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OPTIMALITY CONDITIONS AND DUALITY IN NONDIFFERENTIABLE ROBUST OPTIMIZATION PROBLEMS

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Abstract. We consider a nondifferentiable robust optimization problem, which has a maximum function of continuously differentiable functions and support functions as its objective function, continuously differentiable functions as its constraint functions. We prove optimality conditions for the nondifferentiable robust optimization problem. We formulate a Wolfe type dual problem for the nondifferentiable robust optimization problem and prove duality theorems.

1. Introduction

A standard form of nonlinear programming problem with inequality constraints

$$\text{(P)} \quad \inf_{x\in\mathbb{R}^n}\{f(x) : g_i(x) \leq 0,\ i = 1,\cdots,m\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. The problem in the face of data uncertainty in the constraints can be captured by the following nonlinear programming problem:

$$\text{(UP)} \quad \inf_{x\in\mathbb{R}^n}\{f(x,u) : g_i(x,v_i) \leq 0,\ i = 1,\cdots,m\},$$

where u, v_i are uncertain parameters and $u \in U$, $v_i \in V_i, i = 1,\cdots,m$ for some convex compact sets $U \subset \mathbb{R}^p, V_i \subset \mathbb{R}^q, i = 1,\cdots,m$, respectively and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i = 1,\cdots,m$ are continuously differentiable. Sometimes, $f(x,u)$ in (UP) can be $f(x)$ without the uncertain parameter $u \in U$. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty ([1] – [4], [6]), associates with the uncertain program (UP) its robust counterpart [5],

$$\text{(RP)} \quad \inf_{x\in\mathbb{R}^n}\{\max_{u\in U} f(x,u) : g_i(x,v_i) \leq 0,\ \forall v_i \in V_i,\quad i = 1,\cdots,m\},$$

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where the uncertain constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets U, $V_i,\ i = 1, \cdots, m$.

Recently, Kuroiwa and Lee [9] extend the necessary optimality theorem to a multiobjective robust optimization problem. Furthermore, Kim [8] extend the robust duality theorems to a multiobjective robust optimization problem.

Now, we consider nondifferentiable robust optimization problem:

$$\text{(NRP)} \quad \inf_{x\in\mathbb{R}^n} \{\max_{u\in U} f(x,u) + s(x|C) : g_i(x,v_i) \leq 0,\ \forall v_i \in V_i,\quad i = 1, \cdots, m\}.$$

Let $F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0,\ \forall v_i \in V_i,\ i = 1, \cdots, m\}$ be the robust feasible set.

Let $\bar{x} \in F$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in V_j \text{ s.t. } g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \backslash J_1(\bar{x})$. Let C be a compact convex set of $\mathbb{R}^n$ and $s(x|C) = \max\{x^T y \mid y \in C\}$. Let $k(x) = s(x|C)$. Then k is a convex function and $\partial k(x) = \{w \in C \mid w^T x = s(x|C)\}$, where ∂k is the subdifferential of k. For a continuously differentiable function $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, we use $\nabla_1 g$ to denote the derivative of g with respect to the first variable.

We say that $\bar{x}$ is a robust solution of (NRP) if $\bar{x}$ is a minimizer of (NRP), that is, $\bar{x} \in F$ and $\max_{u\in U} f(x,u) + s(x|C) \geq \max_{u\in U} f(\bar{x},u) + s(\bar{x}|C)\ \forall x \in \mathbb{R}^n,\ u \in U$.

In this paper, we consider a nondifferentiable robust optimization problem, which has a maximum function of continuously differentiable functions and support functions as its objective function, and continuously differentiable functions as its constraint functions. We prove optimality conditions for the nondifferentiable robust optimization problem. We formulate a Wolfe type dual problem for the nondifferentiable robust optimization problem and prove duality theorems.

2. Optimality Theorems

In this section, we give necessary, and sufficient optimality conditions for the nondifferentiable robust optimization problem (NRP).

Lemma 2.1. [11] *Let Θ be a nonempty, compact topological space and let $h : \mathbb{R}^n \times \Theta \to \mathbb{R}$ be such that $h(\cdot, \theta)$ is differentiable for every $\theta \in \Theta$ and $\nabla_1 h(x, \theta)$ is continuous on $\mathbb{R}^n \times \Theta$. Let $\phi(x) = \sup_{\theta\in\Theta} h(x, \theta)$. Define $\bar{\Theta}(x)$ to be $\bar{\Theta}(x) := arg \max_{\theta\in\Theta} h(x, \theta)$. Then the function $\phi(x)$ is locally Lipschitz continuous, directionally differentiable and*

$$\phi'(x, d) = \sup_{\theta\in\bar{\Theta}(x)} \nabla_1 h(x, \theta)^T d,$$

where $\phi'(x, d) = \lim_{t\to 0+} \frac{\phi(x+td)-\phi(x)}{t}$.

Now we say that an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$ for (NRP) if there exists $d \in \mathbb{R}^n$ such that for any

$j \in J_1(\bar{x})$ and any $v_j \in V_j$,

$$\nabla_1 g_j(\bar{x}, v_j)^T d < 0.$$

Now we present a necessary optimality theorem for a solution of (NRP). For the proof of the following theorem, we follow the approaches of proofs for Theorem 3.1 in [7] and Theorem 3.7 in [9].

Theorem 2.2. *Let $\bar{x} \in F$ be a robust solution of (NRP). Suppose that $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$ are concave on $V_j,\ j = 1, \cdots, m$. Then there exist $\mu_j \geq 0,\ j = 1, \cdots, m$, $\bar{u} \in U$, $\bar{v}_j \in V_j,\ j = 1, \cdots, m$ and $w \in C$, such that*

$$\begin{aligned}
&0 \in \lambda\left[\nabla_1 f(\bar{x}, \bar{u}) + w\right] + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j),\\
&f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),\\
&w^T \bar{x} = s(\bar{x}|C),\\
&\mu_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.
\end{aligned}$$

Moreover, if we assume that the Extended Mangasarian-Fromovitz constraint qualification then we have (EMFCQ) holds, then

$$\begin{aligned}
&0 \in \nabla_1 f(\bar{x}, \bar{u}) + w + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j),\\
&f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),\\
&w^T \bar{x} = s(\bar{x}|C),\\
&\mu_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.
\end{aligned}$$

Proof. Assume that $\max_{v_j \in V_j} g_j(\bar{x}, v_j) < 0,\ j = 1, \cdots, m$, and $J_1(\bar{x}) = \emptyset$. Then $\bar{x} \in \mathrm{int}F$, where $\mathrm{int}F$ is the interior of F. Let $k(x) = s(x|C)$. Let $\psi(x) = \max_{u \in U} f(x, u) + k(x)$. Then $U^0 = \{u \in U \mid f(\bar{x}, u) + k(\bar{x}) = \psi(\bar{x})\}$. Then U^0 is convex and compact. By Lemma 2.1, for any $d \in \mathbb{R}^n$,

$$\begin{aligned}
\psi'(\bar{x}, d) &= \max_{u \in U^0} \nabla_1 f(\bar{x}, u)^T d + k'(\bar{x}; d)\\
&= \max_{u \in U^0} \{(\nabla_1 f(\bar{x}, u) + w)^T d \mid u \in U^0,\ w \in \partial k(\bar{x})\}.
\end{aligned}$$

So, we can be proved in the same way as the proof of Theorem 3.3 in [9] $\square$

Theorem 2.3. *Let $\bar{x} \in F$ and assume that $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$ are concave on $V_j,\ j = 1, \cdots, m$. Suppose that there exist $\mu_j \geq 0,\ j = 1, \cdots, m$,*

$\bar{u} \in U$, $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ and $w \in C$ such that

$$\begin{aligned}
&0 \in \nabla_1 f(\bar{x}, \bar{u}) + w + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j),\\
&f(\bar{x}, \bar{u}) = \max_{u \in U} f(\bar{x}, u),\\
&w^T \bar{x} = s(\bar{x}|C),\\
&\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.
\end{aligned}$$

If $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$, are convex on $\mathbb{R}^n$, then $\bar{x} \in F$ is a robust solution of (NRP).

Proof. Let $\bar{x}$ be feasible for (NRP) and assume that $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$ are concave on V_j, $j = 1, \cdots, m$. Suppose that there exist $\mu_j \geq 0$, $j = 1, \cdots, m$, $\bar{u} \in U$, $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ and $w \in C$ such that

$$\nabla_1 f(\bar{x}, \bar{u}) + w + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0. \tag{1}$$

Assume that $\bar{x}$ is not a robust solution of (NRP). Then there exists a feasible solution x of (NRP) such that

$$\max_{u \in U} f(x, u) + s(x|C) < \max_{u \in U} f(\bar{x}, u) + s(\bar{x}|C).$$

Then

$$f(x, \bar{u}) + s(x|C) < f(\bar{x}, \bar{u}) + s(\bar{x}|C).$$

Since $w^T \bar{x} = s(\bar{x}|C)$ and $w \in C$,

$$\begin{aligned}
f(x, \bar{u}) + w^T x &\leq f(x, \bar{u}) + s(x|C)\\
&< f(\bar{x}, \bar{u}) + s(\bar{x}|C)\\
&= f(\bar{x}, \bar{u}) + w^T \bar{x}.
\end{aligned}$$

By the convexity of $f(\cdot, \bar{u})$,

$$\left[\nabla_1 f(\bar{x}, \bar{u}) + w\right]^T (x - \bar{x}) < 0. \tag{2}$$

Since $\mu_j g_j(x, \bar{v}_j) \leq \mu_j g_j(\bar{x}, \bar{v}_j)$, by the convexity of $g_j(\cdot, \bar{v}_j)$,

$$\sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}) \leq 0. \tag{3}$$

From (2) and (3),

$$\left[\nabla_1 f(\bar{x}, \bar{u}) + w + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)\right]^T (x - \bar{x}) < 0.$$

From (1), $\bar{\xi}^T (x - \bar{x}) > 0$, which is a contradiction since (1) holds. $\square$

3. Robust Duality Theorems

In this section, we establish Wolfe type robust duality between (NRP) and (WD).

$$
\begin{array}{lll}
\text{(WD)} & \text{maximize} & f(x,u) + w^T x + \sum_{j=1}^{m} \mu_j g_j(x, v_j) \\
 & \text{subject to} & 0 \in \nabla_1 f(x,u) + w + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(x, v_j), \\
 & & w \in C,\ \mu_j \geq 0, \quad u \in U,\ v_j \in V_j,\ j = 1, \cdots, m.
\end{array}
$$

Let $V = V_1 \times \cdots \times V_m$.

Theorem 3.1. *(Weak Duality) Let $x \in \mathbb{R}^n$ be feasible for (NRP) and $(\bar{x}, \bar{u}, \bar{v}, \bar{w}, \bar{\mu}) \in \mathbb{R}^n \times U \times V \times C \times \mathbb{R}^m$ be feasible for (WD). Suppose that $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j),\ j = 1, \cdots, m$ are convex, $f(\bar{x}, \cdot)$ is concave on U and $g_j(\bar{x}, \cdot)$ are concave on V_j, then*

$$
\max_{u \in U} f(x,u) + s(x|C) \geq f(\bar{x}, \bar{u}) + \bar{w}^T \bar{x} + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j).
$$

Proof. Let x be feasible for (NRP) and $(\bar{x}, \bar{u}, \bar{v}, \bar{w}, \bar{\mu})$ be feasible for (WD). Then $\nabla_1 f(\bar{x}, \bar{u}) + \bar{w} + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0$. Now suppose, contrary to the result. Then we have

$$
\begin{aligned}
f(x, \bar{u}) + \bar{w}^T x &\leq \max_{u \in U} f(x,u) + s(x|C) \\
&< f(\bar{x}, \bar{u}) + \bar{w}^T \bar{x} + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j).
\end{aligned}
$$

Since $\bar{\mu}_j g_j(x, \bar{v}_j) \leq 0$,

$$
f(x, \bar{u}) + \bar{w}^T x + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) < f(\bar{x}, \bar{u}) + \bar{w}^T \bar{x} + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j).
$$

By the convexity of $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j)$,

$$
\left[\nabla_1 f(\bar{x}, \bar{u}) + \bar{w} + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \right]^T (x - \bar{x}) < 0.
$$

This is a contradiction. □

Theorem 3.2. *(Strong Duality) Let $\bar{x}$ be a solution of (NRP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds. Then, there exist $(\bar{u}, \bar{v}, \bar{w}, \bar{\mu})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{w}, \bar{\mu})$ is feasible for (WD) and the objective values of (NRP) and (WD) are equal. If $f(\cdot, \bar{u})$ and $g_j(\cdot, \bar{v}_j),\ j = 1, \cdots, m$ are* *convex, $f(\bar{x},\cdot)$ is concave on U and $g_j(\bar{x},\cdot)$ are concave on V_j, then $(\bar{x},\bar{u},\bar{v},\bar{w},\bar{\mu})$ is a solution of (WD).*

Proof. Since $\bar{x}$ is a solution of (NRP) at which the Extended Mangasarian-Fromovitz constraint qualification is satisfied, then by Theorem 2.1, there exists $\bar{\mu}_j \geq 0,\ j=1,\cdots,m$, $\bar{u}\in U$, $\bar{v}_j \in V_j,\ \ j=1,\cdots,m$, and $\bar{w}\in C$ such that

$$\begin{aligned}
&0 \in \nabla_1 f(\bar{x},\bar{u}) + \bar{w} + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x},\bar{v}_j),\\
&f(\bar{x},\bar{u}) = \max_{u\in U} f(\bar{x},u),\\
&\bar{w}^T\bar{x} = s(\bar{x}|C),\\
&\bar{\mu}_j g_j(\bar{x},\bar{v}_j) = 0,\ \ j=1,\cdots,m.
\end{aligned}$$

Thus $(\bar{x},\bar{u},\bar{v},\bar{\mu})$ is feasible for (WD) and the objective values of (NRP) and (WD) are equal. Moreover, $\max_{u\in U} f(\bar{x},u)+s(\bar{x}|C) = f(\bar{x},\bar{u})+\bar{w}^T\bar{x}+\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x},\bar{v}_j)$. It follows from a weak duality (Theorem 3.1) holds that for any feasible solution $(\widetilde{x},\widetilde{u},\widetilde{v},\widetilde{w},\widetilde{\mu})$ for (WD),

$$\begin{aligned}
f(\bar{x},\bar{u}) + \bar{w}^T\bar{x} + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x},\bar{v}_j) &= \max_{u\in U} f(\bar{x},u) + s(\bar{x}|C)\\
&\geq f(\widetilde{x},\widetilde{u}) + \widetilde{w}^T\widetilde{x} + \sum_{j=1}^{m} \widetilde{\mu}_j g_j(\widetilde{x},\widetilde{v}_j).
\end{aligned}$$

Hence $(\bar{x},\bar{u},\bar{v},\bar{w},\bar{\mu})$ is a solution of (WD). □

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