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NUMERICAL SOLUTIONS OF BURGERS EQUATION BY REDUCED-ORDER MODELING BASED ON PSEUDO-SPECTRAL COLLOCATION METHOD

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ABSTRACT. In this paper, a reduced-order modeling(ROM) of Burgers equations is studied based on pseudo-spectral collocation method. A ROM basis is obtained by the proper orthogonal decomposition(POD). Crank-Nicolson scheme is applied in time discretization and the pseudo-spectral element collocation method is adopted to solve linearlized equation based on the Newton method in spatial discretization. We deliver POD-based algorithm and present some numerical experiments to show the efficiency of our proposed method.

1. INTRODUCTION

When we try to get the approximate solutions in computing of some high order modeling problems such as fluid dynamics or real time feedback control problems, sometimes we meet very high cost of computations or even we feel a deficiency in our resources of computational environment. For these kinds of situations people have considered reduced order modeling which means deriving low order modeling instead of high order modeling using some tactics or methods, and one of which is the proper orthogonal decomposition method. The reason of why we think of proper orthogonal decomposition method in reduced order modeling is that we suppose the solutions of the main system of dynamics be represented by the linear combinations of a basis, and by POD-basis it is possible that the degrees of freedom of the main system of dynamics be reduced. There are many applications of proper orthogonal decomposition method in numerical simulations for some complex dynamic systems or high order modeling.([1], [2], [3], [4])

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Burgers equation is a simplified version of Navier-Stokes equations, which is considered as a turbulence free cartoon for Navier-Stokes equations([5]). Let Ω be $(0,1)$. We consider the one dimensional homogeneous Burgers equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + u(t,x)\dfrac{\partial u}{\partial x}(t,x) - \nu\dfrac{\partial^2 u}{\partial x^2}(t,x) &= f(t,x) \quad \text{in } \Omega\times[0,T], \\ u(t,x) &= 0 \quad \text{on } \partial\Omega\times[0,T], \\ u(0,x) &= u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where ν is a viscosity diffusion coefficient, $T > 0$, u_0 is a given continuous initial condition and f is a continuous forcing term. There are many researches about Burgers equation by finite element approximations. ([6], [7], [8], [9]) Here we provide pseudo-spectral collocation approximation for Burgers equation and extend it to spectral element collocation method using the idea given in ([6], [10]). The spectral method is known to be a very accurate method so that it is worthwhile to develop the Legendre/Chebyshev pseudo-spectral collocation method for solving Burgers equation. In this paper we provide an algorithm for Legendre case. One may easily get an algorithm for Chebyshev case. Then we investigate reduced order modeling using POD-basis in order to reduce the computational cost of pseudo-spectral element collocation approximation.

Our main goal is focused on how to get a better reduced order solution by questioning how many generators of the POD-basis would be suitable for the best or for the most efficient approximation to the solutions in ROM. In this context also we study about getting 'a smallest but best adequate' sets of snapshots from which we organize POD-basis. Of course, as the much abundant set of snapshots we prepare and start from, the more effective set of POD generators we can obtain. And as the size of the POD-basis is increasing, the same is the size of the reduced order system. For this reason 'choosing how many generators of a given POD-basis' is not less important in ROM.

Burgers equation is a well-known nonlinear partial differential equation. We apply the Newton's method to get the linearized systems and we use implicit time step Crank-Nicolson scheme to discretize time step solution. We examine the numerical efficiency of POD-used ROM tactics and idea. As it is mentioned above, we mainly focus on getting an efficient and compact set of snapshots and choosing reasonable set of POD generator functions. For the organization of POD-basis we apply singular value decomposition(SVD) which have actively been used in many areas of scientific research.([3], [4], [11]) When we apply SVD to derive ROM basis set, we regard the sample space is identified with snapshots which be rearranged as column-wisely into a rectangular matrix. See [2],[3], [4], [11], [12] for more details.

For the numerical simulation of ROM we approximate the system and solutions by pseudo-spectral element collocation method for the spatial discretization and we apply Newton's iteration scheme to get linearized system, which is given in section 2, and section 3 is organized for some short preliminaries about proper orthogonal decomposition and the actual application of ROM to the system of pseudo-spectral method given in section 2. And the numerical experiments are in section 4.

2. PSEUDO-SPECTRAL COLLOCATION METHOD

In this paper, we provide an algorithm to solve Burgers equations (1.1) based on Legendre spectral collocation method using Legendre-Gauss-Lobatto points as the collocation points. Then it will be easily extended to one dimensional spectral element method using the idea given in [13].

Let $\Omega = (0,1)$ and let N be a positive integer. Let P_N^0 be the space of all polynomials defined on $[0,1]$ of degree less than or equal to N satisfying the homogeneous boundary conditions. Denote by $\Pi = \{\xi_i\}_{i=0}^N$ the Legendre-Gauss-Lobatto(LGL) points which are the zeros of $(1-x^2)L_N'(x)$ where L_N is the N-th Legendre polynomial. The spatial semi-discrete pseudo-spectral collocation approximate problem of (1.1) can be written as follows([14], [15], [16]): for each $t \in [0,T]$ find a function $u_N(t,\cdot) \in P_N^0$ such that

$$\begin{cases} \dfrac{d}{dt}u_N(t,\xi) + u_N(t,\xi)\,\partial_N u_N(t,\xi) - \nu\,\partial_N^2 u_N(t,\xi) &= f(t,\xi), \quad \xi \in \Pi, \\ u_N(t,0) = u_N(t,1) &= 0, \\ u_N(0,\xi) &= u_0(\xi), \quad \xi \in \Pi \end{cases} \tag{2.1}$$

where ∂_N denotes by the Legendre pseudo-spectral differentiation.

Now, we apply the implicit time step Crank-Nicolson scheme to solve each time step solution. Let M be a positive integer and set

$$t_n := n\Delta t \quad (n = 0,1,2,\cdots,M) \quad \text{with} \quad \Delta t = \frac{T}{M}.$$

Denote by $u_N^n(\cdot) = u_N(t_n,\cdot)$. Now, we have the full discretized approximation for solving Burgers equation (1.1) as follows: for all $\xi \in \Pi$,

$$\begin{aligned} \frac{u_N^{n+1} - u_N^n}{\Delta t} &= \frac{1}{2}\Big(\nu\,\partial_N^2 u_N^{n+1} - u_N^{n+1}\partial_N u_N^{n+1}\Big) \\ &\quad + \frac{1}{2}\Big(\nu\,\partial_N^2 u_N^n - u_N^n \partial_N u_N^n\Big) + \frac{1}{2}\Big(f^{n+1} + f^n\Big). \end{aligned}$$

On the other hand, considering the linearization of the above equation by Newton iteration method, we define

$$\mathcal{N}_N(u_N^{n+1}) = \Big(-\nu\,\partial_N^2 + \frac{2}{\Delta t}\Big)u_N^{n+1} + u_N^{n+1}\partial_N u_N^{n+1} - \mathcal{F}_N$$

where

$$\mathcal{F}_N = \Big(\nu\,\partial_N^2 + \frac{2}{\Delta t}\Big)u_N^n - u_N^n \partial_N u_N^n + f^{n+1} + f^n.$$

Then the full discretized problem of the Burgers equation (1.1) is to find an approximate solution $u_N^{n+1} \in P_N^0$ for $(n+1)$th time-step of the following nonlinear equation:

$$\mathcal{N}_N(u_N^{n+1}) = 0. \tag{2.2}$$

The Newton iteration to solve the above nonlinear equation is as follow: with $u_{N,0}^{n+1} = u_N^n$

$$\mathcal{A}_N\big(u_{N,k}^{n+1}\big)\,u_{N,k+1}^{n+1} = u_{N,k}^{n+1}\,\partial_N u_{N,k}^{n+1} + \mathcal{F}_N \quad (k = 1,2,\cdots) \tag{2.3}$$

where $u_{N,k}^n$ denotes the k-th Newton iteration of u_N^n and

$$\mathcal{A}_N\Big(u_{N,k}^{n+1}\Big)\, u_{N,k+1}^{n+1} = \Big(-\nu\,\partial_N^2 + \frac{2}{\Delta t}\Big) u_{N,k+1}^{n+1} + u_{N,k}^{n+1}\Big(\partial_N u_{N,k+1}^{n+1}\Big) + \Big(\partial_N\, u_{N,k}^{n+1}\Big)\, u_{N,k+1}^{n+1}.$$

Let D_N be the Legendre pseudo-spectral differentiation matrix and let $u_N^n(x) = \sum_{j=1}^{N-1} u_N^n(\xi_j)\,\phi_j(x) \in P_N^0$ where ϕ_j $(j = 0, 1, \cdots, N)$ are the Lagrange basis functions with respect to LGL-points Π. We denote by $\mathbf{v} = \big(\, v(\xi_1), v(\xi_2), \cdots, v(\xi_{N-1})\,\big)^T$ the vector containing the nodal values of a continuous function v. Then the discretized problem of the Newton iteration (2.3) is to find $\mathbf{u}_{N,k+1}^{n+1}$ satisfying

$$A_{N,k}^{n+1}\,\mathbf{u}_{N,k+1}^{n+1} = \mathrm{diag}(\mathbf{u}_{N,k}^{n+1})\, D_N\,\mathbf{u}_{N,k}^{n+1} + F_N^{n+1} \tag{2.4}$$

where I_N denotes the identity matrix,

$$A_{N,k}^{n+1} = -\nu\,(D_N)^2 + \frac{2}{\Delta t} I_N + \mathrm{diag}(\mathbf{u}_{N,k}^{n+1})\, D_N + \mathrm{diag}\,\Big(D_N\,\mathbf{u}_{N,k}^{n+1}\Big)$$

and

$$F_N^{n+1} = \Big(\nu\,(D_N)^2 + \frac{2}{\Delta t}\, I_N\Big)\mathbf{u}_N^n - \mathrm{diag}(\mathbf{u}_N^n)\, D_N\,\mathbf{u}_N^n + \mathbf{f}^{n+1} + \mathbf{f}^n.$$

In order to extend spectral collocation method to spectral element collocation method the domain is decomposed into multiple sub-domains and then the collocation solution is sought in each sub-domain. We follow the idea given in [13] and explain briefly for readers. First, we collocate the Burgers equation at the interior LGL-points of each sub-domain and then we impose two interface jump conditions at each interface point $\xi_N^r = \xi_0^{r+1}$ of neighborhood sub-domains Ω_r and Ω_{r+1} :

$$\text{continuity of function } u_N^n: \quad u_{N,r}^n(\xi_N^r) = u_{N,r+1}^n(\xi_N^r),$$

and

$$\text{continuity of normal derivative } u_N^n: \quad \partial_N u_{N,r}^n(\xi_N^r) = \partial_N u_{N,r+1}^n(\xi_N^r),$$

where $u_{N,r}^n$ denotes the restriction of u_N^n on sub-domain Ω_r. See [13] for more details.

3. Reduced order modeling by POD

3.1. **POD and SVD.** The POD provides a basis for the modal decomposition of an ensemble of functions, such as data obtained in the course of experiments or numerical simulations. For example, suppose there exists numerical solutions of a time dependent partial differential equation. The set of numerical solutions of discretized systems composes a snapshot matrix, in which the columns of the matrix are numerical solutions for each time step. From the matrix, we can find the numerical modal basis of which the linear combinations yield the column space of the matrix. See [2], [4], [12], [17] and reference therein.

Here we recall the idea of POD given in [12] as follows. The main idea of the POD is to find a set of ordered orthonormal basis vectors such that the snapshots can be expressed optimally using the selected first p basis vectors. The mean square error can be used as a measure for the optimal problem, i.e., $E\{\parallel \mathbf{x} - \mathbf{x}(p) \parallel^2\}$, where $\mathbf{x}(p)$ is an approximate expression of a random vector $\mathbf{x}$ using the first p basis vectors of the undetermined set of orthonormal basis

vectors in $\mathbb{R}^N$. The objective of POD is to find a set of orthonormal vectors $\{\phi_i\}_{i=1}^m$ in $\mathbb{R}^N$ which minimizes the error $E\{\| \mathbf{x} - \mathbf{x}(p) \|^2\}$, i.e.,

$$\min_{\phi_i} \epsilon^2(l) = E\{\| \mathbf{x} - \mathbf{x}(p) \|^2\}$$

$$s.t. \quad \phi_i^T \phi_j = \delta_{ij},\ i,j = 1,2,...,m,$$

where $\mathbf{x}(p) = \sum_{i=1}^{p} y_i \phi_i$ $(p \le m)$.

There are three kinds of equivalent POD methods: Principal Component Analysis(PCA), Karhunen-Loeve Decomposition(KLD), Singular-Value Decomposition(SVD). We will use SVD in this paper.

Suppose that N_s snapshots $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N_s} \in \mathbb{R}^{N_d}$ are given. We want to find approximations $\mathbf{x}_i(p)$ of $\mathbf{x}_i$ $(i = 1, 2, \cdots, N_s)$ using only $p(\le \min\{N_d, N_s\})$ orthonormal basis vectors. The POD method of order p is a method to find p basis vectors which minimizes the total error

$$\epsilon^2(p) \ = \ \sum_{i=1}^{N_s} \| \mathbf{x}_i - \mathbf{x}_i(p) \|^2 \ .$$

Applying the SVD algorithm to $X = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_{N_s}]$ produces an $N_d \times N_s$ diagonal matrix Σ of singular values in decreasing order and unitary matrices $V \in \mathbb{R}^{N_d} \times \mathbb{R}^{N_d}$ and $U \in \mathbb{R}^{N_s} \times \mathbb{R}^{N_s}$ satisfying $X = V\,\Sigma\,U^T$, where

$$\Sigma = \begin{bmatrix} \Sigma_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \mathbf{0} \\ & & & \sigma_m & \\ \hline & & \mathbf{0} & & \mathbf{0} \end{array}\right], \quad (\sigma_j > 0,\ j = 1, 2, \cdots, m).$$

If we denote the columns of the matrix $\Sigma\,U^T$ as $\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_{N_s} \in \mathbb{R}^{N_d}$, then it is easily shown that $\mathbf{x}_i = V\,\mathbf{d}_i$ $(i = 1, 2, \cdots, N_s)$ and the components $d_{i,m+1}, d_{i,m+2}, \cdots, d_{i,N_s}$ of $\mathbf{d}_i$ are equal to zeros. Hence we have $\mathbf{x}_i = \sum_{k=1}^{m} d_{i,k}\,\mathbf{v}_k$ where $V = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_{N_d}]$. This means that each snapshot $\mathbf{x}_i$ can be represented by the first m columns of the matrix V. The first m columns of the matrix V is called the POD-basis and the first p columns of V, denote by V_p, is called the POD-basis of order p, $(p \le m)$. In this case, the total error is given by $\epsilon(p)^2 = \sum_{j=p+1}^{m} \sigma_j^2$. See [12].

3.2. **Approximate solution by POD-basis of order p.** In this paper we use the discrete version of POD method as follows. Let N_d be the dimension of approximate spatial space to solve Burgers equation, e.g., spectral element space with d sub-domains. Using the spectral element collocation method, the problem (2.4) is reformulated as follows.

Find the $(k+1)$-th Newton iteration $\mathbf{u}_{k+1}^{n+1} \in \mathbb{R}^{N_d}$ at $(n+1)$-th time step satisfying

$$A_{N_d,k}^{n+1}\,\mathbf{u}_{k+1}^{n+1} = \mathrm{diag}(\mathbf{u}_k^{n+1})\,D_{N_d}\,\mathbf{u}_k^{n+1} + F_{N_d}^{n+1} \tag{3.1}$$

where D_{N_d} denotes the differentiation matrix,

$$A_{N_d,k}^{n+1} = -\nu\,(D_{N_d})^2 + \frac{2}{\Delta t} I_{N_d} + \mathrm{diag}(\mathbf{u}_k^{n+1})\,D_{N_d} + \mathrm{diag}\left(D_{N_d}\,\mathbf{u}_k^{n+1}\right)$$

and

$$F_{N_d}^{n+1} = \Big(\nu\,(D_{N_d})^2 + \frac{2}{\Delta t}\,I_{N_d}\Big)\mathbf{u}^n - \mathrm{diag}(\mathbf{u}^n)\,D_{N_d}\,\mathbf{u}^n + \mathbf{f}^{n+1} + \mathbf{f}^n.$$

Suppose that we have N_s snapshots $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N_s} \in \mathbb{R}^{N_d}$, in which the snapshots can be found by equation (3.1) with an appropriate time space Δt. Using the SVD algorithm produces a diagonal matrix Σ and unitary matrices V and U satisfying

$$X = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_{N_s}] = V\,\Sigma\,U^T$$

so that we can take a POD-basis matrix, $V_p = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_p]$, of order p ($p \le \mathrm{rank}(\Sigma)$).

Replacing the approximate solution $\mathbf{u}_{k+1}^{n+1}$ in equation (3.1) by $V_p\,\mathbf{w}_{k+1}^{n+1}$ with $\mathbf{w}_{k+1}^{n+1} \in \mathbb{R}^p$, we are led to POD-basis method of order p: find $\mathbf{w}_{k+1}^{n+1} \in \mathbb{R}^p$ satisfying

$$A_{p,k}^{n+1}\,\mathbf{w}_{k+1}^{n+1} = V_p^T\,\mathrm{diag}(V_p\,\mathbf{w}_k^{n+1})\,\tilde{D}_p\,\mathbf{w}_k^{n+1} + F_p^{n+1} \tag{3.2}$$

where $A_{p,k}^{n+1} \in \mathbb{R}^{p\times p}$ and $F_p^{n+1} \in \mathbb{R}^p$ are given by

$$A_{p,k}^{n+1} = -\nu\,\tilde{D}_{pp} + \frac{2}{\Delta t}\,I_p + V_p^T\,\mathrm{diag}(V_p\,\mathbf{w}_k^{n+1})\,\tilde{D}_p + V_p^T\,\mathrm{diag}\left(\tilde{D}_p\,\mathbf{w}_k^{n+1}\right)V_p$$

and

$$F_p^{n+1} = \Big(\nu\,\tilde{D}_{pp} + \frac{2}{\Delta t}\,I_p\Big)\mathbf{w}^n - V_p^T\,\mathrm{diag}(V_p\,\mathbf{w}^n)\,\tilde{D}_p\,\mathbf{w}^n + V_p^T\left(\mathbf{f}^{n+1} + \mathbf{f}^n\right)$$

with $\tilde{D}_p = D_{N_d}\,V_p$ and $\tilde{D}_{pp} = V_p^T\,D_{N_d}^2\,V_p$.

In short, the algorithm of POD-basis method of order p can be described as follows.

[Algorithm for POD-basis method of order p]

Initialize

N_t the number of time step and K the number of Newton iterations

Set $\mathbf{w}^0 = V^T\,\mathbf{u}^0, \quad \Delta t = T/N_t$

$\tilde{D}_p = D_{N_d}\,V_p, \quad \tilde{D}_{pp} = V_p^T\,D_{N_d}^2\,V_p, \quad A_0 = -\nu\,\tilde{D}_{pp} + 2/\Delta t$

For each time step $n = 0, 1, 2, \cdots, N_t$

Set $F_0 = \big(-A_0 + 4/\Delta t\big)\mathbf{w}^n - V_p^T\,\mathrm{diag}(V_p\,\mathbf{w}^n)\,\tilde{D}_p\,\mathbf{w}^n + V_p^T\left(\mathbf{f}^{n+1} + \mathbf{f}^n\right)$

$\mathbf{w}_0^{n+1} = \mathbf{w}^n$

For each Newton iteration $k = 0, 1, 2, \cdots, K-1$

Set $B_0 = V_p^T\,\mathrm{diag}(V_p\,\mathbf{w}_k^{n+1})\,\tilde{D}_p$

$A_p = A_0 + B_0 + V_p^T\,\mathrm{diag}\left(\tilde{D}_p\,\mathbf{w}_k^{n+1}\right)V_p$

$F_p = F_0 + B_0\,\mathbf{w}_k^{n+1}$

Find

$\mathbf{w}_{k+1}^{n+1}$ satisfying $A_p\,\mathbf{w}_{k+1}^{n+1} = F_p$

End
Set
$$\mathbf{w}^{n+1} = \mathbf{w}_K^{n+1}$$
$$\mathbf{u}^{n+1} = V_p\, \mathbf{w}^{n+1}$$
End

4. NUMERICAL EXPERIMENTS

In this section, we present numerical results of pseudo-spectral element collocation method with comparison between exact solutions and the reduced order approximate solutions. In our numerical experiments, we divide the spatial domain $\Omega = (0, 1)$ into two sub-domains $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ and use polynomials of degree $N = 17$ for each sub-domain to apply pseudo-spectral element collocation method. Then the system size of the pseudo-spectral element collocation method is $N_d = 36$. To show the accuracy, we provide the numerical discrete relative L_2-error between the exact solution $u(t)$ and full-basis approximation $u_N(t)$ or POD-basis approximation $u_p(t)$:

$$E(u, u_p) = \frac{\sqrt{\sum_{n=0}^{N_t} \|u(t_n) - u_p(t_n)\|_N^2 \, \Delta t}}{\sqrt{\sum_{n=0}^{N_t} \|u(t_n)\|_N^2 \, \Delta t}}$$

where $\|\cdot\|_N$ denotes the spectral discrete norm with LGL nodes.

Example 1. We consider a Burgers equation (1.1) with zero force term $f(t, x) = 0$ and initial condition given by:

$$u_0(x) = \begin{cases} 1 & \text{in } \left(0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

By Hopf-Cole transformation([11], [13]) the exact solution is given

$$u(t, x) = 2\pi\nu \frac{\displaystyle\sum_{n=1}^{\infty} n A_n \exp(-\nu n^2 \pi^2 t) \sin(n\pi x)}{\displaystyle A_0 + \sum_{n=1}^{\infty} A_n \exp(-\nu n^2 \pi^2 t) \cos(n\pi x)}$$

where

$$A_0 = -2\nu \exp\left(-\frac{1}{4\nu}\right) + 2\nu + \frac{1}{2} \exp\left(-\frac{1}{4\nu}\right),$$

$$A_n = -\frac{4\nu}{1 + 4\nu^2 n^2 \pi^2} \left(\exp\left(-\frac{1}{4\nu}\right) \cos\left(\frac{n\pi}{2}\right) - 1 - 2\nu n\pi \exp\left(-\frac{1}{4\nu}\right) \sin\left(\frac{n\pi}{2}\right) \right)$$
$$+ \exp\left(-\frac{1}{4\nu}\right) \left(\frac{2}{n\pi} \sin(n\pi) - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right).$$

We set $\nu = 1$ and $T = 0.2$, and take the time-step size $\Delta t = 1/3000$. In this example, we take POD-basis of order $p = 15$ using all approximate solutions for each time step. In Figure 1, one may see that the error between the exact solution and POD-basis approximate solution is very near to the error between the exact solution and full-basis approximate solution. Also the error between full-basis approximate solution and POD-basis approximate solution is very small.

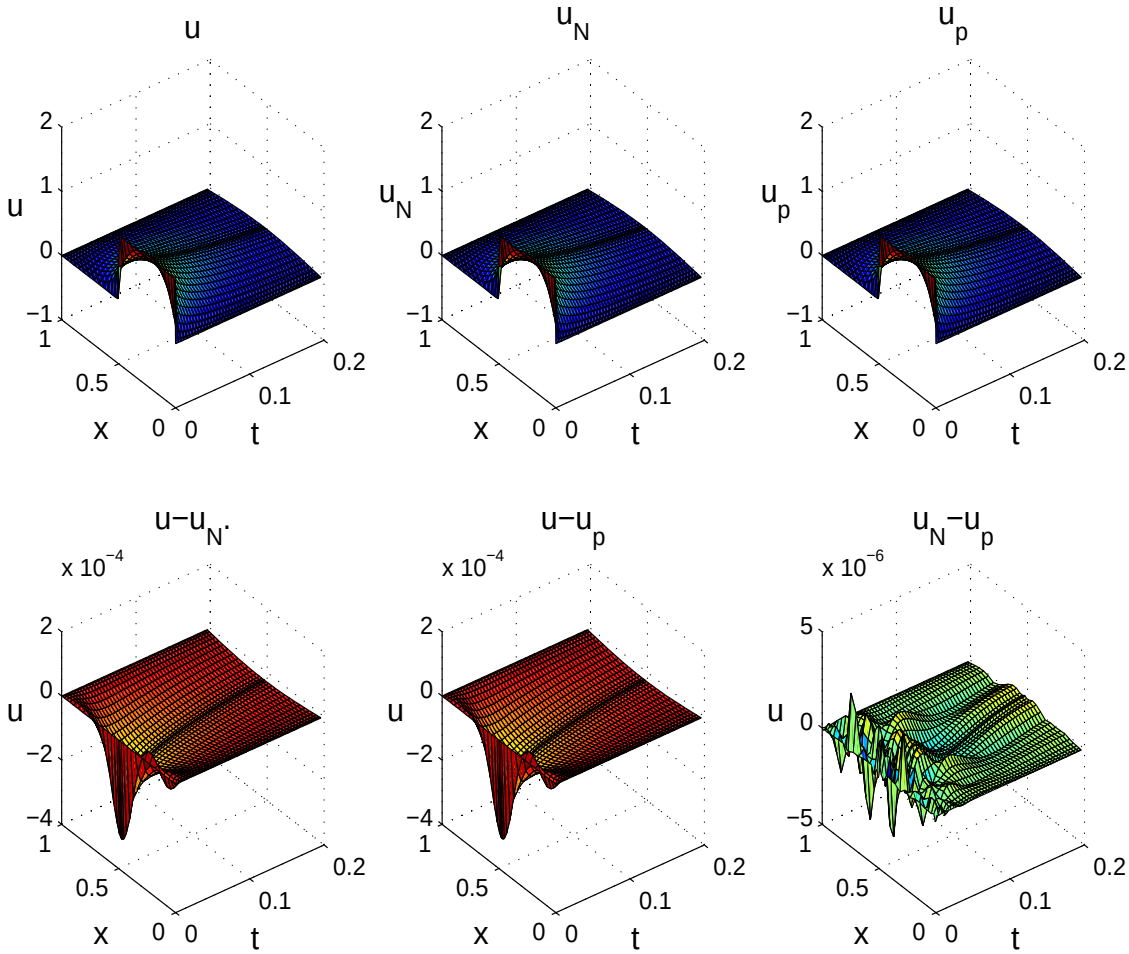

FIGURE 1. Exact, approximate and POD solutions(top), and their errors(bottom).

Example 2. Now, let us address to a forced-Burgers equation (1.1) with following exact solution:

$$u(t, x) = \exp(-ct^2)\sin(\pi x),$$

TABLE 1. $E(u, u_N)$ for various Δt and ν with $T = 1$ and $N = 17$.

	$\nu = 1$	$\nu = 0.5$	$\nu = 0.00001$	$\nu = 0.0000001$
$\Delta t = T/N_t$	$E(u, u_N)$	$E(u, u_N)$	$E(u, u_N)$	$E(u, u_N)$
0.02	$1.20e-005$	$2.13e-005$	$7.09e-005$	$7.09e-005$
0.01	$3.00e-006$	$5.37e-006$	$1.79e-005$	$1.79e-005$
0.005	$7.53e-007$	$1.35e-006$	$4.51e-006$	$4.51e-006$
0.0025	$1.89e-007$	$3.37e-007$	$1.13e-006$	$1.13e-006$
0.0013	$4.72e-008$	$8.43e-008$	$2.83e-007$	$2.83e-007$

where c is a constant. The forcing term is given by

$$f(t,x) = -2ct\exp(-ct^2)\sin(\pi x)+\pi^2\nu\exp(-ct^2)\sin(\pi x)+\pi\{\exp(-ct^2)\}^2\sin(\pi x)\cos(\pi x).$$

We set $c = 1$. The Newton iteration algorithm with Crank-Nicolson scheme given in section 2 is used to approximate the nonlinear problem for a wide range of viscosity ν. In Table 1, we provide the errors between the exact solutions and the full-basis approximations with various Δt and ν for $T = 1$ and $N = 17$.

Now, we are going to approximate the POD-basis solutions. The ROM solution $u_p(t_k)$ of each time step $t_k = k\,\Delta t$ ($k = 1, 2, \cdots, N_t$) with $\Delta t = T/N_t$ is computed as follows. First we take N_s-snapshots of full-based solutions with time step $\Delta t_s = T/N_s$. Then we compute p-POD basis matrix V_p of order p from N_s-snapshots and compute N_t POD-basis solutions using the p-POD basis. The smaller the number N_s is, the smaller the total cost of computing POD-basis solutions is.

In Figure 2, the full ranks of snapshot matrix are increasing as the viscosities are decreasing. However the ranks seem to be bounded as the number of snapshots N_s. It means that it is sufficient to take at most $N_s = 20$ or 30.

If we assume that the snapshot matrix has rank m ($p \le m$) and nonzero singular values σ_j ($j = 1, 2, \cdots m$), then the total error between POD-basis solutions and snapshot solutions is given by

$$\epsilon^2(p) = \sum_{i=1}^{N_s} \|u_{N_s}(t_i) - u_p(t_i)\|^2 = \sum_{j=p+1}^{m} \sigma_j^2.$$

One may easily take an efficient POD-basis of order p considering the size of singular values for the snapshot matrix. In Table 2, the most dominant component of the POD-basis is the first one and an efficient order p of POD-basis can be 2 or 3.

In Table 3-5, we provide numerical experiments for the number of snapshots, $N_s = 5$. with two different viscosities $\nu = 1,\ 0.0001$. Considering the size of singular values given in Table 3, we take $p = 2$, the order of POD-basis, for the viscosity $\nu = 1$ and $p = 3$ for $\nu = 0.0001$. The relative errors $E(u, u_N)$ and $E(u, u_p)$ and CPU-time to compute each solution for various

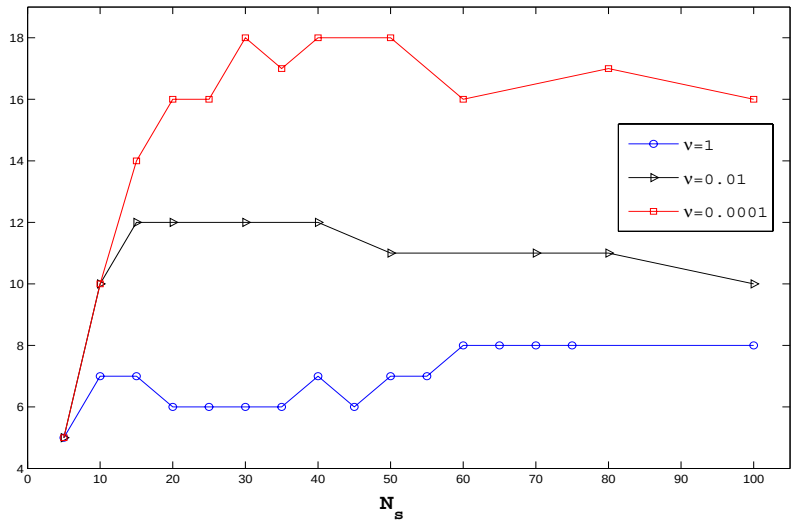

FIGURE 2. The full rank of the snapshot matrix with $T = 1$, $N = 17$ for Example 2.

TABLE 2. Singular values for various N_s with $\nu = 1, T = 1, N = 17$.

	$N_s = 5$	$N_s = 10$	$N_s = 15$	$N_s = 20$
σ_1	$7.85e+000$	$1.07e+001$	$1.30e+001$	$1.49e+001$
σ_2	$2.64e-004$	$8.57e-005$	$4.53e-005$	$2.90e-005$
σ_3	$2.42e-006$	$9.46e-007$	$5.25e-007$	$3.44e-007$
σ_4	$1.18e-008$	$4.90e-009$	$2.77e-009$	$1.83e-009$
σ_5	$1.96e-011$	$2.16e-011$	$1.32e-011$	$9.02e-012$
σ_6		$2.60e-013$	$1.73e-013$	$3.47e-013$
σ_7		$8.80e-014$	$7.34e-014$	

time steps are provided in Table 4 for $\nu = 1$ and in Table 5 for $\nu = 0.0001$. These experiments show that using POD-basis method is very efficient in the point of computing cost.

In Table 6-8, we also provide numerical experiments for the number of snapshots, $N_s = 15$, similarly. From the size of singular values given in Table 6, we also take $p = 2$ for the viscosity $\nu = 1$ and $p = 3$ for $\nu = 0.0001$. The numerical results given in Table 7-8 are similar to those for the case $N_s = 5$. But, increasing the number of snapshots N_s from 5 to 15, we had a little bit improved numerical results in the point of accuracy.

TABLE 3. Singular values σ_j, $j = 1, 2, \cdots$, for $N_s = 5$, $T = 1$, $N = 17$.

$\nu \diagdown \sigma_j$	σ_1	σ_2	σ_3	σ_4	σ_5
1	$7.85e+00$	$2.64e-04$	$2.42e-06$	$1.18e-08$	$1.96e-011$
0.0001	$7.86e+00$	$1.85e-02$	$1.17e-03$	$7.34e-05$	$2.45e-006$

TABLE 4. Relative errors for $\nu = 1$, $T = 1$, $N = 17$, $N_s = 5$, $p = 2$.

$\Delta t = T/N_t$	$[u(t) - u_N(t)]$		$[u(t) - u_p(t)]$	
	$E(u, u_N)$	CPU time	$E(u, u_p)$	CPU time
0.02	$1.20e-005$	1.68(s)	$1.22e-005$	0.41(s)
0.01	$3.01e-006$	3.01(s)	$3.25e-006$	0.50(s)
0.005	$7.54e-007$	5.71(s)	$1.00e-006$	0.39(s)
0.0025	$1.89e-007$	11.31(s)	$4.59e-007$	0.64(s)
0.00125	$4.72e-008$	22.51(s)	$3.32e-007$	0.95(s)

TABLE 5. Relative errors for $\nu = 0.0001$, $T = 1$, $N = 17$, $N_s = 5$, $p = 3$.

$\Delta t = T/N_t$	$[u(t) - u_N(t)]$		$[u(t) - u_p(t)]$	
	$E(u, u_N)$	CPU time	$E(u, u_p)$	CPU time
0.02	$7.31e-005$	2.17(s)	$6.22e-005$	0.36(s)
0.01	$1.82e-005$	4.35(s)	$1.54e-005$	0.58(s)
0.005	$4.54e-006$	8.27(s)	$1.64e-005$	0.44(s)
0.0025	$1.13e-006$	16.58(s)	$1.84e-005$	0.86(s)
0.00125	$2.84e-007$	33.70(s)	$1.90e-005$	0.78(s)

TABLE 6. Singular values σ_j, $j = 1, 2, \cdots$, for $N_s = 15$, $T = 1$, $N = 17$.

$\nu \diagdown \sigma_j$	σ_1	σ_2	σ_3	σ_4	σ_5
1	$1.30e+01$	$4.53e-05$	$5.25e-07$	$2.77e-09$	$1.32e-011$
0.0001	$1.30e+01$	$4.31e-03$	$3.75e-04$	$4.67e-05$	$7.49e-006$

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TABLE 7. Relative errors for $\nu = 1$, $T = 1$, $N = 17$, $N_s = 15$, $p = 2$.

$\Delta t = T/N_t$	$[u(t) - u_N(t)]$		$[u(t) - u_p(t)]$	
	$E(u, u_N)$	CPU time	$E(u, u_p)$	CPU time
0.02	$1.20e-005$	1.68(s)	$1.20e-005$	0.73(s)
0.01	$3.01e-006$	3.01(s)	$3.04e-006$	0.72(s)
0.005	$7.54e-007$	5.71(s)	$7.89e-007$	0.78(s)
0.0025	$1.89e-007$	11.31(s)	$2.25e-007$	0.95(s)
0.00125	$4.72e-008$	22.51(s)	$8.55e-008$	1.23(s)

TABLE 8. Relative errors for $\nu = 0.0001$, $T = 1$, $N = 17$, $N_s = 15$, $p = 3$.

$\Delta t = T/N_t$	$[u(t) - u_N(t)]$		$[u(t) - u_p(t)]$	
	$E(u, u_N)$	CPU time	$E(u, u_p)$	CPU time
0.02	$7.31e-005$	2.17(s)	$6.82e-005$	1.03(s)
0.01	$1.82e-005$	4.35(s)	$1.43e-005$	1.01(s)
0.005	$4.54e-006$	8.27(s)	$6.91e-006$	0.92(s)
0.0025	$1.13e-006$	16.58(s)	$8.24e-006$	1.15(s)
0.00125	$2.84e-007$	33.70(s)	$8.77e-006$	1.23(s)

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