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Explicit Expression for Moment of Waiting Time in a DBR Line Production System with Constant Processing Times Using Max-plus Algebra

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Max-plus 대수를 이용한 상수 공정시간을 갖는 DBR 라인 생산시스템에서의 대기시간에 대한 간결한 표현식

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ABSTRACT

Although systems with finite capacities have been the topic of much study, there are as of yet no analytic expressions for (higher) moment and tail probability of stationary waiting times in systems with even constant processing times. The normal queueing theory cannot properly handle such systems due to the difficulties caused by finite capacity. In this study, for a DBR (Drum-Buffer-Rope) line production system with constant processing times, we introduce analytic expressions by using previous results obtained using a max-plus algebraic approach.

Key words : DBR, Finite-buffer, Max-plus algebra, Waiting time

요 약

유한버퍼를 갖는 시스템에 대한 분석은 광범위하게 연구되어 왔다. 하지만, 상수 공정시간을 갖는 시스템에 대해서도 안정 대기시간에 대한 고차평균과 꼬리확률에 대한 간결한 표현식은 소개된 적이 없다. 유한버퍼로 인한 차단현상으로 유발되는 복잡성 때문에 일반적인 대기행렬이론은 이를 적절히 다루지 못한다. 본 연구에서는 max-plus 대수를 활용한 기존 연구결과로부터 상수 공정시간과 DBR (Drum-Buffer-Rope) 재고규칙을 따르는 라인생산시스템에서의 대기시간에 대한 간결한 표현식을 도출하였다.

주요어 : DBR, 유한버퍼, max-plus 대수, 대기시간

1. Introduction

In order to improve production competitiveness and effectiveness, the investigation of various system performances is crucial. Such analytical results can be utilized in the design and control of the flow of materials and products with the intent of avoiding congestion and improving profits across different systems.

Although finite-capacity systems have been widely studied, research on these systems has provided only a few explicit (analytic) results. Due to the difficulties caused by finite capacities, obtaining analytic solutions is not an easy task; most studies have been limited in the number of nodes and servers, distributions of arrival and service times, and other factors.

WIP (work-in-process) may be either under processing

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or waiting for the next processing. WIP is necessary in most circumstances to pace the flow through imperfectly balanced systems. WIP is also necessary as it improves throughput, QoS (Quality of Service), tardiness, cycle time, and so on. In a DBR (Drum-Buffer-Rope) controlled system, a common finite capacity buffer is shared by the upstream nodes of bottleneck node(s). The DBR system pulls work only once the line begins. Once input enters the system, processes continue without the permission of the following nodes, and WIP moves on to the next nodes.

Under DBR, a drumbeat for the rest of the plant is maintained by sequencing work to be done at the bottleneck operation. The drumbeat is then protected by maintaining a time buffer for parts going to the bottleneck. A rope is tied from the bottleneck to material release points to ensure that material is released only at the rate that is used by the bottleneck, thereby preventing excessive increase in inventory. Radovilsky [10] simply modeled a bottleneck node as an M/M/1/K queue and represented an optimal buffer size with a maximum profit. Louw and Page [9] also proposed an open queueing network model to estimate the size of the time buffers in production systems under the TOC (theory of constraints). Ye and Han [11] developed more simplified methods of determining the sizes of the constraint buffer and assembly buffer by using a machine view's bill of routing instead of a process view's bill of routing.

Unlike the infinite buffer case, the distribution of waiting times in tandem queues with finite buffers is not simply given as a product form due to the blocking between nodes. In response to this problem, some researchers have proposed various approximation methods by decomposition and simulation. In this study, however, we use an exact solution procedure based on max-plus algebra. The advantage of such an approach is that the max-plus linear system now needs only two kinds of operators, "max" and "plus," to represent its performance characteristics.

As is well known, the max-plus linear system (MPL) includes various probabilistic systems commonly found in (tele)communication and computer networks such as tandem queues with blocking and fork-and-join type queues. [1, 3, 7] provided the basic max-plus algebra and some preliminaries on the waiting times in MPLs. Taking into consideration DBR WIP-control policy, we introduce explicit expressions for higher moments of stationary waiting times in m-node tandem systems with constant processing times. We assume that pulling a job between nodes follows a communication blocking policy (blocking before service).

2. Explicit Expression for Moments of Waiting Time

A DBR controlled system consisting of m nodes in a series is shown in Figure 1. Let σ^i $(i=1,2,\ldots,m)$ be a processing time at node i and $K(\geq b)$ be a common buffer capacity. This common buffer is shared by the bottleneck node b and all its upstream nodes. As shown in Figure 1, a dummy node (node 0) with zero processing time ($\sigma^0=0$) and infinite capacity ($K_0=\infty$) is inserted into the foremost node.

In [5], Baccelli and Schmidt introduced characteristics of waiting times in a class of stochastic networks as Taylor series expansions with respect to an arrival rate λ. As they assume that the capacity of the first node is infinite, inserting a dummy node (node 0) in front of node 1 allows for the application of their results to DBR-controlled systems. The assumption that the dummy node has zero processing time can adequately control the movement to node 1 under DBR policy. The series expansions have a sequence of random vectors $\{D_n\}$ as their elements. In max-plus linear systems, the sequence of $\{D_n\}$ can properly capture the dynamic behaviors (the influences of blocking policy and/or network structure) of systems. For a general max-plus linear stochastic system the i^{th} component of the random vector D_k can be interpreted as the longest path from the initial node

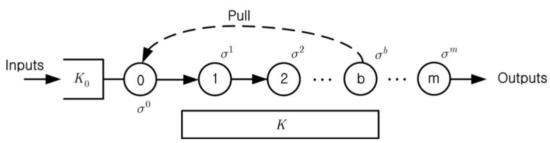

Fig. 1. a DBR system with m-node

to i node in the corresponding task graph.

Once the explicit expression of a sequence $\{D_n\}$ is available, we can compute various characteristics of transient and stationary waiting times by inputting this expression into the series expansions given in [1, 2, 4, 5]. However, because of the difficulty of deriving closed-form expressions for stationary waiting times, work in [1, 2, 4, 5] assumed the i-th element of D_n to be 'ultimately periodic':

$$D_n^i = \begin{cases} \eta_n^i & for\ n = 0,\ldots,\xi_i - 1 \\ \eta_{\xi_i}^i + (n - \xi_i)a_i & for\ n \geq \xi_i \end{cases} \tag{1}$$

for the constant real numbers $0 \leq \eta_0^i \leq \eta_1^i \leq \cdots \leq \eta_{\xi_i}^i$, a_i and some non-negative integers ξ_i.

By placing the explicit expressions of D_n^i satisfying structure (1) into the closed-form formulas given by [1, Corollary 3.1] and [2, Theorem 2.3], we can compute the exact values of higher moment and tail probability of stationary waiting times in a max-plus linear system. In practice, however, computational times vary on the value of ξ_i.

Let node b be a bottleneck node among m nodes and σ^{b_i} be the maximum processing time up to node i, that is $\sigma^{b_i} = max\{\sigma^1,\ldots,\sigma^i\}$. Then the sequence of $\{D_n\}$ for a DBR system is given as follows (see [6, 7]): for node $i(< b)$,

$$D_n^i = \sum_{j=1}^{i-1} \sigma^j + max\{n\sigma^1,\ldots,n\sigma^i\} \quad \text{for } 0 \leq n < K \tag{2}$$

$$D_n^i = \sum_{j=1}^{i-1} \sigma^j + max\left\{n\sigma^1,\ldots,n\sigma^i, \sum_{j=1}^{b} \sigma^j + (n-K)\sigma^{b_m}\right\} \quad \text{for } K \leq n < \infty \tag{3}$$

for node $i(\geq b)$,

$$D_n^i = \sum_{j=1}^{i-1} \sigma^j + n\sigma^{b_m} \text{ for all } n \geq 0 \tag{4}$$

In computing the exact values of characteristics of waiting times, the polynomial $p_k(\cdots)$ introduced in Corollary 3.1 of [1] and Theorem 2.3 of [2] induce computational complexity and difficulty because of the need to calculate the values of polynomials $p_k(\cdots)$ recursively. An alternative (simplified) form of the polynomial $p_k(\cdots)$ is shown in Hasenfuss [6] as follows:

$$\begin{aligned} & p_{k+1}(x_0,x_1,\ldots,x_k) \\ &= \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} \frac{x_n^{k+1}}{(k+1)!} \\ &- \sum_{n=0}^{k-1} \sum_{j=0}^{k-1} \binom{j}{n} (-1)^{j-n} \frac{x_n^{j+1}}{(j+1)!} \\ &\times \{p_{k-j}(x_{n+1},\ldots,x_{n+k-j}) - p_{k-j}(x_n,\ldots,x_{n+k-j-1})\} \end{aligned}$$

where $p_1(x_0) = x_0$.

To avoid this computational difficulty, we eliminate the polynomial terms that resulted. The following theorem shows the explicit expression for moments of stationary waiting times in a DBR line production system with constant processing times. The proof is provided in the next section.

Theorem 1: Let σ^i be a constant service time at node i, $i = 1,\ldots,m$ and λ be an input rate of a stationary Poisson process such that $\lambda \in [0, 1/\sigma^{b_m})$ where $\sigma^{b_i} = max\{\sigma^1,\ldots,\sigma^i\}$. Suppose that the sequence of $\{D_n\}$ satisfies the structure (1). Then moments of stationary waiting times at node i in a DBR line production system can be computed from: for $\xi_i \geq 1$,

$$\begin{aligned} & E[(W^i)^r] \\ &= \sum_{j=0}^{r} \binom{r}{j} \sum_{k=0}^{j} \frac{(-1)^k (\eta_{\xi_i}^i)^{j-k} k!}{\lambda^k} \binom{j}{k} \binom{\xi_i - 1 + k}{k} E[W^{r-j}] \\ &+ (1-\rho_m) \sum_{\ell=0}^{\xi_i - 1} \theta_1 \times \theta_2 \\ &+ (1-\rho_m) \sum_{\ell=0}^{\xi_i - 2} \theta_1 \times \theta_2 \\ &\times \left[\sum_{j=1}^{\xi_i - 1 - \ell} e^{-j\rho_m} \rho_i^j \left\{ \frac{\left(\frac{y_{\ell+j}}{\sigma^{b_i}} - \ell\right)^j}{j!} - \frac{\left(\frac{y_{\ell+j}}{\sigma^{b_i}} - \ell\right)^{j-1}}{(j-1)!} \right\} \right] \end{aligned} \tag{5}$$

with the convention that the summation over an empty set is 0, and for $\xi_i = 0$,

$$E[(W^i)^r] = \sum_{j=0}^{r}\binom{r}{j}(\eta_0^i)^j E[W^{r-j}]$$

where $\rho_i = \lambda\sigma^{b_i}$, $y_{\ell+j} = \eta_{\xi_i}^i + (\ell + j - \xi_i)\sigma^{b_m}$,

$\theta_1 \equiv e^{-\lambda\left(\eta_{\xi_i}^i + (\ell-\xi_i)\sigma^{b_m} - \eta_\ell^i\right)}$,

$\theta_2 \equiv \sum_{k=0}^{r-1}\frac{(-1)^k rk!}{\lambda^{k+1}}\binom{r-1}{k}\binom{\ell+k}{k}(\eta_\ell^i)^{r-k-1}$ and W is the stationary waiting time in an $M/D/1$ queue with service time equaling σ^{b_m} and arrival rate λ.

3. Proof of Theorem 1

The following proof is the same as the proof given in [8]. In [8], explicit expressions are introduced for moments and tail probability of stationary waiting times in two-node tandem queues with blocking where the second node's buffer capacity is finite while the first node has an infinite buffer. Since the proof provided there is still valid for a line production system with m nodes in a series under DBR policy in which a common buffer is completely shared by b nodes (up to the bottle-neck node), we are able to adopt the same procedure.

The proof is based on previous results shown in Corollary 3.1 of [1]. Corollary 3.1 has the following expression (6) instead of the last expression in (5) within the square bracket. The remaining terms are identical to the ones given in Theorem 1 with the exception of a few notations. It is clear that $a_i = \sigma^{b_m}(=\sigma^b)$ from (4).

$$\sum_{j=1}^{\xi_i-1-\ell} e^{-j\rho_m\lambda^j}\left\{p_j\left(\eta_{\xi_i}^i + (\ell+j-\xi_i)\sigma^{b_m}, \eta_{\ell+1}^i, \cdots, \eta_{\ell+j-1}^i\right) - p_j\left(\eta_{\ell+1}^i, \cdots, \eta_{\ell+j-1}^i\right)\right\} \tag{6}$$

Recall that we are assuming that the sequence of $\{D_n\}$ satisfies the structure (1). By letting $\eta_0^i = \sum_{j=1}^{i-1}\sigma^j$ and $D_n^i = \sum_{j=1}^{i-1}\sigma^j + n\sigma^{b_m} = \eta_0^i + (n-0)\sigma^{b_m}$ in (4), we can see that D_n^i, for all n, has $\xi_i = 0$ when $i \ge b$. Thus it is clear that this satisfies the second case of Theorem 1. When $i < b$, on the other hand, there exists a $\xi_i(\ge 1)$ such that

$$max\{n\sigma^1, \cdots, n\sigma^i\} \le \sum_{j=1}^{b}\sigma^j + (n-K)\sigma^{b_m} \text{ for } n \ge \xi_i.$$

For node i, the sequence of $\{D_n\}$ can be written with respect to ξ_i as follows: for $0 \le n < \xi_i$,

$$\begin{aligned} D_n^i &= \sum_{j=1}^{i-1}\sigma^j + max\{n\sigma^1, \ldots, n\sigma^i\} \\ &= \sum_{j=1}^{i-1}\sigma^j + n\sigma^{b_i} = \eta_n^i \end{aligned} \tag{7}$$

for $n \ge \xi_i$,

$$\begin{aligned} D_n^i &= \sum_{j=1}^{i-1}\sigma^j + \sum_{j=1}^{b}\sigma^j + (n-K)\sigma^{b_m} \\ &= \sum_{j=1}^{i-1}\sigma^j + \sum_{j=1}^{b}\sigma^j + (\xi_i - K)\sigma^{b_m} + (n-\xi_i)\sigma^{b_m} \\ &= \eta_{\xi_i}^i + (n-\xi_i)\sigma^{b_m} \end{aligned} \tag{8}$$

These cases satisfy the first case of Theorem 1 in which it is required to calculate the difference of two polynomial functions $p_j(\cdots)$ given in (6). These polynomial functions, however, cause computational difficulties. In the following proof, we use two useful properties of the polynomials introduced in [5, p. 146] to simplify the polynomials. As the polynomial $p_j(\cdots)$ has a 1-invariant property (see Property 1 in [5, p. 146]), the common term $\sum_{j=1}^{i-1}\sigma^j$ in (7) and (8) can be immediately dropped. Moreover, the polynomials can be eliminated from the alternative expression for the difference of the two polynomials given in [4, Theorem 10].

The two ranges in (6), $0 \le \ell \le \xi_i - 2$ and $1 \le j \le \xi_i - 1 - \ell$, lead to $1 \le \ell+1 \le \ell+j \le \xi_i - 1$. Hence, because of $\ell + j \le \xi_i$, we can see that all $\eta_\ell^i, \ldots, \eta_{\ell+j}^i$ are functions of σ^{b_i} (see (7)). By letting $y_{\ell+j} = \eta_{\xi_i}^i + (\ell + j - \xi_i)\sigma^{b_m}$, we can simplify the difference of the two polynomials in (6) as follows:

$$\begin{aligned} &p_j\left(y_{\ell+j},\eta^i_{\ell+1},\cdots,\eta^i_{\ell+j-1}\right)-p_j\left(\eta^i_{\ell+1},\cdots,\eta^i_{\ell+j-1}\right) \\ &=p_j\left(y_{\ell+j},(\ell+1)\sigma^{b_i},(\ell+2)\sigma^{b_i},\cdots,(\ell+j-1)\sigma^{b_i}\right) \\ &\quad -p_j\left((\ell+1)\sigma^{b_i},\cdots,(\ell+j)\sigma^{b_i}\right). \end{aligned}$$

From Property 5 in [5, p. 146], we can obtain the following expression (9).

$$\begin{aligned} \left(\sigma^{b_i}\right)^j\Big\{&p_j\left(\frac{y_{\ell+j}}{\sigma^{b_i}},(\ell+1),(\ell+2),\cdots,(\ell+j-1)\right) \\ &-p_j((\ell+1),\cdots,(\ell+j))\Big\} \end{aligned} \tag{9}$$

Then, with the help of Property 1 in [5, p. 146], (9) can be simplified into (10).

$$\left(\sigma^{b_i}\right)^j\left\{p_j\left(\frac{y_{\ell+j}}{\sigma^{b_i}}-\ell,1,2,\cdots,j-1\right)-p_j(1,\cdots,j)\right\} \tag{10}$$

Finally, the application of [4, Theorem 10] with $m=0$ to (10) yields the following explicit expression (11) without the polynomial terms $p_j(\cdots)$.

$$\left(\sigma^{b_i}\right)^j h_j\left(\frac{y_{\ell+j}}{\sigma^{b_i}}-\ell\right) \tag{11}$$

where $h_j(x)=\frac{x^j}{j!}-\frac{x^{j-1}}{(j-1)!}$ for $j\geq 1$. The proof is thus completed.

4. Examples and Simulations

As the explicit expression for higher moments given in Corollary 3.1 of [1] involves polynomial terms (see (6)), [1, Corollary 3.1] requires significantly more computational time compared to our new expression, Theorem 1, in which the polynomial terms are eliminated. Another advantage of our expression is that the computational time using our expression is almost insensitive to the system parameters.

In this section, we consider a DBR-controlled 5-node line production system with a common buffer capacity.

Table 1. The values of ξ_i

ξ_i	$K=10$	$K=20$	$K=30$	$K=40$	$K=50$	$K=100$
ξ_1	11	23	36	48	61	123
ξ_2	12	26	41	55	69	141
ξ_3	14	31	47	64	81	164
ξ_4	0	0	0	0	0	0
ξ_5	0	0	0	0	0	0

Table 2. The values of mean waiting times at each node

$E(W^i)$	$\lambda=0.15$		$\lambda=0.19$	
	Mean	Simulation	Mean	Simulation
$E(W^1)$	0.088587	0.08866 ∓3.2142E-04	7.633950	7.5730 ∓0.39221
$E(W^2)$	1.218083	1.2183 ∓4.8499E-04	8.783734	8.7233 ∓0.39159
$E(W^3)$	2.928894	2.9294 ∓7.9766E-04	10.541521	10.481 ∓0.39058
$E(W^4)$	12.00000	12.002 ∓0.02167	52.00000	51.894 ∓0.62697
$E(W^5)$	17.00000	17.002 ∓0.02167	57.00000	56.894 ∓0.62696

As we assume that the sequence of processing times is $\{1.0,1.5,2.0,5.0,2.5\}$, a bottleneck node is placed at node 4. Recall that a dummy node (node 0) is inserted at the foremost node with zero processing time and infinite capacity.

Table 1 shows the value of ξ_i for each node i when the input rate λ is 0.19 and the buffer capacity varies. From Table 1, it is clear that ξ_4 and ξ_5 are 0 for all K because, in this particular example, the bottleneck node is 4. For other cases, ξ_i increases exponentially along with K. As previously mentioned, because the polynomial function $p_k(\cdots)$ requires us to calculate the value of polynomial $p_{k-1}(\cdots)$ recursively, the large value of ξ_i generally needs much more computational time (see (6)).

Computational time using our proposed expression, however, is insensitive to the value of ξ_i. Table 2 shows the values of expected waiting times at each node when the buffer capacity K is 20 and the input rate λ varies. Compared to the simulation results conducted by ARENA 13, we see that all mean values obtained from our proposed formula are found within the 95% confidence interval.

5. Conclusion

Although we assume constant processing times at all nodes, obtaining closed-form expressions of moments of stationary waiting times in finite-capacity multi-node systems remains difficult. From a computational viewpoint, the previous results given in [1, 2, 4, 5] are helpful but are limited when applied to larger systems. In this study, we introduced an explicit expression for higher moments of stationary waiting times in DBR line production systems with m nodes. Although we did not demonstrate it here, the same approach is applicable for the expression of tail probability of stationary waiting times.

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