset, a copy of $\{J_z\}$ of $\mathbb{T}$ embedded as a closed subset. We identify these ideals in Proposition 5.7 and Lemma 5.12.

Note for now that $\ker\varphi_T$ and $\ker\varphi_{T^*}$ are primitive ideals of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$: the Toeplitz representation T of $\mathcal{T}(\mathbb{Z})$ on $\ell^2(\mathbb{N})$ is irreducible by [8, Theorem 3.13], and φ_T and φ_{T^*} are surjective homomorphisms of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ onto $\mathcal{T}(\mathbb{Z})$, so $T\circ\varphi_T$ and $T\circ\varphi_{T^*}$ are irreducible representations of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ on $\ell^2(\mathbb{N})$. Moreover, irreducibility of the representation $\mathrm{id}\circ q^{\mathcal{K}} : p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \xrightarrow{q^{\mathcal{K}}} \mathcal{K}(\ell^2(\mathbb{N})) \xrightarrow{\mathrm{id}} B(\ell^2(\mathbb{N}))$ implies the kernel $\mathcal{I} = \ker\varphi_T \cap \ker\varphi_{T^*} \simeq q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ of $\mathrm{id}\circ q^{\mathcal{K}}$ is a primitive ideal of $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p \simeq \ker\varphi_T$. Similarly, $\mathcal{I}$ is a primitive ideal of $\ker\varphi_{T^*} \simeq p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$. Although $\mathcal{I} \notin \mathrm{Prim}\,\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$, the ideal $\mathcal{I}$ is essential in $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ by [7, Lemma 6.8], so the space $\mathrm{Prim}\,\mathcal{I} \simeq \mathrm{Prim}\,\mathbf{c}_0$ is dense in $\mathrm{Prim}\,\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$.

Next consider that $\mathcal{K}(\ell^2(\mathbb{N})) = \overline{\mathrm{span}}\{e_{ij} := T_i(1-TT^*)T_j^* : i,j\in\mathbb{N}\}$, and recall that there is a natural isomorphism Λ of $\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c})) \simeq \mathcal{K}(\ell^2(\mathbb{N}))\otimes\mathbf{c}$ onto the algebra

$$C(\mathbb{N}\cup\{\infty\},\mathcal{K}(\ell^2(\mathbb{N}))) := \{f:\mathbb{N}\to\mathcal{K}(\ell^2(\mathbb{N})) : \lim_n f(n) \text{ exists in } \mathcal{K}(\ell^2(\mathbb{N}))\}$$

given by $\Lambda(e_{ij}\otimes 1_k)(n) = 1_k(n)e_{ij}$ for $i,j,k,n\in\mathbb{N}$. Then $\Lambda(p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p) \subset \mathcal{A}$ because

$$\begin{aligned}[\Lambda(p(e_{ij}\otimes 1_m)p)](n) &= [\Lambda(e_{ij}\otimes 1_{m\vee i\vee j})](n)\\ &= \begin{cases} e_{ij} & \text{if } n\ge m\vee i\vee j\\ 0 & \text{otherwise}\end{cases}\\ &= \pi_n(f^m_{i,j}) = \pi_n^*(g^m_{i,j}).\end{aligned}$$

Since $\Lambda = \pi\circ\Theta_* = \pi^*\circ\Theta$, Λ maps the corners $p\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}))p$ and $q\mathcal{K}(\ell^2(\mathbb{N},\mathbf{c}_0))q$ isomorphically onto the algebra $\mathcal{A}$ and $\mathcal{A}_0$ respectively. Construction of this isomorphism in [7, §6] involves the representations π_n and π_n^*, for each $n\in\mathbb{N}$, of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ on $\ell^2(\mathbb{N})$ that are associated to the partial-isometric representations $k\mapsto P_nT_kP_n$ and $k\mapsto P_nT_k^*P_n$ respectively, where $P_n := 1 - T_{n+1}T_{n+1}^*$ is the projection onto $H_n := \mathrm{span}\{e_i : i = 0,1,2,\ldots,n\}$. For every $a\in\ker\varphi_{T^*}$, the sequence $\{\pi_n(a)\}_{n\in\mathbb{N}}$ is convergent in $\mathcal{K}(\ell^2(\mathbb{N}))$, and then the map $a\in\ker\varphi_{T^*}\mapsto\pi(a) := \{\pi_n(a)\}_{n\in\mathbb{N}}\in\mathcal{A}$ defines the isomorphism.

These observations suggest that an extension of π should give a representation of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ in the algebra $C_{\mathrm{b}}(\mathbb{N},B(\ell^2(\mathbb{N})))$, and then primitive ideals are the kernels of evaluation maps. But we can consider a smaller algebra which gives more information on the image of π. Note that the algebra $C(\mathbb{N}\cup\{\infty\},B(\ell^2(\mathbb{N})))$ is too small to consider, because the sequence $(P_nT_kP_n)_{n\in\mathbb{N}}$ as we see, does not converge to T_k in the operator norm on $B(\ell^2(\mathbb{N}))$, but it converges strongly to T_k. Therefore we consider the set $C_{\mathrm{b}}(\mathbb{N}\cup\{\infty\},B(\ell^2(\mathbb{N}))_{*-\mathrm{s}})$ of functions $\xi:\mathbb{N}\to B(\ell^2(\mathbb{N}))$ such that $\lim_n\xi_n$ exists in the *-strong topology on $B(\ell^2(\mathbb{N}))$, and which satisfies $\|\xi\|_\infty := \sup_n\|\xi_n\| < \infty$. By [9, Lemma 2.56], it is a C^*-algebra with the pointwise operation from $B(\ell^2(\mathbb{N}))$ and the norm $\|\cdot\|_\infty$. Then let

$$\mathcal{B} := \{f:\mathbb{N}\to B(\ell^2(\mathbb{N})) : \sup_{n\in\mathbb{N}}\|f(n)\|_{B(\ell^2(\mathbb{N}))} < \infty,\ f(n)\in P_nB(\ell^2(\mathbb{N}))P_n \text{ and}$$

$\lim_{n\to\infty} f(n)$ exists in the $*$-strong topology on $B(\ell^2(\mathbb{N}))\}$.

Note that $\mathcal{B}$ is a subalgebra of $C_{\rm b}(\mathbb{N}\cup\{\infty\}, B(\ell^2(\mathbb{N}))_{*-{\rm s}})$ because $P_nB(\ell^2(\mathbb{N}))P_n \simeq B(H_n)$ is closed in $B(\ell^2(\mathbb{N}))$ for every $n \in \mathbb{N}$, and $\mathcal{B}$ has an identity $1_{\mathcal{B}} = (P_0, P_1, P_2, \ldots)$.

Proposition 5.5. *There are faithful representations π and π^* of $\mathbf{c} \times_\tau^{\rm piso} \mathbb{N}$ in the algebra $\mathcal{B}$, which defined on each generator $v_k \in \mathbf{c} \times_\tau^{\rm piso} \mathbb{N}$ by*

$$\pi(v_k)(n) := \pi_n(v_k) = P_nT_kP_n \text{ and } \pi^*(v_k)(n) := \pi_n^*(v_k) = P_nT_k^*P_n \text{ for } n \in \mathbb{N}.$$

These representations π and π^* are the extension of isomorphisms $\pi : \ker\varphi_{T^*} \to \mathcal{A}$ and $\pi^* : \ker\varphi_T \to \mathcal{A}$ of [7, Theorem 6.1].

Proof. The map π is induced by the partial-isometric representation $k \mapsto W_k$ where $W_k(n) = P_nT_kP_n$, and similarly for π^* by $k \mapsto S_k$ where $S_k(n) = P_nT_k^*P_n$ for $n \in \mathbb{N}$. These are unital representations: $\pi(1) = \pi(v_0) = (P_0, P_1, P_2, \ldots) = \pi^*(1)$.

By [7, Proposition 5.4], the representation π is faithful if and only if for any $r > 0$ and $i < j$ in $\mathbb{N}$, we have $\xi_{i,j}^r \in \mathcal{B}$ for which

$$\xi_{i,j}^r := (\pi(1) - \pi(v_r)^*\pi(v_r))(\pi(v_i)\pi(v_i)^* - \pi(v_j)\pi(v_j)^*)$$

is a nonzero element of $\mathcal{B}$. Let $r > 0$ and $i < j \in \mathbb{N}$, then we consider the three cases $0 < r \le i < j$, $i < r < j$ and $i < j \le r$ separately. If $0 < r \le i < j$, then

$$\begin{aligned}\xi_{i,j}^r(i) &= (P_i - \pi_i(v_r)^*\pi_i(v_r))(\pi_i(v_i)\pi_i(v_i)^* - \pi_i(v_j)\pi_i(v_j)^*)\\ &= (P_i - P_iT_r^*P_iT_rP_i)(P_iT_iP_iT_i^*P_i - P_iT_jP_iT_j^*P_i)\\ &= (P_i - P_iT_r^*T_rP_{i-r}P_i)(P_iT_iT_i^*P_i - 0)\\ &= (P_i - P_{i-r})(P_iT_iT_i^*P_i)\end{aligned}$$

and that $[\xi_{i,j}^r(i)](e_i) = (P_i - P_{i-r})(e_i) = e_i$. If $i < j \le r$, then similar computations show that $[\xi_{i,j}^r(i)](e_i) = [P_i(P_iT_iT_i^*P_i)](e_i) = e_i$, and for $i < r < j$ we have $[\xi_{i,j}^r(r)](e_r) = (P_r - P_0)(e_r) = e_r$. Thus $\xi_{i,j}^r \neq 0$ in $\mathcal{B}$. The same outline of arguments is valid to show the representation π^* is also faithful. □

So we have for every $n \in \mathbb{N}$ the representations $\pi_n = \varepsilon_n\circ\pi$ and $\pi_n^* = \varepsilon_n\circ\pi^*$ of $\mathbf{c}\times_\tau^{\rm piso}\mathbb{N}$ on H_n, where ε_n are the evaluation map of $C_{\rm b}(\mathbb{N}\cup\{\infty\}, B(\ell^2(\mathbb{N}))_{*-{\rm s}})$. Hence they are irreducible, indeed every nonzero vector of the subspace H_n of $\ell^2(\mathbb{N})$ is cyclic for π_n^*: if $(h_0, h_1, \ldots, h_n) \in H_n$ with $h_j \neq 0$ for some j, then for every $i \in \{0, 1, 2, \ldots, n\}$, we have

$$\begin{aligned}(\pi_n^*(g_{i,j}^n))(h_0, h_1, \ldots, h_n) &= [T_i(1 - TT^*)T_j^*](h_0, h_1, \ldots, h_n)\\ &= (0, \ldots, h_j, \ldots, 0), \text{ where } h_j \text{ is in the } i\text{-th slot},\end{aligned}$$

so $\pi_n^*(\frac{1}{h_j}\, g_{i,j}^n)(h) = e_i$, and therefore $H_n = \operatorname{span}\{\pi_n^*(\xi)h : \xi \in \mathbf{c} \times_\tau^{\rm piso} \mathbb{N}\}$. Same arguments work for π_n.

Note for every $n \in \mathbb{N}$ that $\pi_n(f_{i,j}^m) = e_{ij} = \pi_n(g_{n-i,n-j}^k)$ for all $0 \le i, j, m, k \le n$, and similarly $\pi_n^*(g_{i,j}^m) = e_{ij} = \pi_n^*(f_{n-i,n-j}^k)$ for all $0 \le i, j, m, k \le$

n. Thus every $f^m_{i,j} - g^k_{n-i,n-j}$ is contained in $\ker \pi_n$, and similarly $(g^m_{i,j} - f^k_{n-i,n-j}) \in \ker \pi^*_n$. We shall see many more elements of $\ker \pi_n$ as well as $\ker \pi^*_n$ in Proposition 5.7.

But now we recall that for $n \in \mathbb{N}$ the partial-isometric representation $J^n : \mathbb{N} \to B(H_n)$ in [7, §3] defined by $J^n_t(e_r) = \begin{cases} e_{t+r} & \text{if } r+t \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise,} \end{cases}$ induces the representation $\pi^{\mathbb{N}}_{J^n} \times J^n$ of $(\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}, v)$ on H_n. In fact $\pi^{\mathbb{N}}_{J^n} \times J^n = \pi_n$, because for every $k \in \mathbb{N}$ we have $(\pi^{\mathbb{N}}_{J^n} \times J^n(v_k))(e_r) = J^n_k(e_r) = P_n T_k P_n(e_r)$ where $r \in \{0,1,2,\ldots,n\}$.

The ideal $\ker \oplus^n_{r=0} \pi^{\mathbb{N}}_{J^r} \times J^r$ appears in the structure of $\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}$ [7, Lemma 5.7]. To be more precise about it, we need some results in [7, §5] related to the system $(\mathbb{C}^{n+1}, \tau, \mathbb{N})$. The crossed product $\mathbb{C}^{n+1} \times^{\text{piso}}_\tau \mathbb{N}$ is the universal C^*-algebra generated by a canonical partial-isometric representation w of $\mathbb{N}$ such that $w_r = 0$ for $r \geq n+1$. Let $q_n : (\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}, v) \to (\mathbb{C}^{n+1} \times^{\text{piso}}_\tau \mathbb{N}, w)$ be the homomorphism induced by $w : \mathbb{N} \to \mathbb{C}^{n+1} \times^{\text{piso}}_\tau \mathbb{N}$, and note that it is surjective. Then Lemma 5.7 of [7] shows that $\ker q_n = \ker(\oplus^n_{r=0} \pi^{\mathbb{N}}_{J^r} \times J^r) = \bigcap^n_{r=0} \ker(\pi^{\mathbb{N}}_{J^r} \times J^r)$. So by these arguments we obtain the following equation

$$\ker q_n = \bigcap_{r=0}^{n} \ker \pi_r \text{ for every } n \in \mathbb{N}. \tag{5.2}$$

Lemma 5.6. *For $n \in \mathbb{N}$, let L_n be the ideal of $(\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}, v)$ generated by $\{v_r : r \geq n+1\}$. Then $L_n = \ker q_n$, and it is isomorphic to*

$$\{\xi \in \pi(\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}) \subset C_b(\mathbb{N} \cup \{\infty\}, B(\ell^2(\mathbb{N}))_{*-s}) : \xi \equiv 0 \textit{ on } \{0,1,2,\ldots,n\}\}. \tag{5.3}$$

Proof. We have $L_n \subset \ker q_n$ because $q_n(v_k) = 0$ for all $k \geq n+1$. To see $\ker q_n \subset L_n$, let ρ be a representation of $\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}$ on H_ρ where $\ker \rho = L_n$. Since $\rho(v_t) = 0$ for every $t \geq n+1$, by the universal property of $\mathbb{C}^{n+1} \times^{\text{piso}}_\tau \mathbb{N}$, there exists a representation $\tilde{\rho}$ of $\mathbb{C}^{n+1} \times^{\text{piso}}_\tau \mathbb{N}$ on H_ρ which satisfies $\tilde{\rho} \circ q_n = \rho$. Thus $\ker q_n \subset \ker \rho = L_n$.

Next we show that $\pi(L_n)$ and (5.3) are equal. Let $r \geq n+1$, and consider $\pi(v_r)$ is the sequence $(P_i T_r P_i)_{i \in \mathbb{N}}$. If $0 \leq i \leq n$, then $0 \leq i+1 \leq n+1 \leq r$ and

$$P_i T_r P_i = (1 - T_{i+1} T^*_{i+1}) T_r P_i = (1 - T_{i+1} T^*_{i+1}) T_{i+1} T_{r-(i+1)} P_i = 0.$$

So $\pi(L_n)$ is a subset of (5.3). For the other inclusion, suppose $f \in \pi(\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N})$ in which $f(i) = 0$ for all $0 \leq i \leq n$. Since $f = \pi(\xi)$ for some $\xi \in \mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}$, and $\pi(\xi)(i) = \pi_i(\xi) = f(i)$ for all $i \in \mathbb{N}$, we therefore have $\pi_i(\xi) = f(i) = 0$ for all $0 \leq i \leq n$. Thus $\xi \in \cap^n_{i=0} \ker \pi_i = \ker q_n$, and hence $f = \pi(\xi) \in \pi(L_n)$. □

Let $\pi_\infty := \lim_n \pi_n$ and $\pi^*_\infty := \lim_n \pi^*_n$ where the limits are taken with respect to the strong topology of $B(\ell^2(\mathbb{N}))$. Then π_∞ and π^*_∞ are the irreducible representations $\varphi_T : v_k \mapsto T_k$ and $\varphi_{T^*} : v_k \mapsto T^*_k$ of $\mathbf{c} \times^{\text{piso}}_\tau \mathbb{N}$ on $H_\infty := \ell^2(\mathbb{N})$. Thus by [7, Lemma 6.2] we have

$$\ker \pi_\infty = \ker \varphi_T = \overline{\text{span}}\{g^m_{i,j} := v^*_i v_m v^*_m (1 - v^* v) v_j : i,j,m \in \mathbb{N}\},$$

$$\ker\pi_\infty^* = \ker\varphi_{T^*} = \overline{\operatorname{span}}\{f_{i,j}^m := v_i v_m^* v_m(1-vv^*)v_j^* : i,j,m\in\mathbb{N}\}.$$

For $n\in\mathbb{N}$, let π_n and π_n^* be the irreducible representations of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ on the subspace H_n of $\ell^2(\mathbb{N})$, that are induced by the partial-isometric representations $k\mapsto P_nT_kP_n$ and $k\mapsto P_nT_k^*P_n$. Let L_n be the ideal of $(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}, v)$ generated by $\{v_r : r\geq n+1\}$. Then π_n is the representation

$$\varepsilon_n\circ\pi : \mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N} \xrightarrow{\pi} \mathcal{B}\subset C_{\mathrm{b}}(\mathbb{N}\cup\{\infty\}, B(\ell^2(\mathbb{N}))_{*-\mathrm{s}}) \xrightarrow{\varepsilon_n} B(H_n),$$

and similarly $\pi_n^* = \varepsilon_n\circ\pi^*$. So $\ker\pi_n \simeq \ker\varepsilon_n \simeq \ker\pi_n^*$.

Proposition 5.7. *Let $n\in\mathbb{N}$. Then*

(a) $\ker\pi_n = \ker\pi_n^* \simeq \ker\varepsilon_n = \{\xi\in\mathcal{B} : \xi(n)=0\}$;
(b) $\ker\pi_\infty \simeq \ker\pi_\infty^* = \{\xi\in\mathcal{B} : *\text{-strong } \lim_n\xi(n)=0\}$;

Furthermore,

(c) $\ker\pi_n^* = \overline{\operatorname{span}}\{g_{i,j}^m - f_{n-i,n-j}^k + \eta : 0\leq i,j,m,k\leq n,\ \eta\in L_n\}$,
$\ker\pi_n = \overline{\operatorname{span}}\{f_{i,j}^m - g_{n-i,n-j}^k + \eta : 0\leq i,j,m,k\leq n,\ \eta\in L_n\}$, *and*
$\ker\pi_n^* = \ker\pi_n$ *for* $n\in\mathbb{N}$*, in particular we have* $\ker\pi_0 = \ker\pi_0^* = L_0$;
(d) $\ker\pi_n|_{\ker\varphi_{T^*}} = \overline{\operatorname{span}}\{(f_{i,j}^m - f_{i,j}^k) + f_{x,y}^z : 0\leq i,j,m,k\leq n,$ *one of* $x,y,z\geq n+1\}$,
$\ker\pi_n^*|_{\ker\varphi_T} = \overline{\operatorname{span}}\{(g_{i,j}^m - g_{i,j}^k) + g_{x,y}^z : 0\leq i,j,m,k\leq n,$ *one of* $x,y,z\geq n+1\}$,
$\Theta_*^{-1}(\ker\pi_n|_{\ker\varphi_{T^*}}) = \Theta^{-1}(\ker\pi_n^*|_{\ker\varphi_T})$, *and*
$\ker\pi_n^*|_{\ker\varphi_T} \simeq \{\mathsf{a}\in\mathcal{A} : \mathsf{a}(n)=0\} \simeq \ker\pi_n|_{\ker\varphi_{T^*}}$;
(e) $\ker\pi_n^*|_{\mathcal{I}} = \overline{\operatorname{span}}\{g_{i,j}^m - g_{i,j}^{m+1} : 0\leq i,j\leq m$ *in* $\mathbb{N}$*, and* $m\neq n\} = \ker\pi_n|_{\mathcal{I}} = \overline{\operatorname{span}}\{f_{i,j}^m - f_{i,j}^{m+1} : 0\leq i,j\leq m$ *in* $\mathbb{N}$*, and* $m\neq n\}$ *is isomorphic to the ideal* $\{\mathsf{a}\in\mathcal{A}_0 : \mathsf{a}(n)=0\}$.

Remark 5.8. Note that the representations $\pi_n|_{\ker\varphi_{T^*}}$ and $\pi_n^*|_{\ker\varphi_T}$ are equivalent to the evaluation map $\varepsilon_n : f\in\mathcal{A}\mapsto f(n)\in B(H_n)$ of $\mathcal{A}$ on H_n, so we have $\ker\pi_n|_{\ker\varphi_{T^*}} \simeq \ker\pi_n^*|_{\ker\varphi_T}$ is isomorphic to $\{f\in\mathcal{A} : f(n)=0\}$, and $\ker\pi_n|_{\mathcal{I}} = \ker\pi_n^*|_{\mathcal{I}} \simeq \{f\in\mathcal{A}_0 : f(n)=0\}$; and $\ker\pi_\infty\simeq\ker\pi_\infty^*\simeq\mathcal{A}$.

Proof of Proposition 5.7. Fix $n\in\mathbb{N}$. We show for $\ker\pi_n$, and skip the proof for $\ker\pi_n^*$ because it contains the same arguments. We clarify firstly that the space

$$\mathcal{J} := \overline{\operatorname{span}}\{f_{i,j}^m - g_{n-i,n-j}^k + \eta : 0\leq i,j,m,k\leq n, \eta\in L_n\}$$

is an ideal of $(\mathbf{c}\times_\tau\mathbb{N}, v)$ by showing $v\mathcal{J}\subset\mathcal{J}$ and $v^*\mathcal{J}\subset\mathcal{J}$. Let $i=n$, then

$$\begin{aligned} vv_kv_k^*(1-v^*v)v_{n-j} &= v(v^*vv_kv_k^*)(1-v^*v)v_{n-j} \\ &= vv_kv_k^*v^*v(1-v^*v)v_{n-j} \\ &= v_{k+1}v_{k+1}^*(v-vv^*v)v_{n-j} = 0, \end{aligned}$$

therefore $v(f_{n,j}^m - g_{0,n-j}^k + \eta) = vv_nv_m^*v_m(1-vv^*)v_j^* - vv_kv_k^*(1-v^*v)v_{n-j} + v\eta = f_{n+1,j}^m + v\eta$ belongs to $\mathcal{J}$ because $f_{n+1,j}^m\in L_n$. If $0\leq i\leq n-1$, then $1\leq i+1\leq n$

and $n-i \geq 1$, and we have

$$\begin{aligned} vv_{n-i}^* v_k v_k^* &= vv^* v_{n-i-1}^* v_{n-i-1} v_{n-i-1}^* v_k v_k^* \\ &= v_{n-i-1}^* v_{n-i-1} vv^* v_{n-i-1}^* v_k v_k^* \\ &= v_{n-i-1}^* v_{n-i} v_{n-i}^* v_k v_k^* \\ &= v_{n-i-1}^* v_{\max\{n-i,k\}} v_{\max\{n-i,k\}}^*, \end{aligned}$$

so $v(f_{i,j}^m - g_{n-i,n-j}^k + \eta) = f_{i+1,j}^m - g_{n-(i+1),n-j}^{\max\{n-i,k\}} + v\eta \in \mathcal{J}$.

Now we check for $v^*\mathcal{J}$, and assume $i=0$, then

$$\begin{aligned} v^*[f_{0,j}^m - g_{n,n-j}^k + \eta] &= v^*[v_m^* v_m (1-vv^*) v_j^* - v_n^* v_k v_k^* (1-v^*v) v_{n-j} + \eta] \\ &= 0 - g_{n+1,n-j}^k + v^*\eta \in \mathcal{J} \end{aligned}$$

because $g_{n+1,n-j}^k \in L_n$. It follows by similar computations for $1 \leq i \leq n$ that

$$v^*[f_{i,j}^m - g_{n-i,n-j}^k + \eta] = f_{i-1,j}^{\max\{i,m\}} - g_{n-(i-1),n-j}^k + v^*\eta \in \mathcal{J}.$$

Next we show that $\mathcal{J} = \ker \pi_n$, one inclusion $\mathcal{J} \subset \ker \pi_n$ is clear because $\pi_n(f_{i,j}^m) = \pi_n(g_{n-i,n-j}^k) = T_i(1-TT^*)T_j^*$ and $L_n \subset \ker \pi_n$. For the other inclusion, let $\sigma : \mathbf{c} \times_\tau^{\text{piso}} \mathbb{N} \to B(H_\sigma)$ be a nondegenerate representation with $\ker \sigma = \mathcal{J}$. Note that $B(H_n) = \text{span}\{e_{ij} := T_i(1-TT^*)T_j^* : 0 \leq i,j \leq n\}$. Since $\{f_{i,j}^n : 0 \leq i,j \leq n\}$ is a matrix-units for $B(H_\sigma)$, there is a homomorphism ψ of $B(H_n)$ into $B(H_\sigma)$ which satisfies $e_{ij} \mapsto \sigma(f_{i,j}^n)$. Therefore $\sigma = \psi \circ \pi_n$, and hence $\ker \pi_n \subset \ker \sigma = \mathcal{J}$.

Using the spanning elements of $\ker \pi_n$ and $\ker \pi_n^*$, and the equation $f_{i,j}^m - g_{n-i,n-j}^k = -(g_{n-i,n-j}^k - f_{n-(n-i),n-(n-j)}^m)$, we see that they contain each other, therefore $\ker \pi_n = \ker \pi_n^*$ for every $n \in \mathbb{N}$. The ideal L_0 is $\ker \pi_0 = \ker \pi_0^*$ because $f_{0,0}^0 - g_{0,0}^0 = v^*v - vv^* \in L_0$.

For (d), let now $\mathcal{J}$ be $\overline{\text{span}}\{(f_{i,j}^m - f_{i,j}^k) + f_{x,y}^z : 0 \leq i,j,m,k \leq n$, one of $x,y,z \geq n+1\}$. Then the same idea of calculations shows that $\mathcal{J}$ is an ideal of $\ker \varphi_{T^*}$, and it is contained in $\ker \pi_n|_{\ker \varphi_{T^*}}$, then for the other inclusion let σ be a nondegenerate representation of $\ker \varphi_{T^*}$ such that $\ker \sigma = \mathcal{J}$, get the homomorphism $\psi : B(H_n) \to B(H_\sigma)$ defined by $\psi(e_{ij}) = \sigma(f_{i,j}^n)$, and hence the equation $\psi \circ \pi_n = \sigma$ implies that $\ker \pi_n|_{\ker \varphi_{T^*}} = \mathcal{J}$. By computations on the spanning elements we see that the equation $\Theta_*^{-1}(\ker \pi_n|_{\ker \varphi_{T^*}}) = \Theta^{-1}(\ker \pi_n^*|_{\ker \varphi_T})$ is hold. The same arguments work for the proof of (e), and we skip this. □

Remark 5.9. The map $n \in \mathbb{N} \cup \{\infty\} \mapsto I_n := \ker \pi_n^* \in \text{Prim}(\mathbf{c} \times_\tau^{\text{piso}} \mathbb{N})$ parameterizes the open subset $\{P \in \text{Prim}(\mathbf{c} \times_\tau^{\text{piso}} \mathbb{N}) : \ker \varphi_T \simeq \mathcal{A} \not\subset P\}$ of $\text{Prim}(\mathbf{c} \times_\tau^{\text{piso}} \mathbb{N})$ homeomorphic to $\text{Prim}\,\mathcal{A}$. Note that the ∞ corresponds to the ideal $\ker \pi_\infty^* = \ker \varphi_{T^*} \in \text{Prim}(\mathbf{c} \times_\tau^{\text{piso}} \mathbb{N})$, and it corresponds to $\mathcal{I} = \ker \varphi_{T^*}|_{\ker \varphi_T} \in \text{Prim}\,\mathcal{A}$.

Lemma 5.10. (i) $\bigcap_{n=0}^m I_n = L_m$ *for every* $m \in \mathbb{N}$;

(ii) $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$;

(iii) $\{0\} \subsetneq (\bigcap_{n>m} I_n) \subset \ker \pi_\infty^* \cap \ker \pi_\infty$ *for every* $m \in \mathbb{N}$.

Proof. Part (i) follows from (5.2) and Lemma 5.6. For (ii), note that q_∞ is the identity map on $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$, and that $\oplus_{i\in\mathbb{N}}\pi_i = (\oplus_{i\in\mathbb{N}}(\pi_{J^i}^{\mathbb{N}} \times J^i)) \circ \mathrm{id}$. So $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$ by faithfulness of $\oplus_{i\in\mathbb{N}}(\pi_{J^i}^{\mathbb{N}} \times J^i)$ [7, Corollary 5.5].

The inclusion $\bigcap_{n>m} I_n \subset \ker \pi_\infty^*$ for every $m \in \mathbb{N}$ follows from the next arguments:

$$\begin{aligned}\bigcap_{n>m} \ker(\pi_n^*|_{\ker \pi_\infty}) &\simeq \{f \in \mathcal{A} : f(n) = 0 \ \forall\ n > m\} \\ &\subset \{f \in A : \lim_{n\to\infty} f(n) = 0\} \\ &= \mathcal{A}_0 \simeq \ker \pi_\infty^*|_{\ker \pi_\infty} \subset \ker \pi_\infty^* \in \operatorname{Prim} \mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N},\end{aligned}$$

so the two ideals $J := \bigcap_{n>m} I_n$ and $L := \ker \pi_\infty$ of $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ satisfy $J \cap L \subset \ker \pi_\infty^*$, therefore either $J \subset \ker \pi_\infty^*$ or $L \subset \ker \pi_\infty^*$, but the latter is not possible. To show $J \subset \ker \pi_\infty$, since $\ker \pi_n = \ker \pi_n^*$ for each n, we act similarly using the fact that

$$\bigcap_{n>m} \ker(\pi_n|_{\ker \pi_\infty^*}) \simeq \{f \in \mathcal{A} : f(n) = 0 \ \forall\ n > m\} \subset \ker \pi_\infty \in \operatorname{Prim} \mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}.$$

Therefore, $J \subset \ker \pi_\infty^* \cap \ker \pi_\infty$. Moreover, since $g_{0,0}^0 - g_{0,0}^1 \neq 0$ which satisfies $\pi_n^*(g_{0,0}^0 - g_{0,0}^1) = 0$ for all $n \geq 1$, it follows that $\{0\} \subsetneq (\bigcap_{n>m} I_n)$. □

Remark 5.11. Part (ii) of Lemma 5.10 confirms with the fact that $\mathcal{I}$ is an essential ideal of $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ [7, Lemma 6.8].

Next consider for $z \in \mathbb{T}$, the character $\gamma_z \in \hat{\mathbb{Z}} \simeq \mathbb{T}$ defined by $\gamma_z : m \mapsto \overline{z}^m$. Note that the map $\gamma_z : k \in \mathbb{N} \mapsto \gamma_z(k)$ is a partial-isometric representation of $\mathbb{N}$ in $\mathbb{C} \simeq B(\mathbb{C})$. Consequently for each $z \in \mathbb{T}$, we have a representation $\pi_{\gamma_z} \times \gamma_z$ of $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ on $\mathbb{C}$ such that $\pi_{\gamma_z} \times \gamma_z(v_k) = \gamma_z(k) = \overline{z}^k$ for $k \in \mathbb{N}$, and it is irreducible. Moreover we know that the homomorphism $\Psi : \mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N} \to C(\mathbb{T})$ is the composition of the Fourier transform $\mathbb{C} \times_{\mathrm{id}} \mathbb{Z} \simeq C^*(\mathbb{Z}) \simeq C(\mathbb{T})$ with $\ell \times \delta^* : \mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N} \to \mathbb{C} \times_{\mathrm{id}} \mathbb{Z}$, in which $\ell : (x_n) \in \mathbf{c} \mapsto \lim_n x_n \in \mathbb{C}$ and δ is the unitary representation of $\mathbb{Z}$ on $\mathbb{C} \times_{\mathrm{id}} \mathbb{Z}$.

Lemma 5.12. *For* $z \in \mathbb{T}$, *the character* $\gamma_z : k \mapsto \overline{z}^k$ *in* $\hat{\mathbb{Z}} \simeq \mathbb{T}$ *gives an irreducible representation* $\pi_{\gamma_z} \times \gamma_z$ *of* $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ *on* $\mathbb{C}$ *such that* $\pi_{\gamma_z} \times \gamma_z = \varepsilon_z \circ (\ell \times \delta^*)$. *Denote by* J_z *the primitive ideal* $\ker \pi_{\gamma_z} \times \gamma_z$ *of* $\mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$. *Then* $\ker \pi_\infty$ *and* $\ker \pi_\infty^*$ *are contained in* J_z *for every* $z \in \mathbb{T}$. *Moreover every ideal* I_n *for* $n \in \mathbb{N}$ *is not contained in any* J_z.

Proof. By using the Fourier transform we can view $\mathbb{C} \times_{\mathrm{id}} \mathbb{Z} \simeq C^*(\mathbb{Z})$ as $C(\mathbb{T})$, and it follows that $v_k \in \mathbf{c} \times_\tau^{\mathrm{piso}} \mathbb{N}$ is mapped into the function $\iota_k : t \mapsto \overline{t}^k \in C(\mathbb{T})$.

We know that primitive ideals of $C(\mathbb{T})$ are given by the kernels of evaluation maps $\varepsilon_t(f) = f(t)$ for $t \in \mathbb{T}$, and the character γ_z is a partial-isometric representation of $\mathbb{N}$ in $\mathbb{C}$ for $z \in \mathbb{T}$. Then by inspection on the generators, we see that the representation $\pi_{\gamma_z}\times\gamma_z$ of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ on $\mathbb{C}$ satisfies $\pi_{\gamma_z}\times\gamma_z=\varepsilon_z\circ(\ell\times\delta^*)$. So the primitive ideal $J_z:=\ker\pi_{\gamma_z}\times\gamma_z$ of $\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}$ is lifted from the quotient $(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N})/J\simeq C(\mathbb{T})$.

Since $\pi_{\gamma_z}\times\gamma_z(f_{i,j}^m)=0=\pi_{\gamma_z}\times\gamma_z(g_{i,j}^m)$, $\ker\pi_\infty=\ker\varphi_T$ and $\ker\pi_\infty^*=\ker\varphi_{T^*}$ are contained in J_z for every $z\in\mathbb{T}$. Finally, since $\pi_{\gamma_z}\times\gamma_z(v_{n+1})=\overline{z}^{n+1}\neq 0$ for $n\in\mathbb{N}$, $I_n\not\subset J_z$ for any $z\in\mathbb{T}$. □

Theorem 5.13. *The maps* $n\in\mathbb{N}\cup\{\infty\}\cup\{\infty^*\}\mapsto I_n$ *and* $z\in\mathbb{T}\mapsto J_z$ *combine to give a bijection of the disjoint union* $\mathbb{N}\cup\{\infty\}\cup\{\infty^*\}\cup\mathbb{T}$ *onto* $\mathrm{Prim}(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N})$*, where* $I_{\infty^*}:=\ker\varphi_T$*. Then the hull-kernel closure of a nonempty subset* F *of*

$$\mathbb{N}\cup\{\infty\}\cup\{\infty^*\}\cup\mathbb{T}$$

is given by

(a) *the usual closure of* F *in* $\mathbb{T}$ *if* $F\subset\mathbb{T}$;
(b) F *if* F *is a finite subset of* $\mathbb{N}$;
(c) $F\cup\mathbb{T}$ *if* $F\subset(\{\infty\}\cup\{\infty^*\})$;
(d) $F\cup(\{\infty\}\cup\{\infty^*\}\cup\mathbb{T})$ *if* $F\neq\mathbb{N}$ *is an infinite subset of* $\mathbb{N}$;
(e) $\mathbb{N}\cup\{\infty\}\cup\{\infty^*\}\cup\mathbb{T}$ *if* $\mathbb{N}\subseteq F$.

Proof. The diagram 5.1 together with Proposition 5.7 gives a bijection map of $\mathbb{N}\cup\{\infty\}\cup\{\infty^*\}\cup\mathbb{T}$ onto $\mathrm{Prim}(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N})$.

Lemma 5.10(ii) gives the closure of the subset F in (e), and Lemma 5.10(iii) gives the closure of the subset F in (d). If $F\subset(\{\infty\}\cup\{\infty^*\})$, then $\overline{F}=F\cup\mathbb{T}$ because $\ker\pi_\infty^*,\ker\pi_\infty\subset J_z$ for every $z\in\mathbb{T}$ by Lemma 5.12.

To see that $\overline{F}=F$ for a finite subset $F=\{n_1,n_2,\dots,n_j\}$ of $\mathbb{N}$, we note that if an ideal $P\in\mathrm{Prim}(\mathbf{c}\times_\tau\mathbb{N})$ satisfies $\bigcap_{i=1}^j I_{n_i}\subset P$, then

- $P\neq J_z$ for any $z\in\mathbb{T}$ because $v_{n_j+1}\in\bigcap_{i=1}^j I_{n_i}$ but $v_{n_j+1}\notin J_z$;
- $P\neq I_\infty, I_{\infty^*}$ because $v_{n_j+1}\in\bigcap_{i=1}^j I_{n_i}$ but $v_{n_j+1}\notin I_\infty, I_{\infty^*}$;
- $P\neq I_n$ for $n\notin F$ because $(g_{0,0}^n-g_{0,0}^{n+1})\in\bigcap_{i=1}^j I_{n_i}$ but $(g_{0,0}^n-g_{0,0}^{n+1})\notin I_n$ for $n\notin F$.

So it can only be $P=I_j$ for some $j\in F$. Finally the usual closure of F in $\mathbb{T}$ is followed by the fact that the map $z\mapsto J_z$ is a homeomorphism of $\mathbb{T}$ onto the closed set $\mathrm{Prim}\,C(\mathbb{T})$. □

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