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ON THE $(n,d)^{th}$ f-IDEALS

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Abstract. For a field K, a square-free monomial ideal I of $K[x_1,\ldots,x_n]$ is called an *f-ideal*, if both its facet complex and Stanley-Reisner complex have the same f-vector. Furthermore, for an f-ideal I, if all monomials in the minimal generating set $G(I)$ have the same degree d, then I is called an $(n,d)^{th}$ f-ideal. In this paper, we prove the existence of $(n,d)^{th}$ f-ideal for $d\geq 2$ and $n\geq d+2$, and we also give some algorithms to construct $(n,d)^{th}$ f-ideals.

1. Introduction

Throughout the paper, for a set A, we use A_d to denote the set of the subsets of A with cardinality d. For a field K, let $S=K[x_1,\ldots,x_n]$, and let I be a monomial ideal of S. Denote by $sm(S)$ ($sm(I)$, respectively) the set of square-free monomials in S (in I, respectively). As we know, there is a natural bijection between $sm(S)$ and $2^{[n]}$, denoted by

$$\sigma:\, x_{i_1}x_{i_2}\cdots x_{i_k}\mapsto\{i_1,i_2,\ldots,i_k\},$$

where $[n]=\{1,2,\ldots,n\}$ for a positive integer n. For other concepts and notations, see references [3, 5, 7, 8, 10, 11].

Constructing free resolutions of a monomial ideal is one of the core problems in combinatorial commutative algebra. A main approach to the problem is by taking advantage of properties of a simplicial complex, so it is important to have a research on properties of the complex corresponding to an ideals, see, e.g., references [4, 6, 9, 12]. There is an important class of ideals called f-ideals, whose facet complex $\delta_{\mathcal{F}}(I)$ and Stanley-Reisner complex $\delta_{\mathcal{N}}(I)$ have the same f-vector, where $\delta_{\mathcal{F}}(I)$ is generated by the set $\sigma(G(I))$, and $\delta_{\mathcal{N}}(I)=\{\sigma(g)\mid g\in sm(S)\setminus sm(I)\}$. Note that the f-vector of a complex $\delta_{\mathcal{N}}(I)$, which is not easy to compute in general, is essential in the computation of the Hilbert series of S/I. Since the correspondence of the complex $\delta_{\mathcal{F}}(I)$ and an ideal I is direct and clear, it is more easier to calculate the f-vector of $\delta_{\mathcal{F}}(I)$. So, it is convenient

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to calculate the Hilbert series and study other corresponding properties of S/I while I is an f-ideal.

The formal definition of an f-ideal appeared first in [1], and it is then studied in [2]. In [7], a monomial ideal I of $K[x_1, \dots, x_n]$ is called an $(n,d)^{th}$ ideal if the monomials in the minimal generating set $G(I)$ have the same degree d, and the $(n,d)^{th}$ f-ideals are characterized. General f-ideals are also studied in [7]. In [7], a bijection is introduced between square-free monomial ideals of degree 2 and simple graphs, and it is shown that $V(n,2) \neq \emptyset$ holds for each $n \geq 4$, where $V(n,d)$ is the set of $(n,d)^{th}$ f-ideals. The structure of $V(n,2)$ is determined, and the characterization of the unmixed f-ideals is also studied in [7]. Recall that an ideal I is called *unmixed*, if $\text{codim}(P) = \text{codim}(I)$ holds for every prime ideal P minimal over I.

In this paper, we give another characterization of unmixed f-ideals in part two. In Section 3, we generalize the aforementioned result of [7] by showing that $V(n,d) \neq \emptyset$ for general $d \geq 2$ and $n \geq d+2$. In Section 4, we introduce some algorithms to construct $(n,d)^{th}$ f-ideals, and we show an upper bound of the $(n,d)^{th}$ perfect number in Section 5. In Section 6, we show some examples of nonhomogeneous f-ideals, the existence of which was still open in [7].

The following notations, definitions and propositions are needed in this paper.

Let A be a set of square-free monomials in $K[x_1, \dots, x_n]$. The sets $\sqcup(A)$ and $\sqcap(A)$ are defined respectively by

$$\sqcup(A) = \{gx_i \,|\, g \in A,\ x_i \nmid g,\ 1 \leq i \leq n\}$$

and

$$\sqcap(A) = \{h \mid 1 \neq h,\ h = g/x_i \text{ for some } g \in A \text{ and some } x_i \text{ with } x_i \mid g\}.$$

Definition 1.1 ([7, Definition 2.1]). Let $S = K[x_1, \dots, x_n]$, and let $A \subseteq sm(S)_d$, where $1 < d < n$. A is called an $(n,d)^{th}$ *upper perfect set*, if $\sqcup(A) = sm(S)_{d+1}$ holds. Dually, A is called an $(n,d)^{th}$ *lower perfect set*, if $\sqcap(A) = sm(S)_{d-1}$ holds. If A is both $(n,d)^{th}$ upper perfect and $(n,d)^{th}$ lower perfect, then A is called an $(n,d)^{th}$ *perfect set*, or alternatively, a *perfect subset* of $sm(S)_d$. For a given pair of numbers (n,d), the smallest number among cardinalities of $(n,d)^{th}$ perfect sets is called the $(n,d)^{th}$ *perfect number*, and is denoted by $N_{(n,d)}$.

Proposition 1.2 ([7, Theorem 2.3]). *Let $S = K[x_1, \dots, x_n]$, and let I be an $(n,d)^{th}$ square-free monomial ideal of S with the minimal generating set $G(I)$. Then I is an f-ideal if and only if $G(I)$ is $(n,d)^{th}$ perfect and $|G(I)| = \frac{1}{2}\binom{n}{d}$ holds true.*

Proposition 1.3 ([7, Proposition 3.3]). *$V(n,2) \neq \emptyset$ if and only if $n = 4k$ or $n = 4k+1$ for some positive integer k.*

Proposition 1.4 ([7, Proposition 5.3])**.** *Let $S = K[x_1,\dots,x_n]$. If I is an $(n,d)^{th}$ f-ideal, then I is unmixed if and only if $sm(S)_d \setminus G(I)$ is lower perfect in $sm(S)_d$.*

In [7], a method for finding an $(n,2)^{th}$ perfect set with the smallest cardinality is provided in the following: First, decompose the set $[n]$ into a disjoint union of two subsets B and $\overline{B}$ *uniformly*, i.e., such that $||B| - |\overline{B}|| \leq 1$ holds true. Second, for each such subset B, set

$$A = \{x_i x_j \mid \text{either}\, \{i,j\} \subseteq B, \text{ or}\, \{i,j\} \subseteq \overline{B}\}.$$

Then, A is an $(n,2)^{th}$ perfect set whose cardinality is equal to the $(n,2)^{th}$ perfect number $N_{(n,2)}$, where

$$N_{(n,2)} = \begin{cases} k^2 - k, & \text{if } n = 2k; \\ k^2, & \text{if } n = 2k+1. \end{cases} \tag{1.1}$$

Note that for any such subset A, a set D with $A \subseteq D \subseteq sm(S)_2$ is also an $(n,2)^{th}$ perfect set.

2. $(n,d)^{th}$ unmixed f-ideals

For a positive integer d greater than 2, an $(n,d)^{th}$ f-ideal may be not unmixed, see Example 5.1 of [7] for a counterexample. So, it is interesting to characterize the unmixed f-ideals. In this section, we show a characterization of unmixed f-ideals by the corresponding simplicial complex, by taking advantage of the bijection σ between square-free monomial ideals and simplicial complexes.

A simplicial complex Δ on $[n]$ is called a *d-flag complex* if every minimal nonface of Δ consists of d elements of $[n]$. Note that a flag complex (see, e.g., [8, page 155]) is a 2-flag complex, as is just defined. For a simplicial complex Δ on $[n]$, the Alexander dual of Δ, denoted by $\Delta^\vee$, is defined by $\Delta^\vee = \{[n] \setminus F \mid F \notin \Delta\}$, see [8] for details.

Proposition 2.1. *Let $S = K[x_1,\dots,x_n]$, and let I be an $(n,d)^{th}$ square-free monomial ideal of S. Then I is an $(n,d)^{th}$ unmixed f-ideal if and only if the following conditions hold*:
(1) $|G(I)| = \binom{n}{d}/2$.
(2) $\dim \delta_{\mathcal{F}}(I)^\vee = n - d - 1$.
(3) $\langle \sigma(u) \mid u \in sm(S)_d \setminus G(I)\rangle$ *is a d-flag complex.*

Proof. We claim that the following two results hold true: First, the condition (2) holds if and only if $G(I)$ is lower perfect. Second, the condition (3) holds if and only if $G(I)$ is upper perfect and $sm(S)_d \setminus G(I)$ is lower perfect. If the above two results hold true, then it is easy to see that the conclusion holds by Propositions 1.2 and 1.4.

For the first claim, if $G(I)$ is lower perfect, then for each minimal nonface F of $\delta_{\mathcal{F}}(I)$, $|F| \geq d$ holds. By the definition of the Alexander dual, H is a face of $\delta_{\mathcal{F}}(I)^\vee$ if and only if $[n] \setminus H$ is a nonface of $\delta_{\mathcal{F}}(I)$. So, for each facet L of $\delta_{\mathcal{F}}(I)^\vee$, $|L| \leq n-d$ holds true. Since $|G(I)| \neq \binom{n}{d}$, there exists some nonface of $\delta_{\mathcal{F}}(I)$ with cardinality d, or equivalently, there exists some facet of $\delta_{\mathcal{F}}(I)^\vee$ with cardinality $n-d$. Thus $\dim(\delta_{\mathcal{F}}(I)^\vee) = n-d-1$ holds.

Conversely, assume $\dim(\delta_{\mathcal{F}}(I)^\vee) = n-d-1$. By a similar argument, one can see that the smallest cardinality of nonfaces of $\delta_{\mathcal{F}}(I)$ is d, hence $G(I)$ is lower perfect.

For the second claim, if $sm(S)_d \setminus G(I)$ is lower perfect, then for the complex $\Delta = \langle \sigma(u) \,|\, u \in sm(S)_d \setminus G(I) \rangle$, the cardinality of a nonface is not less than d. Since $G(I)$ is upper perfect, for each nonface F of Δ, there exists $v \in G(I)$ such that $\sigma(v) \subseteq F$. Note that $\sigma(v)$ is a nonface of Δ, so all the minimal nonfaces of Δ have cardinality d. Hence Δ is a d-flag complex.

Conversely, assume that $\Delta = \langle \sigma(u) \,|\, u \in sm(S)_d \setminus G(I) \rangle$ is a d-flag complex. In a similar way, one can see that $G(I)$ is upper perfect and $sm(S)_d \setminus G(I)$ is lower perfect. □

3. Existence of $(n,d)^{th}$ f-ideals

Let $x_{[n]} = x_1 x_2 \cdots x_n$. For a subset M of $sm(S)_d$, denote $M' = \{x_{[n]}/u \,|\, u \in M\}$. The following lemma is essential in the proof of our main result in this section.

Lemma 3.1. *M is a lower (an upper, respectively) perfect subset of $sm(S)_d$ if and only if M' is an upper (a lower, respectively) perfect subset of $sm(S)_{n-d}$.*

Proof. For the necessary part, if M is a lower perfect subset of $sm(S)_d$, then it follows from definition that M' is a subset of $sm(S)_{n-d}$. In order to check that M' is upper perfect, we will show for each monomial $u \in sm(S)_{n-d+1}$ that $u \in \sqcup(M')$ holds. This is equivalent to showing that there exists some $v \in M'$, such that $v \,|\, u$ holds. In fact, since M is lower perfect, for the monomial $u' = x_{[n]}/u \in sm(S)_{d-1}$, there exists some $w \in M$ such that $u' \,|\, w$ holds. Let $v = x_{[n]}/w$. It is easy to see that $v \,|\, u$. Note that $v \in M'$, this shows that M' is upper perfect. In a similar way, one can prove that M' is lower perfect when M is upper perfect. The sufficient part follows from the easy observation that $M'' = M$. □

Corollary 3.2. *If I is an $(n,d)^{th}$ square-free monomial ideal of S, then I is an f-ideal if and only if $|G(I)| = \binom{n}{d}/2$ and $G(I)'$ is a perfect subset of $sm(S)_{n-d}$.*

Denote $sm(S\{\check{k}\})_d = \{u \in sm(S)_d \,|\, x_k \nmid u\}$, and $sm(S\{k\})_d = \{u \in sm(S)_d \,|\, x_k | u\}$. For a subset $X = \{i_1, \ldots, i_j\}$ of $[n]$, denote

$$sm(S\{\check{X}\})_d = \{u \in sm(S)_d \,|\, \; x_k \nmid u \text{ for every } \; k \in X\},$$

and let $sm(S\{X\})_d = \{u \in sm(S)_d \,|\, \; x_k \mid u \text{ for every } k \in X\}$.

Definition 3.3. For a subset M of $sm(S\{\check{k}\})_d$, if $sm(S\{\check{k}\})_{d+1} \subseteq \sqcup(M)$ holds, then M is called *upper perfect without k*. Dually, a subset M of $sm(S\{\check{k}\})_d$ is called *lower perfect without* k, if $sm(S\{\check{k}\})_{d-1} \subseteq \sqcap(M)$ holds. A subset M of $sm(S\{k\})_d$ is called *upper perfect containing* k, if $sm(S\{k\})_{d+1} \subseteq \sqcup(M)$ holds; a subset M of $sm(S\{k\})_d$ is called *lower perfect containing* k, if $sm(S\{k\})_{d-1} \subseteq \sqcap(M)$ holds. If M is not only upper but also lower perfect without k, then M is called *perfect without* k. Similarly, if M is both upper and lower perfect containing k, then M is called *perfect containing* k.

For a subset X of $[n]$, we can define the upper perfect (lower perfect, perfect, respectively) set without X (containing X) similarly. For a subset A of $sm(S)_d$, let $A\{\check{X}\} = A \cap sm(S\{\check{X}\})_d$, and let $A\{X\} = A \cap sm(S\{X\})_d$.

Proposition 3.4. *Let A be a subset of $sm(S)_d$, and let $X = \{i_1, \ldots, i_j\}$ be a subset of $[n]$. Then the following statements hold*:

(1) $A\{\check{X}\} = A\{\check{i_1}\}\{\check{i_2}\}\cdots\{\check{i_j}\}$, *and* $A\{X\} = A\{i_1\}\{i_2\}\cdots\{i_j\}$;

(2) *If A is upper perfect, then $A\{\check{X}\}$ is upper perfect without X*;

(3) *If A is lower perfect, then $A\{X\}$ is lower perfect containing X*;

(4) *If A is upper* (*lower, respectively*) *perfect without X, then A' is lower* (*upper, respectively*) *perfect containing X. Furthermore, the converse also holds true.*

Proof. (1) and (2) are easy to see by the corresponding definitions.

In order to prove (3), it is sufficient to show that $A\{k\}$ is a lower perfect set containing k for each $k \in [n]$. In fact, since A is lower perfect, for each monomial $u \in sm(S\{k\})_{d-1}$, there exists a monomial v in A such that $u \mid v$. Note that $x_k \mid u$ holds, so $x_k \mid v$ also holds, which implies that $v \in sm(S\{k\})_d$ holds. Hence $A\{k\}$ is a lower perfect set containing k.

For (4), we only show that A' is lower perfect containing k when A is upper perfect without k, and the remaining implications are similar to prove. In fact, for each monomial $u \in sm(S\{k\})_{n-d-1} \subseteq sm(S)_{n-d-1}$, $u' = x_{[n]}/u \in sm(S)_{d+1}$. Note that $x_k \mid u$ implies $x_k \nmid u'$ holds true, hence $u' \in sm(S\{\check{k}\})_{d+1}$ also hold. Since A is upper perfect without k, there exists a monomial $v \in A$ such that $v \mid u'$ holds, hence $u \mid v'$ holds, where $v' = x_{[n]}/v \in A'$. This completes the proof. □

Remark 3.5. For a perfect subset A of $sm(S)_d$, $A\{\check{X}\}$ needs not to be a lower perfect set without X, and $A\{X\}$ needs not to be an upper perfect set containing X, see the following for counterexamples:

Example 3.6. Let $S = K[x_1, \ldots, x_6]$, let

$$A = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\},$$

and let $B = A \setminus \{x_1x_2x_6\}$. It is easy to see

$$A\{\check{6}\} = B\{\check{6}\} = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5\},$$
$$A\{6\} = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}, \text{ and}$$
$$B\{6\} = \{x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}.$$

Also, it is direct to check that both A and B are perfect sets, and that both $A\{\check{6}\}$ and $B\{\check{6}\}$ are perfect sets without 6. Note that $A\{6\}$ is a perfect set containing 6, but $B\{6\}$ is not upper perfect.

By Proposition 3.4, we have the following example by mapping A, B to A', B' respectively.

Example 3.7. Let $S = K[x_1, \ldots, x_6]$, and let

$$A' = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5, x_1x_2x_6, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\},$$

and $B' = A' \setminus \{x_3x_4x_5\}$. It is easy to see that

$$A'\{\check{6}\}=\{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5\},\ B'\{\check{6}\}=\{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5\},$$

and $A'\{6\} = B'\{6\} = \{x_1x_2x_6, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\}$. It is direct to check that both A' and B' are perfect sets, and that both $A'\{6\}$ and $A'\{6\}$ are perfect sets containing 6. Note that $A'\{\check{6}\}$ is a perfect set without 6, but $B'\{\check{6}\}$ is not lower perfect.

In order to obtain the main result of this section, we need a further fact.

Lemma 3.8. *Let $S = K[x_1, \ldots, x_n]$, and let A be a subset of $sm(S)_d$. If $A\{\check{k}\}$ is a perfect subset of $sm(S\{\check{k}\})_d$ without k, and $A\{k\}$ is a perfect subset of $sm(S\{k\})_d$ containing k for some $k \in [n]$, then A is a perfect subset of $sm(S)_d$.*

Proof. In order to show A is an upper perfect subset of $sm(S)_d$, it suffice to show that $sm(S)_{d+1} \subseteq \sqcup(A)$. Note that $sm(S)_{d+1} = sm(S\{\check{k}\})_{d+1} + sm(S\{k\})_{d+1}$, it suffice to show $sm(S\{\check{k}\})_{d+1} \subseteq \sqcup(A)$ and $sm(S\{k\})_{d+1} \subseteq \sqcup(A)$. Since $A\{\check{k}\}$ is a perfect subset of $sm(S\{\check{k}\})_d$ without k, we have

$$sm(S\{\check{k}\})_{d+1} \subseteq \sqcup(A\{\check{k}\}) \subseteq \sqcup(A).$$

Similarly, $sm(S\{k\})_{d+1} \subseteq \sqcup(A\{k\}) \subseteq \sqcup(A)$. This shows A is upper perfect. By a similar way, one can check that A is lower perfect. □

Theorem 3.9. *For any integer $d \geq 2$ and any integer $n \geq d+2$, there exists an $(n, d)^{th}$ perfect set with cardinality less than or equal to $\binom{n}{d}/2$.*

Proof. We prove the result by induction on d.

If $d = 2$, the conclusion holds true for any integer $n \geq 4$ by Proposition 1.3. In the following, assume $d > 2$.

Assume that the conclusion holds true for any integer less than d. For d, we claim that the conclusion holds true for any integer $n \geq d+2$. We will show the result by induction on n.

If $n = d+2$, then $\binom{n}{d} = \binom{n}{2}$. Note that for any integer $n \geq 4$, there exists an $(n, 2)^{th}$ perfect set M, such that $|M| \leq \binom{n}{2}/2$. By Lemma 3.1, M' is an $(n, d)^{th}$ perfect set. It is clear that $|M'| = |M| \leq \binom{n}{2}/2 = \binom{n}{d}/2$.

Now assume that the conclusion holds true for any integer less than n. Then by Lemma 3.8, it will suffice to show that there is a perfect subset A

of $sm(S\{\check{n}\})_d$ without n and a perfect subset B of $sm(S\{n\})_d$ containing n, such that $|A| \le |sm(S\{\check{n}\})_d|/2 = \binom{n-1}{d}/2$ and $|B| \le |sm(S\{n\})_d|/2 = \binom{n-1}{d-1}/2$ hold.

Let $L = K[x_1, \dots, x_{n-1}]$. Then clearly, $sm(S\{\check{n}\})_d = sm(L)_d$ holds. By induction on n, there exists an $(n-1,d)^{th}$ perfect subset A of $sm(L)_d$, such that $|A| \le \binom{n-1}{d}/2$. It is easy to see that A is a perfect subset of $sm(S\{\check{n}\})_d$ without n. By induction on d, there exists an $(n-1,d-1)^{th}$ perfect subset B_1 of $sm(L)_{d-1}$, such that $|B_1| \le \binom{n-1}{d-1}/2$ holds. Let $B = \{ux_n \mid u \in B_1\}$. It is easy to see that B is a perfect subset of $sm(S\{n\})_d$ containing n, and $|B| = |B_1| \le \binom{n-1}{d-1}/2$.

Let $D = A \cup B$. Note that $A = D\{\check{n}\}$ and $B = D\{n\}$, by Lemma 3.8, D is a perfect subset of $sm(S)_d$, and $|D| = |A| + |B| \le \binom{n-1}{d}/2 + \binom{n-1}{d-1}/2 = \binom{n}{d}/2$. This completes the proof. □

By Proposition 1.2 and Theorem 3.9, the following corollary is clear.

Corollary 3.10. *For any integer $d \ge 2$ and any integer $n \ge d+2$, $V(n,d) \ne \emptyset$ if and only if $2 \mid \binom{n}{d}$.*

4. Algorithms for constructing examples of $(n,d)^{th}$ f-ideals

In this section, we will show some algorithms to construct $(n,d)^{th}$ f-ideals when $2 \mid \binom{n}{d}$. We discuss the following cases:

Case 1: $d = 2$. An $(n,2)^{th}$ f-ideal is easy to construct by [7]. For reader's convenience, we repeat it as the following: Decompose the set $[n]$ into a disjoint union of two subsets B and $\overline{B}$ *uniformly*, namely, $||B| - |\overline{B}|| \le 1$. Then set $A = \{x_ix_j \mid i,j \in B, \text{ or } i,j \in \overline{B}\}$ to obtain an $(n,2)^{th}$ perfect set. Note that $|A| = N_{(n,2)} \le \binom{n}{2}/2$, choose a subset D of $sm(S)_2 \setminus A$ randomly, such that $|D| = \binom{n}{2}/2 - N_{(n,2)}$ holds. It is easy to see that $A \cup D$ is still a perfect set, and $|A \cup D| = \binom{n}{2}/2$. By Proposition 1.2, the ideal generated by $A \cup D$ is an $(n,2)^{th}$ f-ideal. Note that each $(n,2)^{th}$ f-ideal can be obtained in this way except C_5 by [7].

Case 2: $d > 2$ and $n = d+2$.

Algorithm 4.1. In order to build an f-ideal $I \in V(d+2,d)$, we obey the following steps:

Step 1: Calculate $\binom{d+2}{d}/2$. Note that $\binom{d+2}{d}/2 = \binom{d+2}{2}/2$.

Step 2: As in the case 1, find a perfect subset B of $sm(S)_2$ such that $|B| \le \binom{d+2}{2}/2$, where $S = K[x_1, \dots, x_{d+2}]$.

Step 3: Let $A = B'$. Then A is a perfect subset of $sm(S)_d$ by Lemma 3.1, and $|A| = |B| \le \binom{d+2}{2}/2 = \binom{d+2}{d}/2$.

Step 4: Choose a subset D of $sm(S)_d \setminus A$ randomly, such that $|D| = \binom{d+2}{d}/2 - |A|$ holds. It is easy to see that $M = A \cup D$ is still a perfect set, and $|A \cup D| = \binom{d+2}{d}/2$.

Step 5: Let I be the ideal generated by $A \cup D$. By Proposition 1.2 again, I is a $(d+2,d)^{th}$ f-ideal.

Note that in this way, we construct almost all $(d+2,d)^{th}$ f-ideals.

Example 4.2. *Show an f-ideal $I \in V(8,6)$.*

Note that $8 = 6+2$, we obey the Algorithm 4.1.

Note that $\binom{8}{6}/2 = 14$. Find a perfect subset B of $sm(S)_2$ such that $|B| \le \binom{8}{2}/2 = 14$, where $S = K[x_1,\dots,x_8]$. It is easy to see that

$$B = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_5x_6, x_5x_7, x_5x_8, x_6x_7, x_6x_8, x_7x_8\}$$

is a perfect subset of $sm(S)_2$, with $|B| = 12$. Let

$$\begin{aligned} A = B' = \{\ & x_3x_4x_5x_6x_7x_8, x_2x_4x_5x_6x_7x_8, x_2x_3x_5x_6x_7x_8, x_1x_4x_5x_6x_7x_8,\\ & x_1x_3x_5x_6x_7x_8, x_1x_2x_5x_6x_7x_8, x_1x_2x_3x_4x_7x_8, x_1x_2x_3x_4x_6x_8,\\ & x_1x_2x_3x_4x_6x_7, x_1x_2x_3x_4x_5x_8, x_1x_2x_3x_4x_5x_7, x_1x_2x_3x_4x_5x_6\}. \end{aligned}$$

A is a perfect subset of $sm(S)_6$. Choose $D = \{x_1x_2x_3x_5x_6x_7, x_1x_2x_4x_5x_6x_8\}$, then the ideal I generated by $A \cup D$ is an $(8,6)^{th}$ f-ideal.

Case 3: $d > 2$ and $n > d+2$. Let $S^{[k]} = K[x_1,\dots,x_k]$, and let $S = S^{[n]} = K[x_1,\dots,x_n]$.

Algorithm 4.3. For an integer $n > d+2$, we construct an $(n,d)^{th}$ f-ideal by using the following steps:

Step 1: Let $t = n$, $l = d$ and $E = \emptyset$. Set $\mathcal{B} = \{B(t,l,E)\}$.

Step 2: Assign $\mathcal{C} = \mathcal{B}$, and denote $i = |\mathcal{C}|$.

Step 3: Choose each $B(t,l,E) \in \mathcal{C}$ one by one, deal with each one obeying the following rules:

If $l = 2$ or $t = l+2$, don't change anything.

If $l \ne 2$ and $t > l+2$, then cancel $B(t,l,E)$ from $\mathcal{B}$, and add $B(t-1,l,E)$ and $B(t-1,l-1,E\cup\{t\})$ into $\mathcal{B}$.

After i times, i.e., when $B(t,l,E)$ goes through all the element of $\mathcal{C}$, make a judgement:

If $l = 2$ or $t = l+2$ for each $B(t,l,E) \in \mathcal{B}$, then go to Step 4, else return to Step 2.

Step 4: Choose $B(t,l,E) \in \mathcal{B}$ one by one, deal with each one obeying the following rules:

If $l = 2$, assign $B(t,l,E)$ a perfect subset of $sm(S^{[t]})_l$ as Case 1.

If $l \ne 2$ and $t = l+2$, assign $B(t,l,E)$ a perfect subset of $sm(S^{[t]})_l$ as Case 2.

Step 5: For each $B(t,l,E) \in \mathcal{B}$, denote $B^*(t,l,E) = \{ux_E \mid u \in B(t,l,E)\}$, where $x_E = \prod_{j\in E} x_j$. Denote $\mathcal{B}^* = \cup_{B(t,l,E)\in\mathcal{B}} B^*(t,l,E)$. It is direct to check that $\mathcal{B}^*$ is a perfect subset of $sm(S)_d$, and $|\mathcal{B}^*| \le \binom{n}{d}/2$. Choose a subset D of $sm(S)_d \setminus \mathcal{B}^*$ randomly, such that $|D| = \binom{n}{d}/2 - |\mathcal{B}^*|$ holds.

Step 6: Let I be the ideal generated by $\mathcal{B}^* \cup D$. By Proposition 1.2 again, I is an $(n,d)^{th}$ f-ideal.

Example 4.4. *Show a $(6,3)^{th}$ f-ideal.*

Let $S = K[x_1,\dots,x_6]$. By the above algorithm, we will choose a perfect subset $B(5,3,\emptyset)$ of $sm(S^{[5]})_3$ and a perfect subset $B(5,2,\{6\})$ of $sm(S^{[5]})_2$. Set

$$B(5,3,\emptyset) = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3\} \text{ and}$$
$$B(5,2,\{6\}) = \{x_1x_2, x_1x_3, x_2x_3, x_4x_5\}.$$

Correspondingly,

$$B^*(5,3,\emptyset) = B(5,3,\emptyset) \text{ and}$$
$$B^*(5,2,\{6\}) = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}.$$

Hence

$$\mathcal{B}^* = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}$$

is a perfect subset of $sm(S)_3$. Note that $\binom{6}{3}/2 = 10$, and $|\mathcal{B}^*| = 8$. Set $D = \{x_1x_2x_4, x_1x_2x_5\}$. The ideal I generated by $\mathcal{B}^* \cup D$ is a $(6,3)^{th}$ f-ideal.

Note that the $(6,3)^{th}$ f-ideal given in the above example is not unmixed. In fact, consider the simplicial complex $\sigma(sm(S)_3 \setminus G(I))$, and note that $\{1,2\}$ is a nonface of $\sigma(sm(S)_3 \setminus G(I))$, which implies that $\sigma(sm(S)_3 \setminus G(I))$ is not a 3-flag complex. So, I is not unmixed by Proposition 2.1.

5. An upper bound of the perfect number $N_{(n,d)}$

For a positive integer k and a pair of positive integers $i \le j$, denote by $Q^k_{[i,j]}$ the set of square-free monomials of degree k in the polynomial ring $K[x_i, x_{i+1}, \dots, x_j]$. Note that $Q^k_{[i,j]} = \emptyset$ holds for $i > j$. For a pair of monomial subsets A and B, denote by $A \bullet B = \{uv \mid u \in A, v \in B\}$. If $B = \emptyset$, then assume $A \bullet B = A$. The following theorem gives an upper bound of the $(n,d)^{th}$ perfect number for $n > d+2$.

Theorem 5.1. *Given a integer $d > 2$, and a integer $n \ge d+2$. The following statements about the perfect number $N_{(n,d)}$ hold:*

(1) *If $n = d+2$, then*

$$N_{(n,d)} = N_{(n,2)} = \begin{cases} k^2 - k, & \text{if } n = 2k; \\ k^2, & \text{if } n = 2k+1. \end{cases} \tag{5.1}$$

(2) *If $n > d+2$, then*

$$N_{(n,d)} \le \sum_{i=5}^{n-d+2} N_{(i,2)} \binom{n-i-1}{d-3} + \sum_{j=3}^{d} N_{(j+2,2)} \binom{n-j-3}{d-j}, \tag{5.2}$$

where $\binom{0}{0} = 1$.

Proof. By Lemma 3.1 and the equation (1.1) in Section 1, (1) is clear.

In order to prove (2), it will suffice to show that there exists a perfect set with cardinality $t = \sum_{i=5}^{n-d+2} N_{(i,2)}\binom{n-i-1}{d-3} + \sum_{j=3}^{d} N_{(j+2,2)}\binom{n-j-3}{d-j}$.

Let $P_{(i,2)}$ be an $(i,2)^{th}$ perfect set with cardinality $N_{(i,2)}$ for $5 \le i \le n-d+2$, and let $P_{(j+2,j)}$ be a $(j+2,j)^{th}$ perfect set with cardinality $N_{(j+2,j)}$ for $3 \le j \le d$. We claim that the set

$$M = (\cup_{i=5}^{n-d+2} P_{(i,2)} \bullet x_{i+1} \bullet Q^{d-3}_{[i+2,n]}) \cup (\cup_{j=3}^{d} P_{(j+2,j)} \bullet Q^{d-j}_{[j+4,n]})$$

is an $(n,d)^{th}$ perfect set, with cardinality t. It is easy to check that the cardinality of M is t. It is only necessary to prove that M is perfect.

For each $w \in sm(S)_{d+1}$, denote by $n_k(w)$ the cardinality of the set $\{x_i \mid i \le k$ and $x_i \mid w\}$. If $n_5(w) \ge 4$, then choose the smallest k such that $n_{k+3}(w) = n_{k+2}(w) = k+1$. Clearly, $3 \le k \le d$. It is direct to check that w is divided by some monomial in $P_{(k+2,k)} \bullet Q^{d-k}_{[k+4,n]}$. If $n_5(w) \le 3$, then choose the smallest k such that $n_k(w) = 3$ and $n_{k+1}(w) = 4$. Clearly, $5 \le k \le n-d+2$. It is not hard to check that w is divided by some monomial in $P_{(k,2)} \bullet x_{k+1} \bullet Q^{d-3}_{[k+2,n]}$. Hence M is upper perfect.

For each $w \in sm(S)_{d-1}$, if $n_5(w) \ge 2$, then choose the smallest k such that $n_{k+3}(w) = n_{k+2}(w) = k-1$. Clearly, $3 \le k \le d$. It is direct to check that w divides some monomial in $P_{(k+2,k)} \bullet Q^{d-k}_{[k+4,n]}$. If $n_5(w) \le 1$, then choose the smallest k such that $n_k(w) = 1$ and $n_{k+1}(w) = 2$. Clearly, $5 \le k \le n-d+2$ holds. It is not hard to check that w divides some monomial in $P_{(k,2)} \bullet x_{k+1} \bullet Q^{d-3}_{[k+2,n]}$. Hence M is lower perfect. □

Figure 1 may help to interpret the above theorem intuitively. In this figure, there is a boundary consisting of the line $l = 2$ and the line $t = l+2$. From the point (d,n) to a point of the boundary, every directed chain $\mathcal{C}$ denotes a set of monomials $M(\mathcal{C})$ by the following rules:

(1) Every arrow of $\mathcal{C}$ is from (l,t) to either $(l,t-1)$ or $(l-1,t-1)$.

(2) If the arrow is from (l,t) to $(l,t-1)$, then each monomial in $M(\mathcal{C})$ is not divided by x_t. Correspondingly, if it is from (l,t) to $(l-1,t-1)$, then each monomial in $M(\mathcal{C})$ is divided by x_t.

(3) Each point (l,t) of the boundary is a $(t,l)^{th}$ perfect set.

Actually, the figure shows us a class of $(n,d)^{th}$ perfect sets. If we choose each point (l,t) of the boundary to be a $(t,l)^{th}$ perfect set with cardinality $N_{(t,l)}$, then the cardinality of the $(n,d)^{th}$ perfect set corresponding to the figure is exactly $\sum_{i=5}^{n-d+2} N_{(i,2)}\binom{n-i-1}{d-3} + \sum_{j=3}^{d} N_{(j+2,2)}\binom{n-j-3}{d-j}$.

Example 5.2. *Calculation of the* $(6,3)^{th}$ *perfect number.*

Let A be a $(6,3)^{th}$ perfect set. By Proposition 3.4(3), $A\{6\}$ is a lower perfect set containing 6. Note that for the monomials of $\{x_1, x_2, x_3, x_4, x_5\}$, each monomial in $A\{6\}$ is divided by at most two of them. So, $|A\{6\}| \ge 3$. By Proposition 3.4(2), $A\{\check{6}\}$ is an upper perfect set without 6. As the discussion

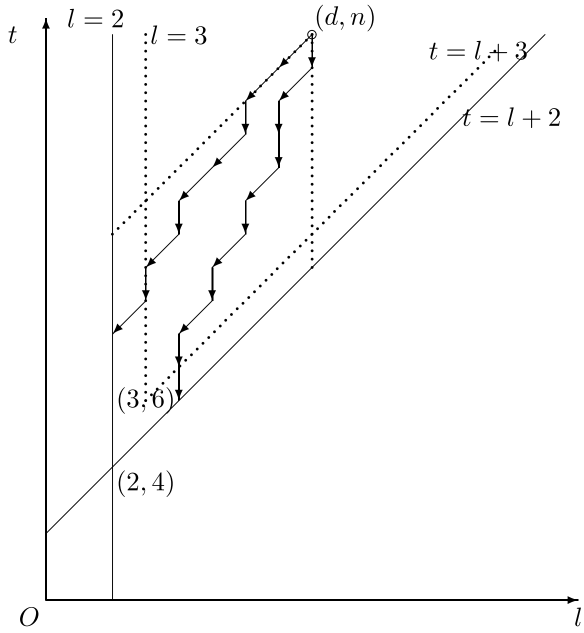

FIGURE 1. Upper bound.

above, $|A\{\check{6}\}| \geq 3$. Hence $|A| \geq |A\{\check{6}\}| + |A\{6\}| \geq 6$. Actually, it is direct to check that the following set

$$B = \{x_1x_2x_3, x_1x_2x_4, x_3x_4x_5, x_1x_5x_6, x_2x_5x_6, x_3x_4x_6\}$$

is a $(6, 3)^{th}$ perfect set with cardinality 6. Thus $N_{(6,3)} = 6$. Note that the upper bound given by Proposition 5.1(2) is 8, and is not bad for the perfect number in the case.

6. Nonhomogeneous f-ideal

In [7], a characterization of f-ideals in general case is shown, but it is still not easy to show an example of nonhomogeneous f-ideal, i.e., the f-ideal I with the property that monomials in $G(I)$ do not have a same degree. In fact, the interference from monomials of different degree makes the computation complicated. Anyway, we finally worked out the following example:

Example 6.1. Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let

$$I = \langle x_1x_2, x_3x_4, x_1x_3x_5, x_2x_4x_5 \rangle.$$

It is direct to check that

$$\delta_{\mathcal{F}}(I) = \langle \{1,2\}, \{3,4\}, \{1,3,5\}, \{2,4,5\} \rangle$$

and

$$\delta_{\mathcal{N}}(I) = \langle \{1,3\}, \{2,4\}, \{1,4,5\}, \{2,3,5\} \rangle.$$

It is easy to see they have the same f-vector, and hence I is an f-ideal, which is clearly nonhomogeneous.

After this nontrivial example, clearly there are a lot of nonhomogeneous f-ideals. We will show another example to end this paper.

Example 6.2. Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6]$, and let

$$I = \langle x_1x_2, x_2x_3, x_1x_3, x_4x_5, x_1x_4x_6, x_1x_5x_6, x_2x_4x_6 \rangle.$$

Note that

$$\delta_{\mathcal{N}}(I) = \langle \{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5,6\}, \{3,4,6\}, \{3,5,6\} \rangle.$$

It is direct to check that I is also a nonhomogeneous f-ideal.

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