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SUFFICIENT CONDITIONS FOR STARLIKENESS

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Abstract. We obtain the conditions on β so that $1+\beta zp'(z) \prec 1+4z/3+2z^2/3$ implies $p(z) \prec (2+z)/(2-z)$, $1+(1-\alpha)z$, $(1+(1-2\alpha)z)/(1-z)$, $(0 \leq \alpha < 1)$, $\exp(z)$ or $\sqrt{1+z}$. Similar results are obtained by considering the expressions $1+\beta zp'(z)/p(z)$, $1+\beta zp'(z)/p^2(z)$ and $p(z)+\beta zp'(z)/p(z)$. These results are applied to obtain sufficient conditions for normalized analytic function f to belong to various subclasses of starlike functions, or to satisfy the condition $|\log(zf'(z)/f(z))| < 1$ or $|(zf'(z)/f(z))^2-1| < 1$ or $zf'(z)/f(z)$ lying in the region bounded by the cardioid $(9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1)=0$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. An analytic function $p(z) = 1 + cz + \cdots$ is a function with a positive real part if $\operatorname{Re} p(z) > 0$. The class of all such functions is denoted by $\mathcal{P}$. For two functions f and g analytic in $\mathbb{D}$, f is *subordinate* to g, denoted by $f \prec g$, if there is an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Noticing that several subclasses of univalent functions are characterized by the quantities $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ lying in a region in the right-half plane, Ma and Minda [6] gave a unified presentation of various subclasses of convex and starlike functions. They considered analytic functions φ with positive real part in $\mathbb{D}$ that map the unit disc $\mathbb{D}$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$. Ma and Minda [6] introduced the following classes:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

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and

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

For special choices of φ, $\mathcal{S}^*(\varphi)$ reduces to well-known subclasses of starlike functions. For example, when $-1 \leq B < A \leq 1$, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1+Az)/(1+Bz))$ is the class of Janowski starlike function [4, 10] and $\mathcal{S}^*[1-2\alpha, -1]$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, introduced by Robertson [12] and $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions. Similarly, $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ is the subclass of $\mathcal{S}^*$ introduced by Sokól and Stankiewicz [18], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. More results regarding these classes can be found in [1, 3, 5, 11, 13, 16, 17]. Recently, Sharma *et al.* [14] introduced and studied the properties of the class

$$\mathcal{S}^*(1 + (4/3)z + (2/3)z^2) = \mathcal{S}^*_C.$$

Precisely, $f \in \mathcal{S}^*_C$ provided $zf'(z)/f(z)$ lies in the region bounded by the cardioid $(9x^2+9y^2-18x+5)^2-16(9x^2+9y^2-6x+1) = 0$. The class $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$, introduced recently by Mendiratta *et al.* [7], consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$.

Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$. Recently Ali *et al.* [2] determined the condition on β for $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p^n(z)$ with $n = 0, 1, 2$ or $(1-\beta)p(z) + \beta p^2(z) + \beta zp'(z)$ is subordinated to $\sqrt{1+z}$. Motivated by the works in [1, 2, 3, 9, 15, 17], in Section 2, we determine the sharp conditions on β so that $p(z) \prec (2+z)/(2-z)$ or $1 + (1-\alpha)z$ or $(1 + (1-2\alpha)z)/(1-z)$, $(0 \leq \alpha < 1)$ when $1 + \beta zp'(z) \prec 1 + 4z/3 + 2z^2/3$. Conditions on β so that $1 + \beta zp'(z)/p(z) \prec 1 + 4z/3 + 2z^2/3$ implies $p(z) \prec (1+z)/(1-z)$ or $1+z$ are also discussed. Conditions on β are derived so that the subordination $1+\beta zp'(z)/p^2(z) \prec 1+4z/3+2z^2/3$ implies $p(z) \prec (1+z)/(1-z)$ or $(2+z)/(2-z)$ or $1+z$. We also determine the conditions on β so that $p(z) \prec (1+z)/(1-z)$ or $1+4z/3+2z^2/3$, when $p(z)+\beta zp'(z)/p(z) \prec 1+4z/3+2z^2/3$. Section 3 of the paper investigates the sharp conditions on β so that $1+\beta zp'(z)/p^n(z) \prec 1+4z/3+2z^2/3$ $(n = 0, 1, 2)$ implies $p(z) \prec e^z$. Similarly, in Section 4, we consider differential implications with the superordinate function e^z replaced by the superordinate function $\sqrt{1+z}$. In addition to this, condition on β is determined so that $p(z) \prec \sqrt{1+z}$ when $p(z)+\beta zp'(z)/p(z) \prec 1+4z/3+2z^2/3$. In Section 5, we give applications of our results which will yield sufficient conditions for $f \in \mathcal{A}$ to belong to the various subclasses of starlike functions.

The following results will be required in our investigation.

Lemma 1.1 ([8, Corollary 3.4h, p. 135])**.** *Let q be univalent in $\mathbb{D}$, and let φ be analytic in a domain D containing $q(\mathbb{D})$. Let $zq'(z)\varphi(q(z))$ be starlike. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and satisfies $zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z))$, then $p \prec q$ and q is the best dominant.*

The following is a more general version of the above lemma.

Lemma 1.2 ([8, Theorem 3.4i, p. 134])**.** *Let q be univalent in $\mathbb{D}$ and let φ and ν be analytic in a domain D containing $q(\mathbb{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\varphi(q(z))$, $h(z) := \nu(q(z)) + Q(z)$. Suppose that* (i) *either h is convex or $Q(z)$ is starlike univalent in $\mathbb{D}$ and* (ii) $\operatorname{Re}(zh'(z)/Q(z)) > 0$ *for $z \in \mathbb{D}$. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and satisfies*

$$\nu(p(z)) + zp'(z)\varphi(p(z)) \prec \nu(q(z)) + zq'(z)\varphi(q(z)), \tag{1}$$

then $p \prec q$ and q is the best dominant.

Lemma 1.3 ([8, Corollary 3.4a, p. 120])**.** *Let q be analytic in $\mathbb{D}$ and ϕ be analytic in a domain D containing $q(\mathbb{D})$ and suppose* (i) $\operatorname{Re}\phi(q(z)) > 0$ *and either* (ii) *q is convex, or* (iii) *$Q(z) = zq'(z)\phi(q(z))$ is starlike. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and $p(z) + zp'(z)\phi(p(z)) \prec q(z)$, then $p \prec q$.*

2. Results associated with starlikeness

Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. In the first result, conditions on β are obtained so that the subordination

$$1 + \beta zp'(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}$$

implies $p(z) \prec (2 + z)/(2 - z)$ or $1 + (1 - \alpha)z$ or $(1 + (1 - 2\alpha)z)/(1 - z)$, $(0 \leq \alpha < 1)$.

Theorem 2.1. *Let $\beta_0 \approx 1.90987$ be the root of the equation $9 + 47\beta + 90\beta^2 - 216\beta^3 + 81\beta^4 = 0$. Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$1 + \beta zp'(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then the following sharp results hold:

(a) *If $\beta \leq -4.5$ or $\beta \geq \beta_0$, then $p(z) \prec (2 + z)/(2 - z)$.*
(b) *If $|\beta| \geq 2/(1 - \alpha), (0 \leq \alpha < 1)$, then $p(z) \prec 1 + (1 - \alpha)z$.*
(c) *If $\beta \leq -4/(1 - \alpha)$ or $\beta \geq 4/3(1 - \alpha)$, $(0 \leq \alpha < 1)$, then $p(z) \prec (1 + (1 - 2\alpha)z)/(1 - z)$.*

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ with $q(0) = 1$. Let us define $\varphi(w) = \beta$ and $Q(z) = zq'(z)\varphi(q(z))$. Since q is the convex univalent function, Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1, that the subordination

$$1 + \beta zp'(z) \prec 1 + \beta zq'(z)$$

implies $p(z) \prec q(z)$. The theorem is proved by computing β so that

$$1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta zq'(z) = 1 + \frac{\beta(A - B)z}{(1 + Bz)^2} := h(z). \tag{2}$$

Set $\psi(z) = 1 + 4z/3 + 2z^2/3$. Clearly, $\psi(\mathbb{D}) = \left\{w \in \mathbb{C} : |-2 + \sqrt{6w - 2}| < 2\right\}$. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$. Thus, by using the definition of h as given in (2), the subordination $\psi(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, we have

$$\left|\left(\sqrt{4+\frac{6\beta(A-B)e^{it}}{(1+Be^{it})^2}}-2\right)\right| \geq 2. \tag{3}$$

Set

$$w = u + iv = 4 + (6\beta(A-B)e^{it})/(1+Be^{it})^2. \tag{4}$$

Then, condition (3) holds if $|\sqrt{w}-2| \geq 2$ which is same as $|w| \geq 4\,\mathrm{Re}(\sqrt{w})$. On further simplification, we get

$$(u^2+v^2-8u)^2 - 64(u^2+v^2) \geq 0. \tag{5}$$

(a) Take $A = 1/2, B = -1/2$ in (4). Then

$$u = 4 + \frac{24\beta(5\cos t - 4)}{(5-4\cos t)^2}, \quad v = \frac{72\beta \sin t}{(5-4\cos t)^2}.$$

So, (5) reduces to

$$\frac{-768}{(5-4\cos t)^4}(1921 - 3712\beta + 2376\beta^2 - 432\beta^4 - 80(37-69\beta+36\beta^2)\cos t$$
$$+ 16(83-132\beta+36\beta^2)\cos 2t - 320\cos 3t + 320\beta\cos 3t + 32\cos 4t) \geq 0.$$

We need to find the values of β for which $f(x) \geq 0$ in the interval $-1 \leq x \leq 1$, where $x = \cos t$ and

$$\begin{aligned} f(x) = -(&1921 - 3712\beta + 2376\beta^2 - 432\beta^4 - 80(37-69\beta+36\beta^2)x \\ &+ 16(83-132\beta+36\beta^2)(2x^2-1) - 320(4x^3-3x) \\ &+ 320\beta(4x^3-3x) + 32(8x^4-8x^2+1)). \end{aligned}$$

A calculation shows that

$$f'(x) = -16(-5+4x)(25+16x^2-57\beta+36\beta^2+20x(-2+3\beta)) = 0$$

if $x = x_1 = 5/4$ or $x = x_2 = (10-15\beta-3\sqrt{-8\beta+9\beta^2})/8$ or $x = x_3 = (10-15\beta+3\sqrt{-8\beta+9\beta^2})/8$. Note that $-1 \leq x_2, x_3 \leq 1$ if and only if $\beta > 8/9$. These observations lead to two cases:

Case 1: $\beta > 8/9$. In this case, $f''(x_2) < 0$ and $f''(x_3) > 0$. Thus $f(x)$ attains its minimum value at $x = x_3$, it follows that $f(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if

$$f(x_3) = \frac{27\beta^2}{2}\left(24 + 153\beta^2 + 40\sqrt{-8\beta+9\beta^2} - 3\beta(68+15\sqrt{-8\beta+9\beta^2})\right) \geq 0,$$

which is possible if $\beta \geq \beta_0$. Hence $p(z) \prec q(z)$ if $\beta \geq \beta_0 \approx 1.90987$.

Case 2: $\beta \leq 8/9$. In this case, $f'(1) \geq 0$, $f'(-1) \geq 0$ and $f'(x)$ has no zero in $]-1,1[$. Hence by Intermediate Value Theorem, $f'(x) \geq 0$ for $-1 \leq x \leq 1$. Thus, $f(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if

$$f(-1) = 27(-3+2\beta)^3(9+2\beta) \geq 0,$$

which is possible if $\beta \leq -4.5$. Hence $p(z) \prec q(z)$ if $\beta \leq -4.5$. This completes the proof for part (a).

(b) Take $A = 1-\alpha$, $B = 0$, $(0 \leq \alpha < 1)$ in (4). Then

$$u = 4 + 6\beta(1-\alpha)\cos t, \quad v = 6\beta(1-\alpha)\sin t.$$

So, (5) takes the following form

$$g(t) := 48(27\beta^4(1-\alpha)^4 - 72\beta^2(1-\alpha)^2 - 16 - 64\beta(1-\alpha)\cos t) \geq 0.$$

We need to find all possible values of β for which $g(t)$ is non negative for $t \in [-\pi, \pi]$. Clearly, $g(t)$ attains its minimum value at $t = 0$ if $\beta > 0$ and $t = \pm\pi$ if $\beta < 0$. If $\beta > 0$, then $g(t) \geq 0$ if and only if

$$g(0) = 48(-2 + \beta(1-\alpha))(2 + 3\beta(1-\alpha))^3 \geq 0$$

which is true if $\beta \geq 2/(1-\alpha)$. Next if $\beta < 0$, then $g(t) \geq 0$ if and only if

$$g(\pi) = 48(2 + \beta(1-\alpha))(-2 + 3\beta(1-\alpha))^3 \geq 0$$

which is possible if $\beta \leq -2/(1-\alpha)$. Hence $p(z) \prec q(z)$ if $|\beta| \geq 2/(1-\alpha)$.

(c) Take $A = 1-2\alpha$, $B = -1$, $(0 \leq \alpha < 1)$ in (4). Then, we get

$$u = 4 - \frac{3\beta(1-\alpha)}{\sin^2 t/2}, \quad v = 0.$$

So, (5) reduces to

$$(u^2 - 8u)^2 - 64u^2 \geq 0,$$

which on further simplification becomes $u(u-16) \geq 0$ which implies that

$$(-4\sin^2 t/2 + 3\beta(1-\alpha))(\beta(1-\alpha) + 4\sin^2 t/2) \geq 0$$

which is possible if $\beta \geq 4/3(1-\alpha)$ or $\beta \leq -4/(1-\alpha)$. This completes the proof for (c). □

Next result depicts the conditions on β so that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}$$

implies $p(z) \prec (1+z)/(1-z)$ or $1+z$ where p is an analytic function in $\mathbb{D}$ with $p(0) = 1$.

Theorem 2.2. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then the following sharp results hold:

(a) *If* $|\beta| \geq \sqrt{(4\sqrt{3}+8)/(3\sqrt{3})} \simeq 1.6947$*, then* $p(z) \prec (1+z)/(1-z)$.

(b) *If* $\beta \geq 4$ *or* $\beta \leq -2$*, then* $p(z) \prec 1+z$.

Proof. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ with $q(0) = 1$. Let us define $\varphi(w) = \beta/w$ and $Q(z) = zq'(z)\varphi(q(z)) = \beta(A - B)z/((1 + Az)(1 + Bz))$. A computation shows that

$$\frac{zQ'(z)}{Q(z)} = \frac{1 - ABz^2}{(1 + Az)(1 + Bz)}.$$

Thus with $z = re^{it}$, $r \in (0, 1)$, $t \in [-\pi, \pi]$, yields

$$\operatorname{Re}\left(\frac{1 - ABz^2}{(1 + Az)(1 + Bz)}\right) = \frac{(1 - ABr^2)(1 + (A + B)r\cos t + ABr^2)}{|1 + Are^{it}|^2|1 + Bre^{it}|^2}.$$

Since $1 + ABr^2 + (A + B)r\cos t \geq (1 - Ar)(1 - Br) > 0$ for $A + B \geq 0$ and similarly, $1 + ABr^2 + (A + B)r\cos t \geq (1 + Ar)(1 + Br) > 0$ for $A + B \leq 0$, it follows that $Q(z)$ is starlike in $\mathbb{D}$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. Now our result is established if we prove

$$1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \frac{\beta(A - B)z}{(1 + Az)(1 + Bz)} := h(z). \tag{6}$$

Let $\psi(z) = 1 + 4z/3 + 2z^2/3$. Then $\psi(\mathbb{D}) = \left\{w \in \mathbb{C} : |-2 + \sqrt{6w - 2}| < 2\right\}$. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$. Thus, by using the definition of h as given in (6), the subordination $\psi(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, we have

$$\left|\left(\sqrt{4 + \frac{6\beta(A - B)e^{it}}{(1 + Ae^{it})(1 + Be^{it})}} - 2\right)\right| \geq 2.$$

Set

$$w = u + iv = 4 + (6\beta(A - B)e^{it})/((1 + Ae^{it})(1 + Be^{it})). \tag{7}$$

Then, proceeding as in Theorem 2.1, we have to deduce (5).

(a) Take $A = 1, B = -1$ in (7). Then $u = 4$ and $v = 6\beta/\sin t$. Substituting u and v in (5), we get

$$\left(\frac{36\beta^2}{\sin^2 t} - 16\right)^2 - 64\left(16 + \frac{36\beta^2}{\sin^2 t}\right) \geq 0.$$

Our problem is now to find all possible values of β for which $p(x) \geq 0$ for $x \in [-1, 1]$ where $x = \sin t$ and $p(x) = -16x^4 - 72x^2\beta^2 + 27\beta^4$. Clearly, $p(x) \geq -16 - 72\beta^2 + 27\beta^4 \geq 0$ if $|\beta| \geq \sqrt{(4\sqrt{3} + 8)/(3\sqrt{3})} \simeq 1.6947$.

(b) Take $A = 1, B = 0$ in (7). Then, $u = 4 + 3\beta$ and $v = 3\beta\tan t/2$. So, (5) becomes

$$-3\sec^4\frac{t}{2}(3(32+64\beta+48\beta^2-9\beta^4)+16(8+16\beta+9\beta^2)\cos t+32(1+2\beta)\cos 2t) \geq 0.$$

Now our problem is to find all values of β for which $g(x)$ is non negative in the whole interval $-1 \le x \le 1$ where $x = \cos t$ and

$$g(x) = -3(3(32+64\beta+48\beta^2-9\beta^4)+16(8+16\beta+9\beta^2)x+32(1+2\beta)(2x^2-1)).$$

A calculation shows that $g'(x) = 0$ if $x = x_0 = (-8-16\beta-9\beta^2)/(8(1+2\beta))$ and $g''(x) = -384(1+2\beta)$. Let us first assume that $\beta < -1/2$. In this case, $g''(x_0) > 0$. Thus, $\min g(x) = g(x_0) = 162\beta^4(2+\beta)/(1+2\beta)$. Hence, $g(x)$ is non negative if and only if $g(x_0)$ is non negative which is possible only if $\beta \le -2$. Let us next assume that $\beta \ge -1/2$. In this case, we get $g''(x) \le 0$ so that $g'(x) \le g'(-1) = -432\beta^2 \le 0$ and hence $g(x)$ is decreasing function. Therefore, $g(x) \ge 0$ if and only if $g(1) = 3(-4+\beta)(4+3\beta)^3 \ge 0$ which can happen only when $\beta \ge 4$. Hence we get our required result. □

In the next result, the conditions on β are derived so that the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\frac{4z}{3}+\frac{2z^2}{3}$$

implies $p(z) \prec (1+z)/(1-z)$ or $(2+z)/(2-z)$ or $1+z$ where p is an analytic function in $\mathbb{D}$ with $p(0) = 1$.

Theorem 2.3. *Let $\beta_0 \approx -1.90987$ be the smallest real root of $9-47\beta+90\beta^2+216\beta^3+81\beta^4 = 0$. Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then the following sharp results hold:

(a) *If $\beta \ge 4$ or $\beta \le -4/3$, then $p(z) \prec (1+z)/(1-z)$.*
(b) *If $\beta \ge 9/2$ or $\beta \le \beta_0$, then $p(z) \prec (2+z)/(2-z)$.*
(c) *If $\beta \ge 8$ or $\beta \le -8/3$, then $p(z) \prec 1+z$.*

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1+Az)/(1+Bz)$, $(-1 \le B < A \le 1)$ and consider the function $Q(z) = \beta zq'(z)/q^2(z) = \beta(A-B)z/(1+Az)^2$. Consider

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Az}{1+Az}.$$

Let $z = re^{it}, -\pi \le t \le \pi, 0 < r < 1$. Then

$$\operatorname{Re}\left(\frac{1-Az}{1+Az}\right) = \frac{1-A^2r^2}{|1+Are^{it}|^2} > 0.$$

Hence, Q is starlike in $\mathbb{D}$. Now it is easy to see that the subordination

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$ by Lemma 1.1. So our result will be proved if we can prove

$$\psi(z) := 1+\frac{4z}{3}+\frac{2z^2}{3} \prec 1+\beta\frac{zq'(z)}{q^2(z)} = 1+\frac{\beta(A-B)z}{(1+Az)^2} := h(z). \tag{8}$$

So, we only need to show that for $t \in [-\pi, \pi]$, the following condition holds

$$\left| \left(\sqrt{4 + \frac{6\beta(A-B)e^{it}}{(1+Ae^{it})^2}} - 2 \right) \right| \geq 2.$$

Let

$$w = u + iv = 4 + \frac{6\beta(A-B)e^{it}}{(1+Ae^{it})^2}. \tag{9}$$

Then, proceeding as in Theorem 2.1, we have to get (5).

(a) Take $A = 1, B = -1$ in (9). Then, $u = 4 + 3\beta \sec^2 t/2$ and $v = 0$. So, (5) reduces to $u(u-16) \geq 0$. Now, it is easy to see that our target is to find conditions on β such that $f(x) \geq 0$ for $-1 \leq x \leq 1$, where

$$x = \cos\frac{t}{2}, \quad f(x) = (4x^2 + 3\beta)(\beta - 4x^2).$$

Clearly, $f(x) \geq 0$ if $\beta \leq -4/3$ or $\beta \geq 4$.

(b) Take $A = 1/2, B = -1/2$ in (9). Then,

$$u = 4\left\{\frac{33 + 24\beta + 10(4+3\beta)\cos t + 8\cos 2t}{(5+4\cos t)^2}\right\}, \quad v = \frac{72\beta \sin t}{(5+4\cos t)^2}.$$

So, (5) reduces to

$$\frac{768}{(5+4\cos t)^4}(-1921 + 8\beta(-464 - 297\beta + 54\beta^3) - 80(37 + 69\beta + 36\beta^2)\cos t$$
$$- 16(83 + 12\beta(11+3\beta))\cos 2t - 320(1+\beta)\cos 3t - 32\cos 4t) \geq 0.$$

We need to find the values of β for which $g(x) \geq 0$ in the interval $-1 \leq x \leq 1$, where $x = \cos t$ and

$$g(x) = -(5+4x)^4 - 16(5+4x)^2(4+5x)\beta - 72(5+4x)^2\beta^2 + 432\beta^4.$$

A calculation shows that

$$g'(x) = -16(5+4x)((5+4x)^2 + 3(19+20x)\beta + 36\beta^2) = 0$$

if $x = x_1 = -5/4$ or $x = x_2 = (-10 - 15\beta - 3\sqrt{8\beta + 9\beta^2})/8$ or $x = x_3 = (-10 - 15\beta + 3\sqrt{8\beta + 9\beta^2})/8$. Note that x_2, x_3 are real numbers if and only if $\beta > 0$ or $\beta < -8/9$. These observations lead to three cases:

Case 1: $\beta < -8/9$. In this case, $g''(x_2) > 0$ and $g''(x_3) < 0$. Thus, $g(x)$ attains its minimum value at $x = x_2$, it follows that $g(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if

$$g(x_2) = \frac{27\beta^2}{2}\left(24 + 40\sqrt{8\beta + 9\beta^2} + 3\beta(68 + 51\beta + 15\sqrt{8\beta + 9\beta^2})\right) \geq 0,$$

which is possible if $\beta \leq -1.90987$.

Case 2: $\beta \geq 0$. In this case, we get $g''(x) \leq 0$ so that $g'(x) \leq g'(-1) = -16(1 - 3\beta + 36\beta^2) \leq 0$ and hence $g(x)$ is a decreasing function. Therefore, $g(x) \geq 0$ if and only if $g(1) = 27(-9+2\beta)(3+2\beta)^3 \geq 0$ which can happen only when $\beta \geq 9/2$.

Case 3: $-8/9 < \beta < 0$. In this case, $f'(1) < 0$, $f'(-1) < 0$ and $f'(x)$ has no zero in $]-1,1[$. Hence by Intermediate Value Theorem, $f'(x) < 0$ for $-1 \leq x \leq 1$. Thus $f(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if

$$f(1) = 27(3+2\beta)^3(-9+2\beta) \geq 0,$$

which is possible if $\beta \leq -3/2$ or $\beta \geq 9/2$. But this is not possible as $-8/9 < \beta < 0$. Hence, $p(z) \prec q(z)$ if $\beta \geq 9/2$ or $\beta \leq -1.90987$.

(c) Take $A = 1, B = 0$ in (9). Then,

$$u = 4 + \frac{3\beta}{2\cos^2 t/2}, \quad v = 0.$$

So, (5) reduces to $p(x) \geq 0$, $x \in [-1,1]$, where

$$x = \cos t, \quad p(x) = (-4+\beta-4x)(4+3\beta+4x)^3.$$

Clearly, $p'(x) < 0$. So, $p(x) \geq 0$ if and only if $p(1) = (-8+\beta)(8+3\beta)^3 \geq 0$ which is true if $\beta \geq 8$ or $\beta \leq -8/3$. Hence proved. □

In the following theorem, we find the conditions on β so that $p(z) \prec 1 + 4z/3 + 2z^2/3$, whenever

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}.$$

Theorem 2.4. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \quad \beta > 0.$$

Then $p(z) \prec 1 + 4z/3 + 2z^2/3$.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = 1 + 4z/3 + 2z^2/3$ with $q(0) = 1$. Let us define $\phi(w) = \beta/w$ $(\beta > 0)$. Consider

$$\operatorname{Re}\phi(q(z)) = \beta \operatorname{Re}\left(\frac{1}{q(z)}\right) > 0.$$

Next, define the function Q as

$$Q(z) := zq'(z)\phi(q(z)) = \frac{\beta zq'(z)}{q(z)} = \frac{4\beta z(1+z)}{3+4z+2z^2}.$$

From definition of Q, we have

$$\frac{zQ'(z)}{Q(z)} = \frac{3+6z+2z^2}{3+7z+6z^2+2z^3} =: K(z).$$

For $t \in [-\pi, \pi]$, we have

$$\operatorname{Re}(K(e^{it})) = \frac{1}{2} + \frac{5+4\cos t}{29+40\cos t+12\cos 2t}.$$

Now, we will find minimum value of $f(x)$ for $-1 \leq x \leq 1$, where

$$x = \cos t, \quad f(x) = \frac{5+4x}{29+40x+12(2x^2-1)}.$$

A calculation shows that $f'(x) = 0$ if $x = x_1 = -(5+\sqrt{3})/4$ or $x = x_2 = (-5+\sqrt{3})/4$. Note that $x_1 < -1$ and $f''(x_2) < 0$. Also note that $f(-1) = 1$ and $f(1) = 1/9$. So, $f(x)$, $-1 \leq x \leq 1$ attains its minimum value at $x = 1$. Hence, $\operatorname{Re}(K(e^{it})) \geq 11/18 > 0$, this shows that Q is starlike in $\mathbb{D}$. The result now follows from Lemma 1.3. □

We close this section by obtaining the conditions on β so that $p(z) \prec (1+z)/(1-z)$, whenever

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}.$$

Theorem 2.5. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \text{for} \quad \beta \geq 0.$$

Then $p(z) \prec (1+z)/(1-z)$.

Proof. For $\beta = 0$, result hold obviously. Let us assume that $\beta > 0$. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1+z)/(1-z)$. Also define $\nu(w) = w$ and $\varphi(w) = \beta/w$. Clearly, the functions ν and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Consider the functions Q and h defined as follows:

$$\begin{aligned} Q(z) &:= zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{2\beta z}{1-z^2} \quad \text{and} \\ h(z) &:= \nu(q(z)) + Q(z) = q(z) + Q(z). \end{aligned}$$

Since the mapping $z/(1-z^2)$ maps $\mathbb{D}$ onto the entire plane minus the two half lines $1/2 \leq y < \infty$ and $-\infty < y \leq -1/2$, $Q(z)$ is starlike univalent in $\mathbb{D}$. A computation shows that

$$\frac{zh'(z)}{Q(z)} = \frac{q(z)}{\beta} + \frac{zQ'(z)}{Q(z)} = \frac{1}{\beta}\left(\frac{1+z}{1-z}\right) + \frac{1+z^2}{1-z^2}.$$

Since, the mapping $zh'(z)/Q(z)$ maps $\mathbb{D}$ onto the plane $\operatorname{Re} w > 0$, all the conditions of Lemma 1.2 are fulfilled and hence it follows that $p(z) \prec q(z)$. In order to complete the proof, we need to show that

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec q(z) + \beta \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z} + \frac{2\beta z}{1-z^2} := h(z).$$

So, we only need to show that for $-\pi \leq t \leq \pi$, the following condition holds

$$\left| \left(\sqrt{-2 + \frac{12\beta e^{it}}{(1-e^{2it})} + \frac{6(1+e^{it})}{1-e^{it}}} - 2 \right) \right| \geq 2.$$

Set

$$w = u + iv = -2 + \frac{12\beta e^{it}}{(1-e^{2it})} + \frac{6(1+e^{it})}{1-e^{it}}$$

so that

$$u = -2 \quad \text{and} \quad v = \frac{6(1+\beta+\cos t)}{\sin t}.$$

Then, substituting the values of u and v in (5), we get

$$\frac{144}{(\sin t)^4}\left(4 + 3\beta(2+\beta) + 6(1+\beta)\cos t + 2\cos 2t\right)^2 \geq 0$$

which is possible for any β. Hence, $p(z) \prec q(z)$ if $\beta \geq 0$. □

3. Results associated with the function e^z

In this section, we compute the sharp conditions on β so that $p(z) \prec e^z$, whenever

$$1 + \beta z p'(z) \quad \text{or} \quad 1 + \beta\frac{zp'(z)}{p(z)} \quad \text{or} \quad 1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

where p is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$.

Theorem 3.1. *Let p be an analytic function defined on $\mathbb{D}$ and $p(0) = 1$. Let $\beta \geq 2e/3$ or $\beta \leq -2e$. If the function p satisfies the subordination*

$$1 + \beta z p'(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then p also satisfies the subordination $p(z) \prec e^z$. The result is sharp.

Proof. Let q be the convex univalent function defined by $q(z) = e^z$. Then clearly, $\beta z q'(z)$ is starlike in $\mathbb{D}$. If the subordination

$$1 + \beta z p'(z) \prec 1 + \beta z q'(z)$$

is satisfied, then $p(z) \prec q(z)$ by Lemma 1.1. It suffices to show that

$$1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta z q'(z) = 1 + \beta z e^z := h(z). \tag{10}$$

Set $\psi(z) = 1 + 4z/3 + 2z^2/3$. Clearly, $\psi(\mathbb{D}) = \left\{w \in \mathbb{C} : |-2 + \sqrt{6w-2}| < 2\right\}$. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$. Thus, by using the definition of h as given in (10), the subordination $\psi(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, we have

$$\left|\sqrt{4 + 6\beta e^{it} e^{e^{it}}} - 2\right| \geq 2. \tag{11}$$

Set $w = u + iv = 4 + 6\beta e^{it} e^{e^{it}}$. Then, we only need to show that $|\sqrt{w} - 2| \geq 2$ which is same as $|w| \geq 4\,\mathrm{Re}(\sqrt{w})$. On further simplification, we get

$$(u^2 + v^2 - 8u)^2 - 64(u^2 + v^2) \geq 0. \tag{12}$$

Clearly, $u = 4+6\beta e^{\cos t}\cos(t+\sin t)$ and $v = 6\beta e^{\cos t}\sin(t+\sin t)$. Our problem is now to find all possible values of β for which $f(t) \geq 0$ for $t \in [-\pi, \pi]$, where

$$f(t) = -16 - 72\beta^2 e^{2\cos t} + 27\beta^4 e^{4\cos t} - 64\beta e^{\cos t}\cos(t + \sin t).$$

Since $f(t)$ is an even function of t. It suffices to find the condition on β for which $f(t) \geq 0$ for $t \in [0, \pi]$. Note that

$$f(0) = (-2 + e\beta)(2 + 3e\beta)^3 \quad \text{and} \quad f(\pi) = \frac{-(2e - 3\beta)^3(2e + \beta)}{e^4}.$$

So, $f(0) \geq 0$ and $f(\pi) \geq 0$ if $\beta \leq -2e$ or $\beta \geq 2e/3$. If $\beta \leq -2e$ or $\beta \geq 2e/3$, then f is a decreasing function of t and since $f(\pi) \geq 0$, we conclude that $f(t) \geq 0$ for $t \in [0, \pi]$ if $\beta \leq -2e$ or $\beta \geq 2e/3$. □

Theorem 3.2. *If p is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying the subordination*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \textit{for} \quad |\beta| \geq 2$$

then p also satisfies the subordination $p(z) \prec e^z$. The result is sharp.

Proof. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = e^z$. Let us define $\varphi(w) = \beta/w$ and $Q(z) = zq'(z)\varphi(q(z)) = \beta z$. Clearly, $Q(z)$ is starlike in $\mathbb{D}$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. Now, our result is established if we prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \beta z := h(z).$$

Since the subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$, we only need to show that for $t \in [-\pi, \pi]$,

$$\left|\sqrt{4 + 6\beta e^{it}} - 2\right| \geq 2.$$

Set $w = u + iv = 4 + 6\beta e^{it}$ so that $u = 4 + 6\beta\cos t$ and $v = 6\beta\sin t$. Then, proceeding as in Theorem 3.1, we need to show that (12) holds. After substituting the values of u and v in (12), we need to find the values of β for which $g(t) \geq 0$ for $t \in [-\pi, \pi]$, where

$$g(t) = -16 - 72\beta^2 + 27\beta^4 - 64\beta\cos t.$$

Note that $g(t)$ is an even function of t. So, we only need to consider $g(t)$ for $t \in [0, \pi]$. Also note that $g'(t) = 64\beta\sin t$. Let us first assume that $\beta > 0$. In this case, $g(t)$ is an increasing function. Therefore, $g(t) \geq 0$ if and only if $g(0) = (-2 + \beta)(2 + 3\beta)^3 \geq 0$ which can happen only when $\beta \geq 2$. Let us next assume that $\beta < 0$. In this case, $g(t)$ being decreasing function, is non negative if and only if $g(\pi) = (2+\beta)(-2+3\beta)^3$ is non negative which is possible if $\beta \leq -2$. Hence, $p(z) \prec q(z)$ if $|\beta| \geq 2$. □

Theorem 3.3. *Let p be an analytic function defined on $\mathbb{D}$ and $p(0) = 1$. Let $\beta \geq 2e$ or $\beta \leq -2e/3$. If the function p satisfies the subordination*

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then $p(z) \prec e^z$. The result is sharp.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = e^z$ and consider the function $Q(z) = \beta zq'(z)/q^2(z) = \beta ze^{-z}$. For $z = x + iy \in \mathbb{D}$, we have

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = \operatorname{Re}(1-z) = 1 - x > 0.$$

Hence, Q is starlike in $\mathbb{D}$. Now, it is easy to see that by Lemma 1.1, the subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. So, our result will be proved if we can prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \beta ze^{-z} := h(z).$$

Thus, we only need to show that $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left|\sqrt{4 + 6\beta e^{it} e^{-e^{it}}} - 2\right| \geq 2.$$

Set $w = u + iv = 4 + 6\beta e^{it} e^{e^{i(t+\pi)}}$. Then, proceeding as in Theorem 3.1, we need to prove (12). Clearly, $u = 4 + 6\beta e^{-\cos t} \cos(t - \sin t)$ and $v = 6\beta e^{-\cos t} \sin(t - \sin t)$. Our problem reduces to find all possible values of β for which $k(t)$ is non negative in $[-\pi, \pi]$, where

$$k(t) = -16 - 72\beta^2 e^{-2\cos t} + 27\beta^4 e^{-4\cos t} - 64\beta e^{-\cos t} \cos(t - \sin t).$$

Observe that $k(-t) = k(t)$ for $t \in [-\pi, \pi]$. Thus, it is sufficient to find the values of β for which $k(t)$ is non negative in $[0, \pi]$. Note that

$$k(0) = \frac{(-2e+\beta)(2e+3\beta)^3}{e^4} \quad \text{and} \quad k(\pi) = (2 + e\beta)(-2 + 3e\beta)^3.$$

Clearly, $k(0)$ and $k(\pi)$ both are non negative if $\beta \leq -2e/3$ or $\beta \geq 2e$. Also, if $\beta \leq -2e/3$ or $\beta \geq 2e$, then k is an increasing function of t and $k(0)$ is non negative. Hence, $k(t) \geq 0$ for $t \in [0, \pi]$ if $\beta \leq -2e/3$ or $\beta \geq 2e$. □

4. Results associated with the lemniscate of Bernoulli

In this section, we compute the conditions on β so that $p(z) \prec \sqrt{1+z}$, whenever

$$1+\beta\frac{zp'(z)}{p^k(z)}\,(k=0,1,2) \quad \text{or} \quad p(z)+\beta\frac{zp'(z)}{p(z)} \prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

where p is an analytic function defined on $\mathbb{D}$ with $p(0)=1$.

Theorem 4.1. *Let $\beta \geq 4\sqrt{2}$. Let p be an analytic function defined on $\mathbb{D}$ with $p(0)=1$ satisfying*

$$1+\beta zp'(z) \prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $p(z) \prec \sqrt{1+z}$. The result obtained is sharp.

Proof. Define the function $q:\mathbb{D}\to\mathbb{C}$ by $q(z)=\sqrt{1+z}$ with $q(0)=1$. Since $q(\mathbb{D})=\{w:|w^2-1|<1\}$ is the right half of the lemniscate of Bernoulli, $q(\mathbb{D})$ is a convex set and hence q is convex and $zq'(z)$ is starlike in $\mathbb{D}$. It follows from Lemma 1.1, that the subordination

$$1+\beta zp'(z) \prec 1+\beta zq'(z)$$

implies $p(z) \prec q(z)$. Now, our result is established if we prove the following:

$$\psi(z) := 1+\frac{4z}{3}+\frac{2z^2}{3} \prec 1+\beta zq'(z) = 1+\frac{\beta z}{2\sqrt{1+z}} := h(z).$$

Now, proceeding as in earlier sections, it is enough to show that $\partial h(\mathbb{D}) \subset \mathbb{C}\setminus\overline{\psi(\mathbb{D})}$ which is equivalent to show that for $t\in[-\pi,\pi]$,

$$\left|\sqrt{4+\frac{3\beta e^{it}}{\sqrt{1+e^{it}}}}-2\right| \geq 2.$$

Taking $w=u+iv=4+3\beta e^{it}/(\sqrt{1+e^{it}})$. Then, we only need to show that

$$(u^2+v^2-8u)^2-64(u^2+v^2)\geq 0. \tag{13}$$

A calculation shows that

$$u=4+\frac{3\beta\cos(3t/4)}{\sqrt{2\cos t/2}} \quad \text{and} \quad v=\frac{3\beta\sin(3t/4)}{\sqrt{2\cos t/2}}.$$

Using these values in (13), our problem reduces to find all possible values of β for which $f(t)\geq 0$ for $t\in[-\pi,\pi]$, where

$$f(t) = -\frac{3}{4}\Big(512-27\beta^4+512\cos t \\ +64\beta\big(9\beta\cos(t/2)+16\sqrt{2}\cos^{3/2}(t/2)\cos(3t/4)\big)\Big).$$

Note that $f(t) = f(-t)$ for any t, so it is sufficient to consider the interval $0 \leq t \leq \pi$. Also note that $f'(t) \geq 0$ for $\beta > 0$, so $f(t)$ attains minimum value at $t = 0$. Clearly,

$$f(0) = \frac{-3}{4}(1024 + 1024\sqrt{2}\beta + 576\beta^2 - 27\beta^4) \geq 0 \quad \text{for} \quad \beta \geq 4\sqrt{2}.$$

Thus, $f(t) \geq 0$ if $\beta \geq 4\sqrt{2}$. This completes the proof. □

Theorem 4.2. *Let $\beta \leq -4$ or $\beta \geq 8$. Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then $p(z) \prec \sqrt{1+z}$. The result obtained is sharp.

Proof. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = \sqrt{1+z}$ with $q(0) = 1$. Let us define $\varphi(w) = \beta/w$ and $Q(z) = zq'(z)\varphi(q(z)) = \beta z/2(1+z)$ which maps $\mathbb{D}$ onto $\operatorname{Re} w < \beta/4$. So, $Q(z)$ is starlike in $\mathbb{D}$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq'(z)}{q(z)}$$

implies $p(z) \prec q(z)$. Now, our result is established if we prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta\frac{zq'(z)}{q(z)} = 1 + \frac{\beta z}{2(1+z)} := h(z). \tag{14}$$

Hence, we only need to show that $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$ which is same as to show that for $t \in [-\pi, \pi]$,

$$\left| \sqrt{4 + \frac{3\beta e^{it}}{1+e^{it}}} - 2 \right| \geq 2.$$

Set $w = u + iv = 4 + 3\beta e^{it}/(1+e^{it})$. Then, proceeding as in Theorem 4.1, our target is to prove (13). Clearly,

$$u = 4 + \frac{3\beta}{2} \quad \text{and} \quad v = \frac{3\beta}{2}\tan\frac{t}{2}.$$

On substituting u and v in (13), we get

$$\frac{1}{16}\left(-64 + 9\beta^2 + 9\beta^2\left(\frac{1-x^2}{x^2}\right)\right)^2 - 16\left((8+3\beta)^2 + 9\beta^2\left(\frac{1-x^2}{x^2}\right)\right) \geq 0,$$

where $x = \cos t/2$. So, our problem reduces to find the values of β for which $G(x) \geq 0$ for $x \in [0,1]$, where

$$G(x) = -12288(1+\beta)x^4 - 3456\beta^2x^2 + 81\beta^4.$$

A calculation shows that

$$G'(x) = -768(9x\beta^2 + 64x^3(1+\beta))$$

and hence $G'(0) = G'(\pm 3\beta/(8\sqrt{-1-\beta})) = 0$. Let us first assume that $\beta \geq -1$. Then, $G(x)$ is a decreasing function of $x \in [0,1]$. Consequently, we have $G(x) \geq 0$ for $x \in [0,1]$ provided $G(1) = 3(-8+\beta)(8+3\beta)^3 \geq 0$, which is equivalent to $\beta \geq 8$. Next, assume that $\beta < -1$. In this case, $G''(-3\beta/(8\sqrt{-1-\beta})) = 13824\beta^2 > 0$. Thus $G(x)$ attains its minimum value at $x = -3\beta/(8\sqrt{-1-\beta})$, it follows that $G(x) \geq 0$ for $0 \leq x \leq 1$ if and only if

$$G(-3\beta/(8\sqrt{-1-\beta})) = \frac{81\beta^4(4+\beta)}{1+\beta} \geq 0,$$

provided $\beta \leq -4$. Hence, $p(z) \prec q(z)$ for $\beta \leq -4$ or $\beta \geq 8$. □

Theorem 4.3. *Let p be an analytic function defined on $\mathbb{D}$ and $p(0) = 1$. If the function p satisfies the subordination*

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \quad \textit{for} \quad \beta \geq 8\sqrt{2}$$

then $p(z) \prec \sqrt{1+z}$. The result is sharp.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$ and consider the function $Q(z) = \beta zq'(z)/q^2(z) = \beta z/2(1+z)^{3/2}$. Clearly,

$$\frac{zQ'(z)}{Q(z)} = 1 - \frac{3z}{2(1+z)}$$

which maps $\mathbb{D}$ onto plane $\operatorname{Re} w > 1/4$. Hence, Q is starlike in $\mathbb{D}$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. So, our result will be proved if we can prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta\frac{zq'(z)}{q^2(z)} = 1 + \beta\frac{z}{2(1+z)^{3/2}} := h(z).$$

So, we only need to show that $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left|\sqrt{4 + \frac{3\beta e^{it}}{(1+e^{it})^{3/2}}} - 2\right| \geq 2.$$

Set $w = u + iv = 4 + (3\beta e^{it})/(1+e^{it})^{3/2}$. Then, proceeding as in Theorem 4.1, we have to find β so that (13) holds. Clearly,

$$u = 4 + 3\beta\frac{\cos t/4}{(2\cos t/2)^{3/2}}, \quad v = 3\beta\frac{\sin t/4}{(2\cos t/2)^{3/2}}.$$

Our problem reduces to find all possible values of β for which $k(t)$ is non negative in $[-\pi, \pi]$, where

$$k(t) = \frac{3}{64}\left\{-16384 - 8192\sqrt{2}\beta\cos\frac{t}{4}\sec^{3/2}\frac{t}{2} - 2304\beta^2\sec^3\frac{t}{2} + 27\beta^4\sec^6\frac{t}{2}\right\}.$$

Observe that $k(-t) = k(t)$ for $t \in [-\pi, \pi]$. Thus, it is sufficient to find the values of β for which $k(t)$ is non negative in $[0, \pi]$. For $\beta \geq 8\sqrt{2}$, k is an increasing function of t and $k(0) = -768 - 384\sqrt{2}\beta - 108\beta^2 + 81\beta^4/64$ is non negative. Hence, $k(t) \geq 0$, $t \in [0, \pi]$ for $\beta \geq 8\sqrt{2}$. □

Theorem 4.4. *Let p be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \textit{for} \quad \beta \geq 12$$

then $p(z) \prec \sqrt{1+z}$.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$. Consider the subordination

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec q(z) + \beta \frac{zq'(z)}{q(z)}.$$

Thus, in view of Lemma 1.2, the above subordination can be written as (1) by defining the functions ν and φ as

$$\nu(w) = w \quad \text{and} \quad \varphi(w) = \beta/w, (\beta \neq 0).$$

Clearly, the functions ν and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined as follows:

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta zq'(z)}{q(z)} = \frac{\beta z}{2(1+z)} \quad \text{and}$$
$$h(z) := \nu(q(z)) + Q(z) = \sqrt{1+z} + \frac{\beta z}{2(1+z)}.$$

Since the mapping $Q(z)$ maps $\mathbb{D}$ onto the plane $\operatorname{Re} w < \beta/4$, $Q(z)$ is starlike univalent in $\mathbb{D}$. A computation shows that

$$\frac{zh'(z)}{Q(z)} = \frac{\sqrt{1+z}}{\beta} + \frac{1}{1+z}.$$

Now, the mapping $1/(1+z)$ maps $\mathbb{D}$ onto plane $\operatorname{Re} w > 1/2$ and $\operatorname{Re}(\sqrt{1+z}) > 0, z \in \mathbb{D}$. Therefore, $\operatorname{Re}(zh'(z)/Q(z)) > 0$, $z \in \mathbb{D}$ if $\beta > 0$. Thus, all the conditions of Lemma 1.2 are satisfied and hence, it follows that $p(z) \prec q(z)$. In order to complete the proof, we need to prove that

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec q(z) + \beta \frac{zq'(z)}{q(z)} = \sqrt{1+z} + \frac{\beta z}{2(1+z)} = h(z).$$

So, we only need to show that $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})}$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left| \sqrt{-2 + 6\sqrt{1+e^{it}} + \frac{3\beta e^{it}}{1+e^{it}}} - 2 \right| \geq 2.$$

Thus, we have to show that

$$\left|-2+6\sqrt{1+e^{it}}+\frac{3\beta e^{it}}{1+e^{it}}\right|\geq 16.$$

Now,

$$\begin{aligned}\left|-2+6\sqrt{1+e^{it}}+\frac{3\beta e^{it}}{1+e^{it}}\right| &= \left|6e^{it/4}\sqrt{2\cos\frac{t}{2}}+\frac{3\beta e^{it/2}}{2\cos\frac{t}{2}}-2\right| \\ &\geq \operatorname{Re}\left(6e^{it/4}\sqrt{2\cos\frac{t}{2}}+\frac{3\beta e^{it/2}}{2\cos\frac{t}{2}}-2\right) \\ &= 6\cos\frac{t}{4}\sqrt{2\cos\frac{t}{2}}+\frac{3\beta}{2}-2 \\ &\geq \frac{3\beta}{2}-2\geq 16 \quad \text{for} \quad \beta\geq 12.\end{aligned}$$

Hence, $p(z)\prec q(z)$ and this completes the proof. □

5. Applications

In this section we give sufficient conditions for functions $f\in\mathcal{A}$ to belong to the various subclasses of starlike functions.

Theorem 5.1. *Let $f\in\mathcal{A}$ and $\beta_0=\sqrt{(4\sqrt{3}+8)/(3\sqrt{3})}\simeq 1.6947$. Then following are the sufficient conditions for $f\in\mathcal{S}^*$.*

(1) *The function f satisfies the subordination*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (|\beta|\geq\beta_0).$$

(2) *The function f satisfies the subordination*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta\leq -4/3 \quad \textit{or} \quad \beta\geq 4).$$

(3) *The function f satisfies the subordination*

$$\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta\geq 0).$$

Proof. Let the function $p:\mathbb{D}\to\mathbb{C}$ be defined by $p(z)=zf'(z)/f(z)$. Then p is analytic in $\mathbb{D}$ with $p(0)=1$. A calculation shows that

$$\frac{zp'(z)}{p(z)}=1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}.$$

The results follow respectively from Theorems 2.2(a), 2.3(a) and 2.5. □

Theorem 5.2. *Let $f\in\mathcal{A}$ and $\beta_0=\sqrt{(4\sqrt{3}+8)/(3\sqrt{3})}\simeq 1.6947$. Then following are the sufficient conditions for $z^2f'(z)/f^2(z)\in\mathcal{P}$.*

(1) *The function* f *satisfies the subordination*

$$1+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3}\quad (|\beta|\geq\beta_0).$$

(2) *The function* f *satisfies the subordination*

$$\frac{z^2f'(z)}{f^2(z)}+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3}\quad (\beta\geq 0).$$

Proof. The two parts of the theorem follows by taking $p(z)=z^2f'(z)/f^2(z)$ in Theorems 2.2(a) and 2.5 respectively. □

Theorem 5.3. *Let* $f\in\mathcal{A}$ *and* $0\leq\alpha<1$.

(1) *Let* $\beta\leq -4/(1-\alpha)$ *or* $\beta\geq 4/3(1-\alpha)$. *If the function* f *satisfies the subordination*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*(\alpha)$.

(2) *Let* $\beta\leq -9/2$ *or* $\beta\geq\beta_0$, *where* β_0 *is given by Theorem 2.1. If the function* f *satisfies the subordination*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*[1/2,-1/2]$.

(3) *Let* $\beta\leq\beta_0$ *or* $\beta\geq 9/2$, *where* β_0 *is given by Theorem 2.3. If the function* f *satisfies the subordination*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*[1/2,-1/2]$.

(4) *Let* $|\beta|\geq 2/(1-\alpha)$. *If the function* f *satisfies the subordination*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*[1-\alpha,0]$

(5) *Let* $\beta\leq -2$ *or* $\beta\geq 4$. *If the function* f *satisfies the subordination*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*[1,0]$.

(6) *Let* $\beta\leq -8/3$ *or* $\beta\geq 8$. *If the function* f *satisfies the subordination*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1+\frac{4z}{3}+\frac{2z^2}{3},$$

then $f\in\mathcal{S}^*[1,0]$.

Proof. The parts of the theorem are obtained by taking $p(z) = zf'(z)/f(z)$ in Theorems 2.1(c), 2.1(a), 2.3(b), 2.1(b), 2.2(b) and 2.3(c) respectively. ☐

Theorem 5.4. *Let $f \in \mathcal{A}$ and $0 \le \alpha < 1$.*

(1) *If f satisfies $1 + \beta z f''(z) \prec 1 + 4z/3 + 2z^2/3$ ($\beta \le -4/(1-\alpha)$ or $\beta \ge 4/3(1-\alpha)$), then $f' \prec (1 + (1-2\alpha)z)/(1-z)$.*
(2) *If f satisfies $1 + \beta z f''(z) \prec 1 + 4z/3 + 2z^2/3$ ($\beta \le -9/2$ or $\beta \ge \beta_0$, where β_0 is given by Theorem 2.1), then $f' \prec (2+z)/(2-z)$.*
(3) *If f satisfies $1 + \beta z f''(z) \prec 1 + 4z/3 + 2z^2/3$ ($|\beta| \ge 2/(1-\alpha)$), then $f' \prec 1 + (1-\alpha)z$.*
(4) *If f satisfies*

$$1 + \beta \left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \le -2 \quad \text{or} \quad \beta \ge 4),$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + z.$$

Proof. The first three parts follows from Theorems 2.1(c), 2.1(a) and 2.1(b) respectively by taking $p(z) = f'(z)$. Next, applying Theorem 2.2(b) to the function $p(z) = z^2 f'(z)/f^2(z)$ yields the last part of the theorem. ☐

Next theorem is an application of Theorem 2.4.

Theorem 5.5. *Let $f \in \mathcal{A}$ and $\beta > 0$.*

(1) *If f satisfies the subordination*

$$\frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then $f \in \mathcal{S}^*_C$.

(2) *If f satisfies*

$$\frac{z^2 f'(z)}{f^2(z)} + \beta \left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}.$$

The three parts of the next theorem are application of Theorems 3.1, 3.2 and 3.3 respectively.

Theorem 5.6. *Let $f \in \mathcal{A}$. Then following are the sufficient conditions for $f \in \mathcal{S}^*_e$.*

(1) *Let $\beta \le -2e$ or $\beta \ge 2e/3$. The function f satisfies the subordination*

$$1 + \beta \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}.$$

(2) *Let* $|\beta| \geq 2$. *The function* f *satisfies the subordination*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+\frac{4z}{3}+\frac{2z^2}{3}.$$

(3) *Let* $\beta \leq -2e/3$ *or* $\beta \geq 2e$. *The function* f *satisfies the subordination*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1+\frac{4z}{3}+\frac{2z^2}{3}.$$

The two parts of the next theorem are application of Theorems 3.1 and 3.2 respectively.

Theorem 5.7. *Let* $f \in \mathcal{A}$.

(1) *If* f *satisfies* $1+\beta zf''(z) \prec 1+4z/3+2z^2/3$ $(\beta \leq -2e$ *or* $\beta \geq 2e/3)$, *then* $f' \prec e^z$.

(2) *If* f *satisfies*

$$1+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right) \prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (|\beta| \geq 2),$$

then

$$\frac{z^2f'(z)}{f^2(z)} \prec e^z.$$

The remaining results are application of Section 4.

Theorem 5.8. *Let* $f \in \mathcal{A}$. *Then following are the sufficient conditions for* $f \in \mathcal{S}^*_L$.

(1) *The function* f *satisfies the subordination*

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta \geq 4\sqrt{2}).$$

(2) *The function* f *satisfies the subordination*

$$1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta \leq -4 \quad \text{or} \quad \beta \geq 8).$$

(3) *The function* f *satisfies the subordination*

$$1-\beta+\beta\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta \geq 8\sqrt{2}).$$

(4) *The function* f *satisfies the subordination*

$$\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+\frac{4z}{3}+\frac{2z^2}{3} \quad (\beta \geq 12).$$

Theorem 5.9. *Let* $f \in \mathcal{A}$.

(1) *If the function* f *satisfies* $1+\beta zf''(z) \prec 1+4z/3+2z^2/3$, $\beta \geq 4\sqrt{2}$, *then* $f' \prec \sqrt{1+z}$.

(2) *If the function f satisfies*

$$1+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3}\quad (\beta\leq -4\quad or\quad \beta\geq 8),$$

then

$$\frac{z^2f'(z)}{f^2(z)}\prec\sqrt{1+z}.$$

(3) *If the function f satisfies*

$$\frac{z^2f'(z)}{f^2(z)}+\beta\left(\frac{(zf(z))''}{f'(z)}-\frac{2zf'(z)}{f(z)}\right)\prec 1+\frac{4z}{3}+\frac{2z^2}{3}\quad (\beta\geq 12),$$

then

$$\frac{z^2f'(z)}{f^2(z)}\prec\sqrt{1+z}.$$

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