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A CLASS OF NEW NEAR-PERFECT NUMBERS

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Abstract. Let α be a positive integer, and let p_1, p_2 be two distinct prime numbers with $p_1 < p_2$. By using elementary methods, we give two equivalent conditions of all even near-perfect numbers in the form $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$, and obtain a lot of new near-perfect numbers which involve some special kinds of prime number pairs. One kind is exactly the new Mersenne conjecture's prime number pair. Another kind has the form $p_1 = 2^{\alpha+1} - 1$ and $p_2 = \frac{p_1^2+p_1+1}{3}$, where the former is a Mersenne prime and the latter's behavior is very much like a Fermat number.

1. Introduction

Definition 1.1. Let n be a positive integer. Set $D = \{d : d \mid n,\ 1 \leqslant d \leqslant n\}$ and $\sigma(n) = \sum_{d \in D} d$.

(1) If $\sigma(n) = 2n$, then n is called a perfect number.
(2) (Sierpinski) If there exists some $S \subseteq D - \{n\}$ such that $n = \sum_{d \in S} d$, then n is called a pseudoperfect number.
(3) (Shevelev) If there exists some $d \in D - \{n\}$ such that $\sigma(n) = 2n + d$, then n is called a near-perfect number with the redundant divisor d.
(4) (Pollack & Shevelev) If there exist some r, $0 \leqslant r \leqslant k$ and $S \subseteq D - \{n\}$ with $|S| = r$, such that $\sigma(n) = 2n + \sum_{d \in S} d$, then n is called a k-near-perfect number with the redundant divisors set S.

By Definition 1.1 above, it is easy to see that pseudoperfect numbers are a generalization for perfect numbers, near-perfect numbers are special pseudoperfect numbers, and perfect numbers and near-perfect numbers constitute 1-near-perfect numbers.

People have been interested in perfect numbers for a long time. Euclid and Euler determined all even perfect numbers, which are closely related to

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Mersenne primes. For odd perfect numbers, Euler obtained a necessary condition for the existence (see [5]). In recent years, there have been many papers for odd perfect numbers having to do with the conjecture that there exists no odd perfect numbers (see [2, 3, 4, 7, 12]). Until now, the conjecture has not been proved. Therefore, people study other similar numbers, such as pseudoperfect numbers, near-perfect numbers, k-near-perfect numbers and deficient-perfect numbers, which are closely related to perfect numbers (see [8, 9, 10, 13]).

In 2012, based on the criterion for the existence of even perfect numbers, Paul Pollack and Vladimir Shevelev obtained 3 classes of even near-perfect numbers as follows (see [8]).

Proposition 1.2. *Suppose that $2^p - 1$ is a Mersenne prime. Then $n = 2^{p-1}(2^p-1)^2$ is a near-perfect number with the redundant divisor $2^p - 1$.*

Proposition 1.3. *Let t and k be positive integers with $t \geqslant k+1$. Suppose that $2^t - 2^k - 1$ is an odd prime number. Then $n = 2^{t-1}(2^t - 2^k - 1)$ is near-perfect with the redundant divisor 2^k.*

Proposition 1.4. *Let α be a positive integer and let p be a prime number. Suppose that m is an even perfect number and $2^{p-1} \parallel m$. Then $n = 2^\alpha m$ is near-perfect if and only if $\alpha = 1$ or $\alpha = p$.*

It is easy to see that the near-perfect numbers in Proposition 1.2–1.4 are in the form $2^\alpha p^\beta$, where $\alpha, \beta \geqslant 1$ and p is an odd prime number. Noting that $40 = 2^3 \cdot 5$ which isn't in Propositions 1.2–1.4 is also a near-perfect number with the redundant divisor 10, Chen and Ren improved the above results by proving the following proposition in [9].

Proposition 1.5. *Suppose that n has exactly two distinct prime divisors. Then n is near-perfect if and only if $n = 40$ or n is given by one of Propositions* 1.2–1.4.

Moreover, for near-perfect numbers with at least three distinct prime divisors, there is the following conjecture in [9].

Conjecture 1.6. *For any $k \geqslant 3$, there exists only finitely many near-perfect numbers with exactly k distinct prime divisors.*

In fact, for the generalized case, let α_i be positive integers and let p_i be distinct primes, where $i = 1, 2, \ldots, r$. Suppose that $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is near-perfect with the redundant divisor d, namely, $\sigma(n) = 2n + d$. Note that for any $i = 1, 2, \ldots, r$,

$$\sigma(p_i^{\alpha_i}) = p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \cdots + 1, \ \gcd\left(p_i^{\alpha_i}, \sigma(p_i^{\alpha_i})\right) = 1,$$

and so

$$\sigma(n) = \sum_{d \mid n} d = \prod_{i=1}^{r} \sum_{j=0}^{\alpha_i} p_i^j = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}) = \sigma(p_i^{\alpha_i})\sigma(\frac{n}{p_i^{\alpha_i}}).$$

Therefore we have the following lemma.

Lemma 1.7. *The assumptions are as the above. Then for any $i = 1, 2, \dots, r$, we have $p_i \mid \sigma(n/p_i^{\alpha_i})$ if $p_i \mid d$, or $p_i \mid (\sigma(n/p_i^{\alpha_i}) - d)$ if $p_i \nmid d$.*

In particular, for an even near-perfect number n, from Lemma 1.7, we can get

Lemma 1.8. *Let r, α_i be positive integers, and let p_i be distinct primes $(1 \leqslant i \leqslant r)$. Suppose that $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is an even near-perfect number with the redundant divisor d. Then there exists some i with $1 \leqslant i \leqslant r$, such that α_i is odd if and only if $2 \mid d$.*

Proof. From $\sigma(n/2^{\alpha_0}) = \prod_{i=1}^{r} \sum_{j=0}^{\alpha_i} p_i^j$ and all p_i $(1 \leqslant i \leqslant r)$ are odd primes, we know that there exists some $i \in \{1, \dots, r\}$ such that α_i is odd if and only if $2 \mid \sigma(n/2^{\alpha_0})$.

On the one hand, from $2 \mid d$ and Lemma 1.7, we can get $2 \mid \sigma(n/2^{\alpha_0})$, which means that there exists some i with $1 \leqslant i \leqslant r$ such that α_i is odd.

On the other hand, suppose that there exists some i with $1 \leqslant i \leqslant r$ such that α_i is odd, then $2 \mid \sigma(n/2^{\alpha_0})$. In this case, if $2 \nmid d$, then by Lemma 1.7 we have $2 \mid \big(\sigma(n/2^{\alpha_0}) - d\big)$. Thus from $2 \mid \sigma(n/2^{\alpha_0})$, we have $2 \mid d$, which is a contradiction.

Thus we complete the proof of Lemma 1.8. □

Beyond that, in [8], the authors generalized near-perfect numbers to k-near-perfect numbers and proved the following proposition.

Proposition 1.9. *Let $k \geqslant 2$ and $\alpha + 1 > r_1 > r_2 > \cdots > r_k \geqslant 1$. Suppose that $p = 2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1$ is an odd prime number. Then $n = 2^{\alpha} p$ is a k-near-perfect number with k redundant divisors $2^{r_1}, \dots, 2^{r_k}$.*

In fact, some k-near-perfect numbers in Proposition 1.9 are also near-perfect. Using Proposition 1.5, one can deduce the following conclusion.

Corollary 1.10. *Let $k \geqslant 2$ and $\alpha + 1 > r_1 > r_2 > \cdots > r_k \geqslant 1$. Suppose that $p = 2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1$ is an odd prime number. Then the k-near-perfect number $n = 2^{\alpha} p$ is also near-perfect if and only if one of the following is true.*

(1) *k is a prime number, $\alpha = 2k - 1$ and $r_i = 2k - i$ $(1 \leqslant i \leqslant k)$.*

(2) *$n = 40$.*

Proof. From the definition of near-perfect numbers, the sufficiency is immediate.

Now we prove the necessity. Suppose that $n = 2^{\alpha}(2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1)$ is near-perfect, then from Proposition 1.5, there exists some α_0, such that $0 < \alpha_0 \leqslant \alpha$, and

$$2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1 = 2^{\alpha+1} - 2^{\alpha_0} - 1; \tag{I}$$

or there exists some prime q, such that $\alpha = 2q - 1$ and

$$2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1 = 2^{q} - 1; \tag{II}$$

or $\alpha = 3$ and

$$\text{(III)} \qquad 2^{\alpha+1} - 2^{r_1} - \cdots - 2^{r_k} - 1 = 5.$$

From (I), we know that

$$2^{\alpha_0 - r_k} = 2^{r_1 - r_k} + \cdots + 2^{r_{k-1} - r_k} + 1.$$

The left side is even. From $r_1 > r_2 > \cdots > r_k$ we know that the right side is odd. This is a contradiction.

From (II), we have

$$2^{q-r_k} = 2^{2q-r_k} - 2^{r_1-r_k} - \cdots - 2^{r_{k-1}-r_k} - 1 \geqslant 0,$$

thus $q \geqslant r_k$. From $2q = \alpha + 1 > r_1 > r_2 > \cdots > r_k$ we know that the right side is odd, and so $q = r_k$. Hence

$$1 = 2^{q-1} - 2^{r_1-q-1} - \cdots - 2^{r_{k-1}-q-1}.$$

Now from $r_{k-1} \geqslant r_k + 1 = q + 1$, the left side of the above equation is odd, and then $r_{k-1} = q+1$. Thus we can get $r_{k-2} = q+2, \ldots, r_1 = q+k-1$. Then taking $2q = \alpha+1$, $r_k = q$, $r_{k-1} = q+1, \ldots, r_1 = q+k-1$ to the equation (II), we have

$$2^{2q} - 2^{q+k-1} - \cdots - 2^{q+2} - 2^{q+1} - 2^q = 2^q,$$

i.e.,

$$2^{2q} - 2^q(2^k - 1) = 2^q,$$

thus $k = q,\ r_i = 2q - i\ (1 \leqslant i \leqslant k)$.

From (III), we have

$$2^3 - 2^{r_1-1} - \cdots - 2^{r_k-1} = 3,$$

thus $r_k = 1$, and

$$2^{r_1-1} + \cdots + 2^{r_{k-1}-1} = 4.$$

Note that $r_1 > r_2 > \cdots > r_k$, therefore $k - 1 = 1$, $r_1 = 3$. In this case $n = 40$.

Thus we complete the proof of Corollary 1.10. □

In the present paper, by Lemmas 1.7–1.8, we give two equivalent conditions of all even near-perfect numbers in the form $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$ $(p_1 < p_2)$ (Theorem 2.1 and Theorem 3.1). These near-perfect numbers involve some special kinds of prime number pairs. One kind is exactly the new Mersenne conjecture's prime number pair. Another kind has the form $p_1 = 2^{\alpha+1} - 1$ and $p_2 = \frac{p_1^2 + p_1 + 1}{3}$, where the former is a Mersenne prime and the latter's behavior is very much like a Fermat number.

2. Near-perfect numbers in the form $2^\alpha p_1p_2$

Suppose that $n = 2^\alpha p_1p_2$ is near-perfect, where $\alpha \geqslant 1$, and both p_1 and p_2 are odd primes with $p_1 < p_2$. From Lemma 1.8, taking $r = 2$ and $\alpha_1 = \alpha_2 = 1$, we have that the redundant divisor d is even. Hence $d = 2^\beta,\ 2^\beta p_1,\ 2^\beta p_2$ $(1 \leqslant \beta \leqslant \alpha)$, or $2^\beta p_1p_2$ $(1 \leqslant \beta \leqslant \alpha - 1)$. Thus we obtain an equivalent condition of all even near-perfect numbers in the form $2^\alpha p_1p_2$. In fact we have the following result.

Theorem 2.1. *Let α be a positive integer. Suppose that both p_1 and p_2 are odd primes with $p_1 < p_2$. Then $n = 2^\alpha p_1p_2$ is near-perfect if and only if one of the following conditions is true.*

(1) $p_1 = \frac{2^{\alpha+1}-1+k}{2^\beta+1-k}$, where $k = \frac{2^{\alpha+1}-1}{p_2}$ and $1 \leqslant \beta \leqslant \alpha - 1$. In this case, the redundant divisor is $2^\beta p_1p_2$.

(2) $p_1 = 2^{\alpha+1} - 1 + \frac{2^\alpha-2^{\beta-1}}{k}$, where k is determined by the equation $p_2 = (2^{\alpha+1} - 1)(2k + 1) - 2^\beta$, $1 \leqslant \beta \leqslant \alpha$. In this case, the redundant divisor is $2^\beta p_1$.

(3) $p_2 = 2^{\alpha+1} - 1 + \frac{2^{2\alpha+1}-2^\alpha-2^{\beta-1}}{k}$, where $k = \frac{p_1-(2^{\alpha+1}-1)}{2}$ and $1 \leqslant \beta \leqslant \alpha$. In this case, the redundant divisor is 2^β.

Proof. We first prove the sufficiency.

(1) From the assumption we know that

$$(2^\beta + 1)p_1 = (2^{\alpha+1} - 1) + k + p_1k,\ 2^{\alpha+1} - 1 = p_2k.$$

Note that $n = 2^\alpha p_1p_2$, hence

$$\begin{aligned}\sigma(n) - 2n &= (2^{\alpha+1} - 1)(p_1 + p_2 + 1) - p_1p_2\\ &= (2^{\alpha+1} - 1)\frac{2^\beta + 1}{k}p_1 - p_1\frac{2^{\alpha+1} - 1}{k}\\ &= 2^\beta p_1p_2.\end{aligned}$$

Note that $1 \leqslant \beta \leqslant \alpha - 1$, thus $2^\beta p_1p_2 \mid n$ and $2^\beta p_1p_2 \neq n$. Therefore from the definition of the near-perfect number, $n = 2^\alpha p_1p_2$ is near-perfect with the redundant divisor $2^\beta p_1p_2$.

The sufficiency proofs of (2) and (3) are similar.

Thus we complete the proof of the sufficiency.

Now we prove the necessity. Suppose that $n = 2^\alpha p_1p_2$ is near-perfect with the redundant divisor d.

First, we can conclude that $d \neq 2^\beta p_2$ $(1 \leqslant \beta \leqslant \alpha)$. Otherwise, from $d = 2^\beta p_2$ and Lemma 1.7, we have

$$p_2 \mid \sigma(2^\alpha p_1) = (2^{\alpha+1} - 1)(p_1 + 1).$$

Note that $p_1 < p_2$, and then gcd $(p_2, p_1 + 1) = 1$, thus

$$p_2 \mid (2^{\alpha+1} - 1).$$

Set $2^{\alpha+1} - 1 = kp_2$, then

$$2^\beta p_2 = d = \sigma(n) - 2n$$

$$\begin{aligned}&= (2^{\alpha+1}-1)(p_1+1)(p_2+1) - 2^{\alpha+1}p_1p_2\\&= kp_2(p_1+1)(p_2+1) - (kp_2+1)p_1p_2\\&= p_2[2^{\alpha+1}-1+(k-1)(p_1+1)+1].\end{aligned}$$

Thus

$$2^\beta = 2^{\alpha+1}-1+(k-1)(p_1+1)+1.$$

But from $1 \leqslant \beta \leqslant \alpha$, we know that

$$2^{\alpha+1}-1+(k-1)(p_1+1)+1 \geqslant 2^{\alpha+1} > 2^\beta,$$

which is a contradiction.

Therefore the redundant divisor d must be in the form

$$2^\beta,\ 2^\beta p_1\ (1 \leqslant \beta \leqslant \alpha) \text{ or } 2^\beta p_1p_2\ (1 \leqslant \beta \leqslant \alpha-1).$$

Now we prove the necessity of Theorem 2.1 according to the form of the redundant divisor d.

(1) Suppose that $d = 2^\beta p_1p_2$ $(1 \leqslant \beta \leqslant \alpha-1)$, then from Lemma 1.7,

$$p_2 \mid \sigma(2^\alpha p_1) = (2^{\alpha+1}-1)(p_1+1).$$

Note that gcd $(p_2, p_1+1) = 1$, so

$$p_2 \mid (2^{\alpha+1}-1).$$

Set $2^{\alpha+1}-1 = kp_2$, then

$$\begin{aligned}2^\beta p_1p_2 &= d = \sigma(n) - 2n\\&= kp_2(p_1+1)(p_2+1) - (kp_2+1)p_1p_2,\end{aligned}$$

namely,

$$(2^\beta+1)p_1 = k(p_1+p_2+1).$$

Therefore

$$p_1 = \frac{2^{\alpha+1}-1+k}{2^\beta+1-k}.$$

Thus we complete the proof of (1).

(2) Suppose that $d = 2^\beta p_1$ $(1 \leqslant \beta \leqslant \alpha)$, then from Lemma 1.7 we have

$$p_1 \mid \sigma(2^\alpha p_2) = (2^{\alpha+1}-1)(p_2+1),$$

thus

$$p_1 \mid (2^{\alpha+1}-1), \text{ or } \ p_1 \mid (p_2+1).$$

If $p_1 \mid (2^{\alpha+1}-1)$, we can set $2^{\alpha+1}-1 = kp_1$, then

$$\begin{aligned}2^\beta p_1 &= d = \sigma(n) - 2n\\&= kp_1(p_1+1)(p_2+1) - (kp_1+1)p_1p_2\\&= p_1[2^{\alpha+1}+(k-1)(p_2+1)],\end{aligned}$$

namely,

$$2^\beta = 2^{\alpha+1}+(k-1)(p_2+1).$$

Note that $1 \leqslant \beta \leqslant \alpha$, hence

$$2^{\alpha+1} + (k-1)(p_2+1) \geqslant 2^{\alpha+1} > 2^{\beta},$$

which is a contradiction.

Therefore $p_1 \nmid (2^{\alpha+1}-1)$, and so we must have $p_1 \mid (p_2+1)$. Set $p_2 = 2kp_1-1$, then from

$$\begin{aligned} 2^{\beta}p_1 = d &= \sigma(n) - 2n \\ &= (2^{\alpha+1}-1)(p_1+1)2kp_1 - 2^{\alpha+1}p_1(2kp_1-1) \\ &= 2p_1[k(2^{\alpha+1}-1-p_1) + 2^{\alpha}], \end{aligned}$$

we can get

$$p_1 = 2^{\alpha+1} - 1 + \frac{2^{\alpha} - 2^{\beta-1}}{k}.$$

Thus we complete the proof of (2).

(3) Suppose that $d = 2^{\beta}$ $(1 \leqslant \beta \leqslant \alpha)$, then $p_2 \nmid d$. And so from Lemma 1.7 we have

$$p_2 \mid \big(\sigma(2^{\alpha}p_1) - d\big) = (2^{\alpha+1}-1)(p_1+1) - 2^{\beta}.$$

Set $(2^{\alpha+1}-1)(p_1+1) - 2^{\beta} = 2kp_2$, i.e.,

$$2kp_2 + 2^{\beta} = (2^{\alpha+1}-1)(p_1+1). \tag{2.1}$$

Then

$$\begin{aligned} 2^{\beta} = d &= \sigma(n) - 2n \\ &= (2kp_2 + 2^{\beta})(p_2+1) - 2^{\alpha+1}p_1p_2 \\ &= p_2[2k(p_2+1) + 2^{\beta} - 2^{\alpha+1}p_1] + 2^{\beta}, \end{aligned}$$

hence

$$2k(p_2+1) + 2^{\beta} - 2^{\alpha+1}p_1 = 0.$$

And then by (2.1), we have

$$p_1 = 2^{\alpha+1} - 1 + 2k,\ p_2 = 2^{\alpha+1} - 1 + \frac{2^{2\alpha+1} - 2^{\alpha} - 2^{\beta-1}}{k}.$$

Thus we complete the proof of (3). □

By taking $\alpha = 2\beta - 5,\ k = 2^{\beta-2}+1$ in (1) of Theorem 2.1, and by supposing that both $p_1 = \frac{2^{\beta-2}+1}{3}$ and $p_2 = 2^{\beta-2}-1$ are odd primes, then one can get the near-perfect number $2^{2\beta-5}p_1p_2$ with the redundant divisor $2^{\beta}p_1p_2$. The primes p_1 and p_2 have the form of (3) and (2), respectively, in the following conjecture.

The New Mersenne Conjecture ([1])**.** *If two of the following statements about an odd prime number p are true, the third is also true.*

(1) $p = 2^k \pm 1$ *or* $p = 2^{2k} \pm 3$.

(2) $2^p - 1$ *is a prime number.*

(3) $\frac{2^p+1}{3}$ *is a prime number.*

Searching by computer (Setting $\alpha = 2\beta - 5,\ k = 2^{\beta-2} + 1$), for $\alpha \leqslant 1000$ one can get exactly 8 even near-perfect numbers which satisfy (1) of Theorem 2.1 as follows:

$$2^9 \cdot 11 \cdot 31, \quad 2^{13} \cdot 43 \cdot 127, \quad 2^{25} \cdot 2\,731 \cdot 8\,191, \quad 2^{33} \cdot 43\,691 \cdot 131\,071,$$
$$2^{37} \cdot 174\,763 \cdot 524\,287, \quad 2^{61} \cdot 715\,827\,883 \cdot 2\,147\,483\,647,$$
$$2^{121} \cdot 768\,614\,336\,404\,564\,651 \cdot 2\,305\,843\,009\,213\,693\,951,$$
$$\begin{aligned} 2^{253} \quad &\cdot \quad 56\,713\,727\,820\,156\,410\,577\,229\,101\,238\,628\,035\,243 \\ &\cdot \quad 170\,141\,183\,460\,469\,231\,731\,687\,303\,715\,884\,105\,727, \end{aligned}$$

and the corresponding $\beta = 7,\ 9,\ 15,\ 19,\ 21,\ 33,\ 63$ and 129.

For the case (2) of Theorem 2.1, when $1 \leqslant \alpha \leqslant 10$, there exist exactly 38 even near-perfect numbers; when $11 \leqslant \alpha \leqslant 20$, there exist exactly 148 even near-perfect numbers; when $21 \leqslant \alpha \leqslant 25$, there exist exactly 103 near-perfect numbers. In particular, if $3 \leqslant \alpha \leqslant 6$, then there exist exactly 7 even near-perfect numbers:

$$2^3 \cdot 17 \cdot 101, \quad 2^3 \cdot 19 \cdot 37, \quad 2^3 \cdot 17 \cdot 67, \quad 2^5 \cdot 67 \cdot 937,$$
$$2^5 \cdot 79 \cdot 157, \quad 2^5 \cdot 71 \cdot 283, \quad 2^6 \cdot 131 \cdot 3\,929,$$

and the corresponding $\beta = 2,\ 3,\ 3,\ 3,\ 5,\ 5$ and 3.

Finally, for the case (3) of Theorem 2.1, when $\alpha < 100$ and $k \leqslant 100$, there exist exactly 248 even near-perfect numbers. In particular, if $3 \leqslant \alpha \leqslant 5$, then there exist exactly 10 even near-perfect numbers:

$$2^3 \cdot 17 \cdot 131, \quad 2^3 \cdot 19 \cdot 73, \quad 2^4 \cdot 43 \cdot 113, \quad 2^5 \cdot 67 \cdot 1\,069, \quad 2^5 \cdot 103 \cdot 163,$$
$$2^5 \cdot 83 \cdot 263, \quad 2^5 \cdot 79 \cdot 313, \quad 2^5 \cdot 73 \cdot 463, \quad 2^5 \cdot 71 \cdot 563, \quad 2^5 \cdot 67 \cdot 1\,063,$$

and the corresponding $\beta = 3,\ 3,\ 3,\ 3,\ 5,\ 5,\ 5,\ 5,\ 5$ and 5.

3. Near-perfect numbers in the form $2^\alpha p_1^2 p_2$

Suppose that $n = 2^\alpha p_1^2 p_2$ is near-perfect, where $\alpha \geqslant 1$, and both p_1 and p_2 are odd primes with $p_1 < p_2$. From Lemma 1.8 the redundant divisor d is even, and so $d = 2^\beta,\ 2^\beta p_1,\ 2^\beta p_1^2,\ 2^\beta p_2,\ 2^\beta p_1 p_2$ $(1 \leqslant \beta \leqslant \alpha)$ or $2^\beta p_1^2 p_2$ $(1 \leqslant \beta \leqslant \alpha - 1)$. Thus we obtain an equivalent condition of all even near-perfect numbers in the form $2^\alpha p_1^2 p_2$ $(p_1 < p_2)$. In fact we have:

Theorem 3.1. *Let α be a positive integer. Suppose that both p_1 and p_2 are odd primes with $p_1 < p_2$. Then $n = 2^\alpha p_1^2 p_2$ is near-perfect if and only if one of the following is true.*

(1) *There exist some β and γ, such that $1 \leqslant \beta \leqslant \alpha$, $0 \leqslant \gamma \leqslant 2$, and the redundant divisor $d = 2^\beta p_1^\gamma = (2^{\alpha+1} - 1)(p_1^2 + p_1 + 1) - kp_2$, where $k = p_1^2 - (2^{\alpha+1} - 1)(p_1 + 1)$.*

(2) *There exists some β, such that $1 \leqslant \beta \leqslant \alpha$, and the redundant divisor $d = 2^\beta p_2 = (2^{\alpha+1} - 1)(p_2 + 1) - 2kp_1$, where $2k = p_1 p_2 - (2^{\alpha+1} - 1)(p_1 + p_2 + 1)$.*

(3) *There exist some* β *and* γ, *such that* $1 \leqslant \beta \leqslant \alpha$, $1 \leqslant \gamma \leqslant 2$, *and* $2^\beta p_1^\gamma = (2^{\alpha+1}-1)(p_1+1)+k-p_1^2$, *where* $k = \frac{(2^{\alpha+1}-1)(p_1^2+p_1+1)}{p_2}$. *In this case the redundant divisor* $d = 2^\beta p_1^\gamma p_2$.

Proof. First, we prove the sufficiency.

(1) From the assumption we have

$$\begin{aligned} kp_2 + d + k &= (2^{\alpha+1}-1)(p_1^2+p_1+1) + p_1^2 - (2^{\alpha+1}-1)(p_1+1) \\ &= 2^{\alpha+1}p_1^2. \end{aligned}$$

Note that $n = 2^\alpha p_1^2 p_2$, then

$$\begin{aligned} \sigma(n) - 2n &= (2^{\alpha+1}-1)(p_1^2+p_1+1)(p_2+1) - 2^{\alpha+1}p_1^2p_2 \\ &= (kp_2+d)(p_2+1) - 2^{\alpha+1}p_1^2p_2 \\ &= p_2[k(p_2+1)+d-2^{\alpha+1}p_1^2] + d \\ &= d, \end{aligned}$$

and $d = 2^\beta p_1^\gamma$ $(1 \leqslant \beta \leqslant \alpha,\ 0 \leqslant \gamma \leqslant 2)$, we can get $d \mid n$ and $d \neq n$. Thus from the definition we know that $n = 2^\alpha p_1^2 p_2$ is near-perfect with the redundant divisor d.

The sufficiency proofs of (2) and (3) are similar.

Now we prove the necessity. Suppose that $n = 2^\alpha p_1^2 p_2$ is near-perfect with the redundant divisor d.

(1) If $d = 2^\beta p_1^\gamma$ $(1 \leqslant \beta \leqslant \alpha,\ 0 \leqslant \gamma \leqslant 2)$, then $p_2 \nmid d$. By Lemma 1.7 we have

$$p_2 \mid (\sigma(2^\alpha p_1^2) - d) = (2^{\alpha+1}-1)(p_1^2+p_1+1) - d.$$

Set

$$(2^{\alpha+1}-1)(p_1^2+p_1+1) - d = kp_2. \tag{3.1}$$

Note that n is near-perfect with the redundant divisor d, therefore

$$\begin{aligned} d &= \sigma(n) - 2n \\ &= [(2^{\alpha+1}-1)(p_1^2+p_1+1) - d](p_2+1) + d(p_2+1) - 2^{\alpha+1}p_1^2p_2 \\ &= kp_2(p_2+1) + d(p_2+1) - 2^{\alpha+1}p_1^2p_2 \\ &= p_2[k(p_2+1)+d-2^{\alpha+1}p_1^2] + d, \end{aligned}$$

namely,

$$k(p_2+1) + d - 2^{\alpha+1}p_1^2 = 0,$$

thus

$$kp_2 = 2^{\alpha+1}p_1^2 - k - d.$$

By (3.1) we know that

$$2^{\alpha+1}p_1^2 - k = (2^{\alpha+1}-1)(p_1^2+p_1+1),$$

hence

$$k = p_1^2 - (2^{\alpha+1}-1)(p_1+1).$$

Thus we complete the proof of (1).

(2) If $d = 2^\beta p_2$ $(1 \leqslant \beta \leqslant \alpha)$, then $p_1 \nmid d$. From Lemma 1.7 we have

$$p_1 \mid (\sigma(2^\alpha p_2) - d) = (2^{\alpha+1} - 1)(p_2 + 1) - 2^\beta p_2.$$

Set

$$(2^{\alpha+1} - 1)(p_2 + 1) - 2^\beta p_2 = 2kp_1, \tag{3.2}$$

i.e.,

$$2^\beta p_2 = (2^{\alpha+1} - 1)(p_2 + 1) - 2kp_1.$$

Thus from

$$\begin{aligned} 2^\beta p_2 &= d \\ &= \sigma(n) - 2n \\ &= [(2^{\alpha+1} - 1)(p_2 + 1) - 2^\beta p_2](p_1^2 + p_1 + 1) + 2^\beta p_2(p_1^2 + p_1 + 1) \\ &\quad - 2^{\alpha+1} p_1^2 p_2 \\ &= 2kp_1(p_1^2 + p_1 + 1) + 2^\beta p_2 p_1(p_1 + 1) - 2^{\alpha+1} p_1^2 p_2 + 2^\beta p_2 \\ &= p_1[2k(p_1^2 + p_1 + 1) + 2^\beta p_2(p_1 + 1) - 2^{\alpha+1} p_1 p_2] + 2^\beta p_2, \end{aligned}$$

we know that

$$2k(p_1^2 + p_1 + 1) + 2^\beta p_2(p_1 + 1) - 2^{\alpha+1} p_1 p_2 = 0.$$

And then by (3.2),

$$[(2^{\alpha+1} - 1)(p_2 + 1) - 2^\beta p_2](p_1 + 1) + 2k + 2^\beta p_2(p_1 + 1) - 2^{\alpha+1} p_1 p_2 = 0,$$

hence

$$(2^{\alpha+1} - 1)(p_2 + 1)(p_1 + 1) + 2k - 2^{\alpha+1} p_1 p_2 = 0.$$

Therefore

$$(2^{\alpha+1} - 1)(p_1 + p_2 + 1) + 2k - p_1 p_2 = 0,$$

namely,

$$2k = p_1 p_2 - (2^{\alpha+1} - 1)(p_1 + p_2 + 1).$$

Thus we complete the proof of (2).

(3) If $d = 2^\beta p_1^\gamma p_2$ $(1 \leqslant \beta \leqslant \alpha, 1 \leqslant \gamma \leqslant 2)$, then $p_2 \mid d$. From Lemma 1.7 we have

$$p_2 \mid \sigma(2^\alpha p_1^2) = (2^{\alpha+1} - 1)(p_1^2 + p_1 + 1).$$

Set

$$(2^{\alpha+1} - 1)(p_1^2 + p_1 + 1) = kp_2. \tag{3.3}$$

Noting that

$$\begin{aligned} 2^\beta p_1^\gamma p_2 &= \sigma(n) - 2n \\ &= kp_2(p_2 + 1) - 2^{\alpha+1} p_1^2 p_2, \end{aligned}$$

we have

$$k(p_2 + 1) - 2^{\alpha+1} p_1^2 = 2^\beta p_1^\gamma.$$

Thus from (3.3) we know that

$$(2^{\alpha+1}-1)(p_1^2+p_1+1)=2^{\alpha+1}p_1^2-k+2^{\beta}p_1^{\gamma},$$

namely,

$$2^{\beta}p_1^{\gamma}=(2^{\alpha+1}-1)(p_1+1)+k-p_1^2.$$

Thus we complete the proof of (3). □

Corollary 3.2. *Let α be a positive integer, and let both $p_1=2^{\alpha+1}-1$ and $p_2=p_1^2+p_1+1$ be odd primes. Then $n=2^{\alpha}p_1^2p_2$ is near-perfect if and only if $n=2\cdot 3^2\cdot 13$.*

Proof. From the definition of near-perfect numbers, the sufficiency is clear.

Now we prove the necessity. First, from Lemma 1.8, the redundant divisor d of n must have the form

$$2^{\beta}p_1^{\gamma}\ (1\leqslant\beta\leqslant\alpha,\ 0\leqslant\gamma\leqslant 2),\ 2^{\beta}p_2\ (1\leqslant\beta\leqslant\alpha),$$

or

$$2^{\beta}p_1^{\gamma}p_2\ (1\leqslant\beta\leqslant\alpha,\ 1\leqslant\gamma\leqslant 2).$$

If $d=2^{\beta}p_1^{\gamma}$ $(1\leqslant\beta\leqslant\alpha,\ 0\leqslant\gamma\leqslant 2)$, then from (1) of Theorem 3.1 we have

$$d=2^{\beta}p_1^{\gamma}=(2^{\alpha+1}-1)(p_1^2+p_1+1)-kp_2.$$

Note that $p_1=2^{\alpha+1}-1$ and $p_2=p_1^2+p_1+1$, i.e.,

$$2^{\beta}p_1^{\gamma}=p_1p_2-kp_2=(p_1-k)p_2,$$

thus $p_2\mid 2^{\beta}p_1^{\gamma}$, which is a contradiction to the fact that gcd $(p_2,2p_1)=1$.

If $d=2^{\beta}p_2$ $(1\leqslant\beta\leqslant\alpha)$, then from (2) of Theorem 3.1 and the assumption, we have

$$d=2^{\beta}p_2=(2^{\alpha+1}-1)(p_2+1)-2kp_1=p_1(p_2+1-2k),$$

thus $p_1\mid 2^{\beta}p_2$, which is also a contradiction.

Therefore the redundant divisor d must have the form $2^{\beta}p_1^{\gamma}p_2$ $(1\leqslant\beta\leqslant\alpha,\ 1\leqslant\gamma\leqslant 2)$, namely, it satisfies (3) of Theorem 3.1. Thus we have

$$d=2^{\beta}p_1^{\gamma}p_2=[(2^{\alpha+1}-1)(p_1+1)+k-p_1^2]p_2,$$

and

$$kp_2=(2^{\alpha+1}-1)(p_1^2+p_1+1).$$

Note that $p_1=2^{\alpha+1}-1$ and $p_2=p_1^2+p_1+1$, so that

$$2^{\beta}p_1^{\gamma}p_2=[p_1(p_1+1)+k-p_1^2]p_2=(k+p_1)p_2,\ kp_2=p_1p_2,$$

i.e., $k=p_1$ and $2^{\beta}p_1^{\gamma}=2p_1$, which means that $\gamma=\beta=1$, and therefore $d=2p_1p_2$.

On the other hand, since $p_1=2^{\alpha+1}-1$ is an odd prime number, we have $\alpha+1$ is prime. Set $q=\alpha+1$, then

$$\begin{aligned}p_2&=p_1^2+p_1+1=(2^q-1)^2+2^q=2^{2q}-2^q+1\\&\equiv(-1)^{2q}-(-1)^q+1\pmod 3\end{aligned}$$

$$\equiv 2 - (-1)^q \pmod 3.$$

If q is an odd prime number, then $p_2 \equiv 2 + 1 \equiv 0 \pmod 3$, and thus $p_2 = 3$. This is a contradiction to the assumption $p_2 = p_1^2 + p_1 + 1 > 3$. Therefore $q = 2$, i.e.,

$$\alpha = 1,\ p_1 = 3,\ p_2 = 13,$$

which means that there exists a unique even near-perfect number $n = 2 \cdot 3^2 \cdot 13$, of the desired form, and the redundant divisor is $2 \cdot 3 \cdot 13$.

Thus we complete the proof of Corollary 3.2. □

In the case (1) of Theorem 3.1, by taking

$$\alpha = 1,\ p_1 = 5,\ p_2 = 13,\ k = 7,$$

one can get a near-perfect number $n = 2 \cdot 5^2 \cdot 13$ with the redundant divisor $d = 2$. Similarly, in (3) of Theorem 3.1, by taking

$$p_1 = 2^{\alpha+1} - 1,\ p_2 = \frac{p_1^2 + p_1 + 1}{3},\ k = 3(2^{\alpha+1} - 1),\ \beta = 2,\ \gamma = 1,$$

when p_1, p_2 are prime, one can get the near-perfect number

$$n = 2^\alpha (2^{\alpha+1} - 1)^2 \cdot \frac{2^{2(\alpha+1)} - 2^{\alpha+1} + 1}{3},$$

and the redundant divisor is

$$d = 2^2 (2^{\alpha+1} - 1) \cdot \frac{2^{2(\alpha+1)} - 2^{\alpha+1} + 1}{3}.$$

In particular, in (3) of Theorem 3.1, by taking $q = \alpha + 1 = 3, 5, 7, 13$, respectively, then both $p_1 = 2^q - 1$ and $p_2 = \frac{2^{2q} - 2^q + 1}{3}$ are odd primes. Thus one can get 4 near-perfect numbers as follows.

$$2^2 \cdot 7^2 \cdot 19, \quad 2^4 \cdot 31^2 \cdot 331, \quad 2^6 \cdot 127^2 \cdot 5\,419, \quad 2^{12} \cdot 8\,191^2 \cdot 22\,366\,891.$$

Furthermore, searching by computer, $2^{60} \cdot (2^{61} - 1)^2 \cdot \frac{2^{122} - 2^{61} + 1}{3}$ is near-perfect satisfying the case (3) of Theorem 3.1. A natural question is if there are any other near-perfect numbers in this form, namely, does there exist a pair of primes in the form $(p_1 = 2^{\alpha+1} - 1,\ p_2 = \frac{p_1^2 + p_1 + 1}{3})$?

It is easy to see that p_1 must be a Mersenne prime. In fact, taking p_1 as one of the first twelve Mersenne primes (except the first one), we find that p_2 has no square factor (except 1). This is very much like Fermat numbers.

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