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ON SEMI-ARMENDARIZ MATRIX RINGS

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Abstract. Given a positive integer n, a ring R is said to be n-semi-Armendariz if whenever $f^n = 0$ for a polynomial f in one indeterminate over R, then the product (possibly with repetitions) of any n coefficients of f is equal to zero. A ring R is said to be semi-Armendariz if R is n-semi-Armendariz for every positive integer n. Semi-Armendariz rings are a generalization of Armendariz rings. We characterize when certain important matrix rings are n-semi-Armendariz, generalizing some results of Jeon, Lee and Ryu from their paper (J. Korean Math. Soc. 47 (2010), 719–733), and we answer a problem left open in that paper.

1. Introduction

Throughout this paper, all rings are associative, and all rings have an identity except where explicitly indicated. For a ring R, the ring of polynomials in the indeterminate x over R is denoted by $R[x]$, and if $A \subseteq R$, then $A[x]$ stands for the set of polynomials in $R[x]$ whose all coefficients belong to A.

Recall that a ring R is said to be an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, in the polynomial ring $R[x]$ the following holds:

$$\begin{aligned} &\text{for any } f = \sum_{i=0}^{k} a_i x^i,\ g = \sum_{j=0}^{m} b_j x^j \in R[x], \\ &\text{if } fg = 0, \text{ then } a_i b_j = 0 \text{ for all } i, j. \end{aligned} \tag{1}$$

The name for such rings was chosen to honor E. P. Armendariz, who noted in [2] that all reduced rings (i.e., rings containing no nonzero nilpotent elements) satisfy condition (1). Various interesting properties and constructions of Armendariz rings can be found, e.g., in [1], [4], [6], [8], [11], [13], [14] and [15].

Armendariz rings, as well as many other classes of Armendariz-like rings, have recently been objects of intensive investigation (see [13]). These new

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classes of Armendariz-like rings were defined using generalizations or modifications of condition (1). For example, by replacing in (1) the polynomial ring $R[x]$ with the power series ring $R[[x]]$, one obtains the definition of a powerserieswise Armendariz ring, introduced by N. K. Kim, K. H. Lee, and Y. Lee in [9]. By replacing in (1) the requirement that the products a_ib_j are all zero with the condition that the products a_ib_j are all nilpotent, we obtain the definition of a weak Armendariz ring, introduced by Z. Liu and R. Zhao in [12]. By considering in (1) the square of a single polynomial instead of the product of two polynomials, we obtain the definition of a 2-semi-Armendariz ring, introduced by Y. C. Jeon, Y. Lee and S. J. Ryu in [7], according to which a ring R is said to be 2-*semi-Armendariz* provided for any polynomial $f = \sum_{i=0}^{m} a_ix^i \in R[x]$, if $f^2 = 0$, then $a_ia_j = 0$ for all i, j.

More generally, for a positive integer n, in [7] Jeon, Lee and Ryu define a ring R to be n-*semi-Armendariz* if for any polynomial $f = \sum_{i=0}^{m} a_ix^i \in R[x]$,

$$f^n = 0 \text{ implies } a_{i_1}a_{i_2}\cdots a_{i_n} = 0 \text{ for any subset } \{i_1, i_2, \ldots, i_n\} \subseteq \{0, 1, \ldots, m\},$$

and they call a ring R *a semi-Armendariz ring* if R is n-semi-Armendariz for every positive integer n. The following well-known result of D. D. Anderson and V. Camillo shows that all Armendariz rings are semi-Armendariz.

Proposition 1.1 ([1, Proposition 1]). *Suppose R is an Armendariz ring. If $f_1, f_2, \ldots, f_n \in R[x]$ are such that $f_1f_2\cdots f_n = 0$, then $a_1a_2\cdots a_n = 0$, where a_i is a coefficient of f_i.*

The following proposition summarizes basic properties of n-semi-Armendariz rings and semi-Armendariz rings.

Proposition 1.2 (see [7]).

(a) *Every subring of an n-semi-Armendariz ring is n-semi-Armendariz.*
(b) *Direct sum of n-semi-Armendariz rings is n-semi-Armendariz.*
(c) *A ring R is n-semi-Armendariz if and only if the ring $R[x]$ is n-semi-Armendariz.*
(d) *If a ring R is semi-Armendariz, then $nil(R[x]) \subseteq nil(R)[x]$, where $nil(A)$ denotes the set of nilpotent elements of a ring A.*

The aim of this paper is to characterize when some important matrix rings are n-semi-Armendariz. The motivation for this work were results of Jeon, Lee and Ryu from their paper [7] and a problem left open in [7].

In [7, Theorem 1.2] it was proved that for any $n \geq 2$ the $n \times n$ upper triangular matrix ring $U_n(R)$ over a ring R is n-semi-Armendariz if and only if R is reduced. In Section 2 we extend this and some other results of [7] to the ring of upper triangular $n \times n$ matrices over a ring R whose diagonal entries belong to a given subring S of R (see Proposition 2.4 and Theorem 2.5).

For a ring R and an integer $n \geq 2$, the ring $D_n(R)$ of upper triangular $n \times n$ matrices over R whose diagonal entries are equal is a subring of $U_n(R)$. Hence it follows from the aforementioned result [7, Theorem 1.2] and Proposition 1.2(a) that if R is a reduced ring, then the ring $D_n(R)$ is n-semi-Armendariz. It is natural to ask, whether the implication can be reversed, that is whether a ring R has to be reduced if the ring $D_n(R)$ is n-semi-Armendariz. In [7] the problem was left unsolved. In Section 3 we show that for any integer $n \geq 2$ the answer to the problem is negative (see Example 3.2). The answer follows easily from a general result (Theorem 3.1), which also allows to construct further examples important for the theory of n-semi-Armendariz rings (see Examples 3.4 and 3.6).

In this paper, the full ring of $n \times n$ matrices over a ring R is denoted by $M_n(R)$, and the ring of upper triangular $n \times n$ matrices over R is denoted by $U_n(R)$. For a matrix $A \in M_n(R)$ and any $i, j \in \{1, 2, \ldots, n\}$ the (i, j) entry of A is denoted by $A^{(ij)}$. The symbol E_{ij} stands for the matrix with (i, j) entry equal to 1 and all other entries equal to 0 (dimensions of the matrix E_{ij} will be clear from the context). The canonical ring isomorphism of $M_n(R)[x]$ onto $M_n(R[x])$ is denoted by Φ. Recall that the isomorphism $\Phi : M_n(R)[x] \to M_n(R[x])$ maps a polynomial

$$f = A_0 + A_1x + A_2x^2 + \cdots + A_kx^k \in M_n(R)[x]$$

to the $n \times n$ matrix $\Phi(f)$ over $R[x]$ whose (i, j) entry is the polynomial

$$A_0^{(ij)} + A_1^{(ij)}x + A_2^{(ij)}x^2 + \cdots + A_k^{(ij)}x^k \in R[x]$$

for all $i, j \in \{1, 2, \ldots, n\}$. We will usually consider the isomorphism Φ restricted to a concrete subring of the ring $M_n(R)$; such a restriction will still be denoted by Φ.

2. *n*-semi-Armendariz matrix rings

The aim of this section is to identify n-semi-Armendariz subrings of the full matrix ring $M_m(R)$, where R is a ring and $m \geq 2$. We start by showing that the ring $M_m(R)$ is never n-semi-Armendariz for $n \geq 2$.

Proposition 2.1. *Let R be a ring and let $m, n \geq 2$ be integers. Then the ring $M_m(R)$ is not n-semi-Armendariz.*

Proof. Let $f = A_0 + A_1x + A_2x^2 \in M_m(R)[x]$, where

$$A_0 = E_{1m},\ A_1 = E_{11} - E_{mm},\ A_2 = -E_{m1}.$$

Then $f^2 = 0$ and thus $f^n = 0$. Since $A_1^n = E_{11} + (-1)^n E_{mm} \neq 0$, the ring $M_m(R)$ is not n-semi-Armendariz. □

It is well known that for every $n \geq 2$ and arbitrary ring R, the upper triangular matrix ring $U_n(R)$ is not Armendariz (see [8, Example 1]). However, for any reduced ring R the ring $U_n(R)$ is n-semi-Armendariz, which was proved in [7, Theorem 1.2] (and which shows that the class of n-semi-Armendariz rings is indeed wider than the class of Armendariz rings). In Theorem 2.5 below, we generalize this result by showing that for any reduced subring S of an *arbitrary* ring R, the upper triangular $n \times n$ matrices over R whose diagonal entries belong to S form an n-semi-Armendariz ring. The following observation will be useful in our proofs.

Lemma 2.2. *Let R be a ring, let I be an ideal of R such that the factor ring R/I is reduced, and let m be a positive integer such that $I^m = 0$. Then R is n-semi-Armendariz for every $n \geq m$.*

Proof. Clearly, the set $I[x]$ of polynomials from $R[x]$ with all coefficients in I is an ideal of $R[x]$. Since the ring R/I is reduced, so is the ring $R[x]/I[x] \cong (R/I)[x]$. Therefore, if a polynomial $f = a_0 + a_1x + \cdots + a_kx^k \in R[x]$ satisfies $f^n = 0$, then $f \in I[x]$ and thus $a_i \in I$ for any i. Hence, if $n \geq m$, then for any $i_1, i_2, \ldots, i_n \in \{0, 1, \ldots, k\}$ we have $a_{i_1}a_{i_2}\cdots a_{i_n} \in I^n \subseteq I^m = 0$, which proves that the ring R is n-semi-Armendariz. $\square$

An immediate consequence of Lemma 2.2 is [7, Proposition 2.6]. Example 3.4 shows that the condition that R/I is reduced in Lemma 2.2 is not superfluous.

Remark 2.3. Let R be a ring. Recall that an ideal J of R is said to be *completely prime* if the factor ring R/J is a domain. The intersection of all completely prime ideals of R is called the *generalized nil radical* of R and denoted by $\mathcal{N}_g(R)$ (see [3, Example 3.8.16]). Note that if I is an ideal of R such that the ring R/I is reduced and $I^m = 0$ for some positive integer m, then $I = \mathcal{N}_g(R)$. Indeed, since $I^m = 0$, we deduce that I is contained in every completely prime ideal of R and $I \subseteq \mathcal{N}_g(R)$ follows. The opposite inclusion is an immediate consequence of the fact that any reduced ring is a subdirect product of domains (see [10, Theorem 12.7]). Thus $I = \mathcal{N}_g(R)$, as desired. Therefore, Lemma 2.2 can alternatively be formulated as follows: *If R is a ring such that $\mathcal{N}_g(R)^m = 0$ for some positive integer m, then R is n-semi-Armendariz for every $n \geq m$.*

Let S be a subring of a ring R. For any positive integer n we set

$$U_n(S, R) = \{A \in U_n(R) \mid A^{(ii)} \in S \ \text{ for every } \ i \in \{1, 2, \ldots, n\}\},$$

i.e., $U_n(S, R)$ consists of all upper triangular $n \times n$ matrices over R whose diagonal entries belong to S. Clearly, $U_n(S, R)$ is a subring of $U_n(R)$. Moreover, we set

$$N_n(R) = \{A \in U_n(R) \mid A^{(ii)} = 0 \ \text{ for every } \ i \in \{1, 2, \ldots, n\}\},$$

i.e., $N_n(R)$ is the set of upper triangular $n \times n$ matrices over R with all diagonal entries equal to zero. Obviously, $N_n(R)$ is an ideal of both $U_n(R)$ and $U_n(S, R)$.

The following result will be helpful in proving that some matrix rings of the form $U_n(S, R)$ are k-semi-Armendariz.

Proposition 2.4. *Let S be a subring of a ring R and let n be a positive integer. If there exists an ideal I of R such that $I \subseteq S$, and the ring S/I is reduced, and $I^m = 0$ for some positive integer m, then the ring $U_n(S, R)$ is k-semi-Armendariz for every $k \geq n + m - 1$.*

Proof. Since I is an ideal of R, the set

$$J = \{A \in U_n(S,R) \mid A^{(ii)} \in I \text{ for every } i \in \{1,2,\ldots,n\}\}$$

is an ideal of $U_n(S,R)$. Furthermore, the factor ring $U_n(S,R)/J$ is isomorphic to the direct sum of n copies of the reduced ring S/I and thus also $U_n(S,R)/J$ is reduced. Hence by Lemma 2.2, to complete the proof it suffices to show that $J^{n+m-1} = 0$. For this, it is enough to show that

$$\text{for any } A_1, A_2, \ldots, A_{n+m-1} \in J \text{ we have } A_1A_2\cdots A_{n+m-1} = 0.$$

Note that for any $i,j \in \{1,2,\ldots,n\}$ the (i,j) entry of the matrix $A_1A_2\cdots A_{n+m-1}$ is a sum of products of the form

$$(2)\quad A_1^{(k_1k_2)}A_2^{(k_2k_3)}A_3^{(k_3k_4)}\cdots A_{n+m-1}^{(k_{n+m-1}k_{n+m})}, \quad \text{where } k_1 = i \text{ and } k_{n+m} = j.$$

If $k_l > k_{l+1}$ for some $l \in \{1,2,\ldots,n+m-1\}$, then $A_l^{(k_lk_{l+1})} = 0$ and thus the product (2) is equal to 0 in this case. Otherwise we have

$$1 \le i = k_1 \le k_2 \le k_3 \le \cdots \le k_{n+m-1} \le k_{n+m} = j \le n.$$

Hence in this case there must exist at least m pairs (k_l, k_{l+1}) with $k_l = k_{l+1}$, and since $A_l \in J$, for any such a pair (k_l, k_{l+1}) we have $A_l^{(k_lk_{l+1})} \in I$. Consequently, the product (2) belongs to I^m and thus it is equal to zero. Therefore, $A_1A_2\cdots A_{n+m-1} = 0$. □

The following result extends [7, Theorem 1.2] to matrix rings of the form $U_n(S,R)$, with a different proof than that given in [7].

Theorem 2.5. *Let S be a subring of a ring R and let $n \ge 2$ be an integer. Then*

(a) *The following conditions are equivalent*:
  (i) *$U_n(S,R)$ is n-semi-Armendariz;*
  (ii) *$U_n(S,R)$ is k-semi-Armendariz for every integer $k \ge n$;*
  (iii) *S is reduced.*

(b) *$U_{n+1}(S,R)$ is n-semi-Armendariz if and only if S is reduced and R is Armendariz.*

Proof. (a) (i) ⇒ (iii): Assume $U_n(S,R)$ is n-semi-Armendariz. Then by Proposition 1.2(a), the ring $U_n(S)$ is n-semi-Armendariz. Hence by [7, Theorem 1.2], the ring S is reduced.

(iii) ⇒ (ii): Apply Proposition 2.4 with $I = 0$ and $m = 1$.

(ii) ⇒ (i): Obvious.

(b) We will use the following observation:

For any ring T and matrices $B_1, B_2, \ldots, B_n \in N_{n+1}(T)$ we have

$$(3)\qquad B_1B_2\cdots B_n = B_1^{(1,2)}B_2^{(2,3)}\cdots B_n^{(n,n+1)}E_{1,n+1}.$$

To establish (3), note that for any $i, j \in \{1, 2, \ldots, n+1\}$ the (i, j) entry of $B_1 B_2 \cdots B_n$ is a sum of products of the form

$$B_1^{(k_1 k_2)} B_2^{(k_2 k_3)} B_3^{(k_3 k_4)} \cdots B_n^{(k_n k_{n+1})}, \quad \text{where} \quad k_1 = i \quad \text{and} \quad k_{n+1} = j. \tag{4}$$

For any $l \in \{1, 2, \ldots, n\}$ we have $B_l \in N_{n+1}(T)$ and thus if $k_l \geq k_{l+1}$, then $B_l^{(k_l k_{l+1})} = 0$. Hence the product (4) can be nonzero only in the case when

$$1 \leq i = k_1 < k_2 < k_3 < \cdots < k_n < k_{n+1} = j \leq n+1,$$

that is, when $k_l = l$ for every $l \in \{1, 2, \ldots, n+1\}$. Now (3) follows.

To prove (b), set $V = U_{n+1}(S, R)$ and $N = N_{n+1}(R)$. Assume V is n-semi-Armendariz. Since $U_n(S, R)$ is isomorphic to a subring of V, the ring $U_n(S, R)$ is n-semi-Armendariz by Proposition 1.2(a), and thus by the already proved part (a), the ring S is reduced. To show that R is Armendariz, consider any polynomials $f = a_0 + a_1 x + \cdots + a_k x^k$, $g = b_0 + b_1 x + \cdots + b_l x^l \in R[x]$ such that $fg = 0$. Without loss of generality we can assume that $k = l$. For any $m \in \{0, 1, \ldots, k\}$ we set

$$A_m = a_m E_{12} + b_m E_{23} + E_{34} + \cdots + E_{n,n+1} \in N,$$

and we put

$$h = A_0 + A_1 x + \cdots + A_k x^k \in N[x].$$

Let $\Phi : U_{n+1}(R)[x] \to U_{n+1}(R[x])$ be the canonical isomorphism, and let $H = \Phi(h)$. Since $h \in N[x]$, it follows that $H \in N_{n+1}(R[x])$. Hence (3) implies

$$H^n = H^{(1,2)} H^{(2,3)} H^{(3,4)} \cdots H^{(n,n+1)} E_{1,n+1} = fg(1+x+\cdots+x^k)^{n-2} E_{1,n+1} = 0,$$

and thus $h^n = 0$. Since V is n-semi-Armendariz and $h^n = 0$, it follows that $A_i A_j A_1^{n-2} = 0$ for any i, j. Since $A_i, A_j, A_1 \in N$, (3) implies

$$a_i b_j = A_i^{(1,2)} A_j^{(2,3)} A_1^{(3,4)} \cdots A_1^{(n,n+1)} = 0,$$

which proves that R is Armendariz.

To prove the converse, assume S is reduced and R is Armendariz, and consider any polynomial $q = A_0 + A_1 x + \cdots + A_k x^k \in V[x]$ with $q^n = 0$. We claim that $A_i \in N$ for each i. To see this, note that since S is reduced and N is an ideal of V such that V/N is isomorphic to the direct sum of $n+1$ copies of S, the ring V/N is reduced, and thus so is the ring $V[x]/N[x] \cong (V/N)[x]$. Hence $q^n = 0$ yields $q \in N[x]$, which proves our claim.

Let $\Phi : U_{n+1}(R)[x] \to U_{n+1}(R[x])$ be the canonical isomorphism, and let $Q = \Phi(q)$. Since $q \in N[x]$, it follows that $Q \in N_{n+1}(R[x])$. Furthermore, $Q^n = \Phi(q^n) = \Phi(0) = 0$, and thus (3) implies

$$Q^{(1,2)} Q^{(2,3)} \cdots Q^{(n,n+1)} = 0.$$

Hence, if for any $i \in \{1, 2, \ldots, n\}$ we set

$$f_i = A_0^{(i,i+1)} + A_1^{(i,i+1)} x + \cdots + A_k^{(i,i+1)} x^k \in R[x],$$

then $f_1 f_2 \cdots f_n = 0$. Since R is Armendariz, by Proposition 1.1 we have

$$A_{s_1}^{(1,2)} A_{s_2}^{(2,3)} \cdots A_{s_n}^{(n,n+1)} = 0 \text{ for any } n\text{-tuple } (s_1, s_2, \ldots, s_n),$$

and (3) implies $A_{s_1} A_{s_2} \cdots A_{s_n} = 0$. Thus V is n-semi-Armendariz. □

As an immediate consequence of the above theorem we obtain [7, Theorem 1.2], which is just the first part of the following corollary. The second part says that if R is a reduced ring, then $U_n(R)$ is a maximal n-semi-Armendariz subring of the full matrix ring $M_n(R)$.

Corollary 2.6. *For any ring R and integer $n \geq 2$, the following conditions are equivalent*:

(i) *$U_n(R)$ is n-semi-Armendariz*;
(ii) *$U_n(R)$ is k-semi-Armendariz for every $k \geq n$*;
(iii) *$U_{n+1}(R)$ is n-semi-Armendariz*;
(iv) *R is reduced.*

If any of these equivalent conditions holds, then $U_n(R)$ is a maximal n-semi-Armendariz subring of $M_n(R)$.

Proof. Since reduced rings are Armendariz, Theorem 2.5 implies that (i), (ii), (iii), and (iv) are equivalent. To prove the second part of the corollary, assume R is reduced and T is an n-semi-Armendariz subring of $M_n(R)$ such that $U_n(R) \subsetneq T$. Then there exists a matrix $A \in T$ such that for some $i > j$ the (i, j) entry of A, say a, is non-zero. Since $E_{ii}, E_{jj} \in T$, also $E_{ii} A E_{jj} = a E_{ij} \in T$. Thus for the matrices $A_0 = aE_{ij}$, $A_1 = aE_{ii} - aE_{jj}$ and $A_2 = -aE_{ji}$ we have $f = A_0 + A_1 x + A_2 x^2 \in T[x]$. It is easy to see that $f^2 = 0$, and thus $f^n = 0$. Since T is n-semi-Armendariz, $0 = A_1^n = a^n E_{ii} - a^n E_{jj}$. Hence $a^n = 0$, and since R is reduced, we obtain $a = 0$, a contradiction. □

The following result is an immediate consequence of Corollary 2.6.

Corollary 2.7 (cf. [7, Corollary 1.3])**.** *For any ring R the following conditions are equivalent*:

(i) *R is reduced*;
(ii) *$U_2(R)$ is semi-Armendariz*;
(iii) *$U_3(R)$ is semi-Armendariz.*

Given integers $n \geq 2$ and $m \geq n + 2$, in the context of the first part of Corollary 2.6 it is natural to ask, when for a ring R the ring $U_m(R)$ is n-semi-Armendariz. The following result shows that the answer is *"never"* (cf. [7, Example 1.6]).

Proposition 2.8. *Let R be a ring and let $n \geq 2$ be an integer. Then for any integer $m \geq n + 2$ the ring $U_m(R)$ is not n-semi-Armendariz.*

The proof of Proposition 2.8 will be based on the following lemma. In the statement of the lemma, in an obvious way, we extend the notion of an n-semi-Armendariz ring to rings without unity. Recall that $N_{n+2}(R)$ consists of upper triangular $(n+2)\times(n+2)$ matrices over R with all diagonal entries equal to zero. Clearly, $N_{n+2}(R)$ is a non-unital subring of $U_{n+2}(R)$ (even more: it is an ideal of $U_{n+2}(R)$).

Lemma 2.9. *For any ring R and integer $n \geq 2$ the non-unital ring $N_{n+2}(R)$ is not n-semi-Armendariz.*

Proof. We start the proof with the case $n = 2$. Set $A = E_{12} + E_{34}$ and $B = E_{12} + E_{13} - E_{24} + E_{34}$. Then $f = A + Bx \in N_4(R)[x]$ and $f^2 = 0$, but $AB = -E_{14} \neq 0$ and thus $N_4(R)$ is not 2-semi-Armendariz.

Next assume that $n \geq 3$. In this case we simply repeat arguments of [7, Example 1.6]. We set

$$A = E_{12} + E_{23} + \cdots + E_{n-2,n-1} + E_{n-1,n+1} + E_{n,n+2} \in N_{n+2}(R),$$

$$B = E_{n-1,n} + E_{n-1,n+1} + E_{n,n+2} - E_{n+1,n+2} \in N_{n+2}(R).$$

It is easy to verify that $B^2 = BA^2 = BAB = 0$,

$$A^k = E_{1,k+1} + E_{2,k+2} + \cdots + E_{n-k-1,n-1} + E_{n-k,n+1} \quad \text{for} \quad 2 \leq k \leq n-2,$$

and $A^{n-1} = E_{1,n+1}$. Hence for $f = A + Bx \in N_{n+2}(R)[x]$ we have

$$f^n = A^n + (A^{n-2}BA + A^{n-1}B)x = (E_{1,n+2} - E_{1,n+2})x = 0.$$

Since $A^{n-2}BA = E_{1,n+2} \neq 0$, the ring $N_{n+2}(R)$ is not n-semi-Armendariz. □

Now we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. From Proposition 1.2(a) and Lemma 2.9 we deduce that the ring $U_{n+2}(R)$ is not n-semi-Armendariz. To complete the proof, consider any integer $m \geq n+2$. Then the ring $U_{n+2}(R)$ is isomorphic to a subring of $U_m(R)$. Since the ring $U_{n+2}(R)$ is not n-semi-Armendariz, Proposition 1.2(a) implies that $U_m(R)$ is not n-semi-Armendariz. □

Let R be a ring and let n, k be integers such that $2 \leq n < k$. By Corollary 2.6, R being reduced implies that the ring $U_n(R)$ is k-semi-Armendariz. The following example shows that the opposite implication is not true.

Example 2.10. (*For any $2 \leq n < k$ there exists a nonreduced ring R such that the ring $U_n(R)$ is k-semi-Armendariz.*) Let p be a prime number, and let $m = k - n + 1$. Let $R = \mathbb{Z}_{p^m}$ be the ring of integers modulo p^m and let $I = (p)$ be the ideal of R generated by p. Then the factor ring $R/I \cong \mathbb{Z}_p$ is reduced and $I^m = 0$. Hence by Proposition 2.4 the ring $U_n(R)$ is k-semi-Armendariz. Furthermore, since $m \geq 2$, the ring R is not reduced.

Corollary 2.6 implies that the ring $U_n(R)$ constructed in Example 2.10 is not n-semi-Armendariz. Hence, the same example shows also that *for any $2 \leq n < k$ there exists a k-semi-Armendariz ring that is not n-semi-Armendariz* (cf. [7, p. 725]).

3. Answer to Jeon-Lee-Ryu's problem

Given an integer $n \geq 2$, in [7, Theorem 1.2] Jeon, Lee and Ryu characterized rings R for which the ring $U_n(R)$ is n-semi-Armendariz. Namely, they proved that $U_n(R)$ is n-semi-Armendariz if and only if R is reduced (see also Corollary 2.6). Since the set

$$D_n(R) = \{A \in U_n(R) \mid A^{(1,1)} = A^{(2,2)} = \cdots = A^{(n,n)}\}$$

(i.e., the set of all upper triangular $n \times n$ matrices over R whose diagonal entries are equal) is a subring of $U_n(R)$, it follows from Proposition 1.2(a) that *if R is a reduced ring, then the ring $D_n(R)$ is n-semi-Armendariz.* It is natural to ask, whether the implication can be reversed, which led the authors of [7] to the following problem (see [7, p. 724, the paragraph preceding Proposition 1.5]):

(5) *Problem*: *If $n \geq 2$ and the ring $D_n(R)$ is n-semi-Armendariz, must R be a reduced ring?*

In [7] the problem was left unsolved. In Example 3.2 we will show that the answer to the problem is negative for any integer $n \geq 2$. The answer will easily follow from the general Theorem 3.1.

The main reason why in Theorem 3.1 we focus on rings $D_n(S)$ with S of the form $S = D_2(R)$ is that the ring $S = D_2(R)$ is never reduced, so in the context of Jeon-Lee-Ryu's problem (5) it is natural to ask, when the ring $D_n(S) = D_n(D_2(R))$ is n-semi-Armendariz. In the theorem below we answer this question in the case where the ring R is reduced and commutative.

For a ring R, if n is a positive integer and $r \in R$, then nr denotes the sum $r + r + \cdots + r$ with n summands.

Theorem 3.1. *Let R be a reduced commutative ring and let n be a positive integer. Then the following conditions are equivalent*:

(i) *The ring $D_n(D_2(R))$ is n-semi-Armendariz*;
(ii) *The ring $(D_2(R)[x])/(x^n)$ is n-semi-Armendariz, where (x^n) denotes the ideal of $D_2(R)[x]$ generated by x^n*;
(iii) *$nr = 0 \Rightarrow r = 0$ for any $r \in R$.*

Proof. The case $n = 1$ is trivial, because every ring is obviously 1-semi-Armendariz. Thus to the end of the proof we assume that $n \geq 2$.

(i) $\Rightarrow$ (ii): Assume the ring $D_n(D_2(R))$ is n-semi-Armendariz. The factor ring $(D_2(R)[x])/(x^n)$ is isomorphic to the matrix ring

$$\left\{ \begin{pmatrix} A_1 & A_2 & \cdots & A_{n-1} & A_n \\ 0 & A_1 & \ddots & & A_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & A_1 & A_2 \\ 0 & 0 & \cdots & 0 & A_1 \end{pmatrix} \mid A_1, A_2, \ldots, A_n \in D_2(R) \right\},$$

which is a subring of $D_n(D_2(R))$. Hence Proposition 1.2(a) implies that the ring $(D_2(R)[x])/(x^n)$ is n-semi-Armendariz.

(ii) $\Rightarrow$ (iii): Put $P = (D_2(R)[x])/(x^n)$ and assume P is n-semi-Armendariz. We will use the bar notation for elements of P, that is, for any $h \in D_2(R)[x]$ we will write $\bar{h}$ to denote the coset $h + (x^n) \in P$. Let $r \in R$ be such that $nr = 0$. Set

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in D_2(R)$$

and consider the polynomial $f = \bar{A}\bar{x} + \bar{B}t \in P[t]$. Note that in P we have

$$\bar{A}\bar{B} = \bar{B}\bar{A}, \ \bar{B}^2 = 0, \ \bar{x}^{\,n} = 0, \ \text{and } n\bar{A} = 0.$$

Hence, by the binomial theorem,

$$f^n = (\bar{A}\bar{x} + \bar{B}t)^n = \sum_{i=0}^{n} \binom{n}{i} (\bar{A}\bar{x})^{n-i}(\bar{B}t)^i = \bar{A}^{\,n}\bar{x}^{\,n} + n\bar{A}^{n-1}\bar{x}^{\,n-1}\bar{B}t = 0.$$

Since the ring P is n-semi-Armendariz, it follows that $(\bar{A}\bar{x})^{n-1}\bar{B} = 0$ and thus $A^{n-1}Bx^{n-1} \in (x^n)$. Hence $A^{n-1}B = 0$ and consequently $r^{n-1} = 0$. Since R is reduced, we obtain $r = 0$, as desired.

(iii) $\Rightarrow$ (i): Set $D(R) = D_n(D_2(R))$. *To the end of the proof the ring $D(R)$ is considered as a subring of $U_{2n}(R)$.* For any $B \in D(R)$ and $i, j \in \{1, 2, \ldots, 2n\}$, $B^{(ij)}$ denotes the (i, j) entry of the matrix B.

Obviously $N(R) = \{B \in D(R) : B^{(ii)} = 0 \ \text{ for any } \ i \in \{1, 2, \ldots, 2n\}\}$ is an ideal of $D(R)$. We start with describing the form of any product of n matrices from $N(R)$, which will be crucial in the proof of the implication (iii) $\Rightarrow$ (i). The reader should note that *in this part of the proof we do not put any assumption on the ring R.*

Let $B_1, B_2, \ldots, B_n \in N(R)$ and let $B = B_1 B_2 \cdots B_n$. Then for any $i, j \in \{1, 2, \ldots, 2n\}$, the (i, j) entry of B is a sum of products of the form

$$(6) \qquad b = B_1^{(k_0k_1)} B_2^{(k_1k_2)} B_3^{(k_2k_3)} \cdots B_n^{(k_{n-1}k_n)}, \ \text{ where } \ k_0 = i \ \text{ and } \ k_n = j.$$

Assume $b \neq 0$. Then $B_t^{(k_tk_{t+1})} \neq 0$ for every $t \in \{0, 1, \ldots, n-1\}$, and since $B_t \in N(R)$, it follows that $k_t < k_{t+1}$. Thus

$$1 \leq i = k_0 < k_1 < k_2 < \cdots < k_{n-1} < k_n = j \leq 2n.$$

Notice that for any matrix $A \in D(R) = D_n(D_2(R))$ we have $A^{(pq)} = 0$ for every pair (p, q) with even p and odd q. Hence, since $b \neq 0$, if k_t is even for some $t \in \{0, 1, \ldots, n-1\}$, then so are $k_{t+1}, k_{t+2}, \ldots, k_n$. In particular, if k_0 would be even, then so would be $k_1, k_2, \ldots, k_n$, and we would obtain $n + 1$ even integers in the interval $\langle 1, 2n \rangle$, a contradiction. Thus k_0 is odd. If also $k_1, k_2, \ldots, k_n$ would be odd, then there would be $n + 1$ odd integers between 1 and $2n$, a contradiction. Therefore, there exists $m \in \{1, 2, \ldots, n\}$ such that $k_0, k_1, \ldots, k_{m-1}$ are odd and $k_m, k_{m+1}, \ldots, k_n$ are even. Hence

$$(7) \qquad k_0 + 2(m-1) \leq k_{m-1} \ \text{ and } \ k_m \leq k_n - 2(n-m).$$

Since furthermore $k_{m-1} < k_m$, it follows from (7) that $2n-1 \leq k_n - k_0$, and thus $k_0 = 1$ and $k_n = 2n$. Putting these values of k_0 and k_n into (7), we obtain $k_{m-1} = 2m-1$ and $k_m = 2m$.

By the above, if $b \neq 0$, then $i = 1$, $j = 2n$ and the sequence

$$(k_0, k_1, k_2, \ldots, k_{n-1}, k_n)$$

is of the following form:

$$(8) \qquad (\underbrace{1, 3, \ldots, 2m-1}_{\text{odd integers}}, \underbrace{2m, 2m+2, \ldots, 2n}_{\text{even integers}}), \quad \text{for some } \ m \in \{1, 2, \ldots, n\}.$$

The sequence (8) will be denoted by γ_m and the set of all such sequences will be denoted by $\mathcal{C}_n$, i.e., $\mathcal{C}_n = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. For example, $\mathcal{C}_4$ consists of the following four sequences:

$$\gamma_1 = (1, 2, 4, 6, 8), \quad \gamma_2 = (1, 3, 4, 6, 8), \quad \gamma_3 = (1, 3, 5, 6, 8), \quad \gamma_4 = (1, 3, 5, 7, 8).$$

To simplify notation, for any $\gamma = (k_0, k_1, \ldots, k_{n-1}, k_n) \in \mathcal{C}_n$, the product (6) will be denoted by $\gamma(B_1, B_2, \ldots, B_n)$, i.e.,

$$\gamma(B_1, B_2, \ldots, B_n) = B_1^{(k_0 k_1)} B_2^{(k_1 k_2)} B_3^{(k_2 k_3)} \cdots B_n^{(k_{n-1} k_n)}.$$

Summarizing, and using the just introduced notation, if $B_1, B_2, \ldots, B_n \in N(R)$, then the $(1, 2n)$ entry of the product $B_1 B_2 \cdots B_n$ is equal to the sum $\sum_{m=1}^{n} \gamma_m(B_1, B_2, \ldots, B_n)$, and all other entries are equal to 0, i.e.,

$$(9) \qquad B_1 B_2 \cdots B_n = \left(\sum_{m=1}^{n} \gamma_m(B_1, B_2, \ldots, B_n) \right) E_{1,2n}.$$

Now we are in a position to prove the implication (iii) $\Rightarrow$ (i). Assume R is a reduced commutative ring and $n \geq 2$ is an integer satisfying (iii), i.e., we have $nr = 0 \Rightarrow r = 0$ for any $r \in R$. To prove (i), we consider an arbitrary polynomial

$$f = A_0 + A_1 x + \cdots + A_t x^t \in D(R)[x] \ \text{ such that } f^n = 0,$$

and we have to show that

$$(10) \qquad A_{s_1} A_{s_2} \cdots A_{s_n} = 0 \ \text{ for any } n\text{-tuple } \ (s_1, s_2, \ldots, s_n).$$

Since $N(R)$ is an ideal of $D(R)$ and the ring $D(R)/N(R)$ is isomorphic to R, it follows that $D(R)/N(R)$ is reduced. Hence also the ring

$$D(R)[x]/N(R)[x] \cong (D(R)/N(R))[x]$$

is reduced and from $f^n = 0$ we deduce that $f \in N(R)[x]$. Thus $A_0, A_1, \ldots, A_t \in N(R)$, and (9) implies that to prove (10) it suffices to show that for any $m \in \{1, 2, \ldots, n\}$ and n-tuple $(s_1, s_2, \ldots, s_n)$ we have

$$(11) \qquad \gamma_m(A_{s_1}, A_{s_2}, \ldots, A_{s_n}) = 0.$$

To prove (11), let $\Phi : D(R)[x] \to D(R[x])$ be the canonical isomorphism and let $F = \Phi(f)$. Since $f \in N(R)[x]$ and $f^n = 0$, we have $F \in N(R[x])$ and $F^n = 0$. Hence (9) implies

$$\sum_{m=1}^{n} \gamma_m(\underbrace{F, F, \ldots, F}_{n \text{ times}}) = 0. \tag{12}$$

Recall that γ_m denotes the sequence (8) and thus

$$\begin{aligned} &\gamma_m(F, F, \ldots, F) \\ &= F^{(1,3)}F^{(3,5)} \cdots F^{(2m-3,2m-1)}F^{(2m-1,2m)}F^{(2m,2m+2)} \cdots F^{(2n-2,2n)}. \end{aligned} \tag{13}$$

Note that since $F \in D(R[x]) = D_n(D_2(R[x]))$, for any odd $i \in \{1, 2, \ldots, 2n\}$ we have $F^{(i,i+1)} = F^{(1,2)}$ and if furthermore $i \leq 2n-3$, then $F^{(i,i+2)} = F^{(i+1,i+3)}$. Moreover, since R is commutative, the ring $R[x]$ is commutative. Combining all of these with (13), we obtain

$$\begin{aligned} \gamma_m(F, F, \ldots, F) &= F^{(2,4)}F^{(4,6)} \cdots F^{(2m-2,2m)}F^{(1,2)}F^{(2m,2m+2)} \cdots F^{(2n-2,2n)} \\ &= F^{(1,2)}F^{(2,4)}F^{(4,6)} \cdots F^{(2m-2,2m)}F^{(2m,2m+2)} \cdots F^{(2n-2,2n)} \\ &= \gamma_1(F, F, \ldots, F). \end{aligned}$$

We have shown that

$$\gamma_m(F, F, \ldots, F) = \gamma_1(F, F, \ldots, F) \quad \text{for any } m \in \{1, 2, \ldots, n\}, \tag{14}$$

and thus (12) implies

$$n \cdot \gamma_1(F, F, \ldots, F) = 0. \tag{15}$$

Since by (iii) we have $nr = 0 \Rightarrow r = 0$ for any $r \in R$, it follows from (15) that $\gamma_1(F, F, \ldots, F) = 0$, and thus, by (14), for any $m \in \{1, 2, \ldots, n\}$ we have

$$\gamma_m(F, F, \ldots, F) = 0. \tag{16}$$

We will show that (16) implies (11), which is enough to complete the proof. Let $\gamma_m = (k_0, k_1, k_2, \ldots, k_{n-1}, k_n)$ (at this point, the exact form of γ_m does not matter). From (16) we obtain

$$F^{(k_0k_1)}F^{(k_1k_2)} \cdots F^{(k_{n-1}k_n)} = 0. \tag{17}$$

For any $i \in \{1, 2, \ldots, n\}$, set

$$f_i = A_0^{(k_{i-1}k_i)} + A_1^{(k_{i-1}k_i)}x + \cdots + A_t^{(k_{i-1}k_i)}x^t \in R[x].$$

Since $F = \Phi(f)$, it follows from (17) that

$$f_1 f_2 \cdots f_n = 0. \tag{18}$$

By assumption the ring R is reduced, so R is Armendariz as well, and thus (18) and Proposition 1.1 imply that for any n-tuple $(s_1, s_2, \ldots, s_n)$ we have

$$A_{s_1}^{(k_0k_1)}A_{s_2}^{(k_1k_2)} \cdots A_{s_n}^{(k_{n-1}k_n)} = 0,$$

that is,

$$\gamma_m(A_{s_1}, A_{s_2}, \ldots, A_{s_n}) = 0 \text{ for any } m \in \{1, 2, \ldots, n\}.$$

Since $A_{s_1}, A_{s_2}, \ldots, A_{s_n} \in N(R)$, from (9) we obtain

$$A_{s_1} A_{s_2} \cdots A_{s_n} = 0,$$

which is just condition (10). The proof is complete. □

Now we are in a position to answer Jeon-Lee-Ryu's problem (5).

Example 3.2. (*There exists a nonreduced ring R such that for any $n \geq 2$ the ring $D_n(R)$ is n-semi-Armendariz.*) Let, as usually, $\mathbb{Z}$ denote the ring of integers. The ring $R = D_2(\mathbb{Z})$ is not reduced but, by Theorem 3.1, for any $n \geq 2$ the ring $D_n(R)$ is n-semi-Armendariz.

Remark 3.3. Using the same arguments as in the proof of Theorem 3.1 one can prove the following generalization of Theorem 3.1.

Let S be a reduced subring of a commutative Armendariz ring R and let $n \geq 2$ be an integer. Then the following conditions are equivalent:

(i) *The ring $D_n(D_2(S, R))$ is n-semi-Armendariz*;

(ii) *The ring $(D_2(S, R)[x])/(x^n)$ is n-semi-Armendariz, where (x^n) denotes the ideal of $D_2(S, R)[x]$ generated by x^n*;

(iii) $nr = 0 \Rightarrow r = 0$ *for any $r \in R$.* □

Let R be a ring and let I be a proper ideal of R. Assume R/I and I are n-semi-Armendariz, where I is considered as an n-semi-Amendariz ring without unity. One could conjecture that in this case R has to be n-semi-Armendariz. However, this is not the case as shown in [7, Example 2.2]. Below we use Theorem 3.1 to construct another example of a ring R with an ideal I such that R/I and I are n-semi-Armendariz but R is not n-semi-Armendariz.

Example 3.4. (*For any $n \geq 2$ there exist a ring R and an ideal I of R such that I and R/I are n-semi-Armendariz, but R is not n-semi-Armendariz.*) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n and let $m = p_1 p_2 \cdots p_k$. Note that the ring $\mathbb{Z}_m$ of integers modulo m is reduced. Set $R = (D_2(\mathbb{Z}_m)[x])/(x^n)$ and $I = (x^{n-1})/(x^n)$. Then I is an ideal of R with $I^2 = 0$ and thus I is n-semi-Armendariz. Furthermore, the ring $R/I \cong (D_2(\mathbb{Z}_m)[x])/(x^{n-1})$ is isomorphic to a subring of $U_{n-1}(D_2(\mathbb{Z}_m))$ (see the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 3.1). Since $N = N_2(\mathbb{Z}_m)$ is an ideal of $D_2(\mathbb{Z}_m)$ such that the ring $D_2(\mathbb{Z}_m)/N \cong \mathbb{Z}_m$ is reduced and $N^2 = 0$, Proposition 2.4 implies that the ring $U_{n-1}(D_2(\mathbb{Z}_m))$ is n-semi-Armendariz, and thus the ring R/I is n-semi-Armendariz as well. Finally note that by Theorem 3.1 the ring R is not n-semi-Armendariz.

Recall that a ring R is said to be *abelian* if every idempotent of R is central. In [7, Example 2.4(1)] it was shown that an abelian prime ring need not be semi-Armendariz. Bellow we show the same, constructing a simpler example than this in [7].

Example 3.5. (*An abelian prime ring need not be semi-Armendariz.*) Let S be a domain and let $n \geq 2$. Set $R = D_{n+2}(S)$. Then by [5, Lemma 2] the ring R is abelian. To see that R is prime, consider any non-zero matrices $A, B \in R$. Let a_{ij} and b_{kl} be any nonzero entries of A and B, respectively. Then the (i,l) entry of $AE_{jk}B$ is equal to $a_{ij}b_{kl} \neq 0$. Thus $ARB \neq 0$, proving that R is prime. Finally, Lemma 2.9 implies that R is not n-semi-Armendariz and thus R is not semi-Armendariz.

We close the paper with an example of a commutative ring which is not semi-Armendariz (cf. [7, Example 2.4(2)]).

Example 3.6. (*A commutative ring need not be semi-Armendariz.*) Obviously, the ring $R = D_2(D_2(\mathbb{Z}_2))$ is commutative. However, R is not 2-semi-Armendariz by Theorem 3.1 and thus R is not semi-Armendariz.

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