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NEW AND OLD RESULTS OF COMPUTATIONS OF AUTOMORPHISM GROUP OF DOMAINS IN THE COMPLEX SPACE

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Abstract. The automorphism group of domains is main stream of classification problem coming from E. Cartan's work. In this paper, I introduce classical technique of computations of automorphism group of domains and recent development of automorphism group. Moreover, I suggest new research problems in computations of automorphism group.

1. Introduction

The classification program for domains in the complex Euclidean space $\mathbb{C}^n$ with a noncompact holomorphic automorphism group has its origin in H. Poincaré's observation that the unit ball $B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ and the bidisc $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ are biholomorphically inequivalent.

Theorem 1.1. *There are no bijective holomorphic maps φ between the unit ball $B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ and the bidisc $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$.*

The key part of proof is based on the automorphism groups of the unit ball and the bidisc. The automorphism group is the set of all bijective holomorphic maps from a domain into itself. At that time, E. Cartan completely proved that there are only six types of *bounded symmetric domains*. The domain D is called bounded symmetric domain if for every point $p \in D$, there is bijective holomorphic map φ (call as an automorphism) from D onto itself, such that p is an isolated fixed point of φ and $\varphi \circ \varphi$ is equal to the identity map Id. This automorphism φ of domain D has the same role as the rigid motion in the euclidean space. Hence the automorphis group $\mathrm{Aut}(D)$ of a domain D consists of all automorphisms of D. This automorphism group $\mathrm{Aut}(D)$ is a topological

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group under compact-open topology. Moreover, this is a real Lie group with Lie algebra $\mathfrak{aut}(D)$ if D is bounded.

Since then, the classification problem focus on domains with noncompact automorphism groups. If D is a bounded symmetric domain, then D has noncompact autormphism group. The reverse is not true. Therefore, the classification program of domain with noncompact automorphism group is bigger problem. This program is paused because there are no tools for classifying domains.

B. Wong[12] and J.P. Rosay[10] proved that all strongly pseudoconvex domain with noncompact automorphism group is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$. The key idea of proof is based on the ratio of Kobayashi volume and Eisenman volume. This technique cannot be extended to the general domains.

Theorem 1.2. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. If* $\mathrm{Aut}(D)$ *is noncompact, then D is biholomorphically equivalent to the unit ball B.*

In 1991, E. Bedford and S. Pinchuk introduce the scaling method or streching coordinates in [1] and proved the classification result of finite type domains in $\mathbb{C}^2$.

Theorem 1.3. *Let D be a bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ with real analytic boundary. If* $\mathrm{Aut}(D)$ *is noncompact, then D is biholomorphically equivalent to the Thullen domain* $E_m = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$.

The scaling technique is powerful tools for classification problem of domain with noncompact automorphism group. In [7], K.T. Kim proved that every domain in $\mathbb{C}^2$ with a piecewise Levi flat boundary which possess a noncompact automorphism group is equivalent to the unit bidisc.

Theorem 1.4. *Let D be a convex bounded pseudoconvex domain of piecewise Levi flat boundary in $\mathbb{C}^2$. If* $\mathrm{Aut}(D)$ *is noncompact, then D is biholomorphically equivalent to the bidisc* $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$.

After that, there are lots of results for classification program using scaling technique. The detailed information is refered in [5].

There are three standard model domains, the unit ball, the bidisc, and Thullen domain in the above well-known theorem. Obiviously, the automorphism group of three domains have noncompact automorphism group and the automorphisms of them are well known as explicit form.

In this paper, we do compute the explicit automorphisms of three standard and old domains. Also, there are well-known examples, for example, the Worm Domain, the Kohn-Nirenberg domain, the Fornæss domain, etc. We also introduce new results of these domains and conjectures in computations of automorphism group.

We already know the automorphism group of given is used in Poincaré's Theorem. The author believes that the computation of automorphism group of domain is basic step in classification problem if a domain is given. Recently, the computation of automorphism group affects on the rigidity problem - any proper holomorphic map are automorphism of domain.

The organization of paper is as follows : In Section 2, the useful techniques for computation is introduced. In Section 3, we compute the explicit automorphisms of the unit ball, the bidisc and the Thullen domain. In Section 4, we introduce the recent results of automorphism group and the standard steps for computation of automorphisms.

2. Preliminary

Let D be a bounded domain in $\mathbb{C}^n$. Then $\mathrm{Aut}(D)$ has a group structure under the function composition. We can consider $G := \mathrm{Aut}(D)$ be a group acting on D by

$$\begin{aligned} \Psi :& G \times D \to D \\ & \Psi(f,z) = f(z). \end{aligned}$$

We can consider the stabilizer G_p of a point $p \in D$ by $G_p = \{f \in G \mid f(p) = p\}$. This stabilizer G_p is called by the isotropy group at a point p. The orbit $Gp = \{f(p) \mid f \in G\}$ of a point $p \in D$ is called by automorphism orbit of p. The quotient group G/G_p is naturally identified with the orbit Gp.

The compact-open topology is defined on the automorphism group $G := \mathrm{Aut}(D)$. The sequence $\{f_j\} \subset G$ converges $f_0 \in G$ with respect to the compact-open topology means that f_j uniformly converges to f_0 on the every compact subset $K \subset D$.

Now we introduce Cartan's Uniqueness Theorem.

Theorem 2.1 (Cartan's Uniquess Theorem)**.** *Let D be a bounded domain in $\mathbb{C}^n$ and a point $p \in D$. If f is holomorphic map from D to D satisfying the differential $df(p)$ at p is the identity matrix, then f is the identity map.*

Proof. First of all, we prove Theorm in one variable case. Expecting contradiction, f is not identity map. Without loss of generality, we assume that p is the origin. Since $df(o)$ is the identity, $df(o) = f'(o) = 1$. So we can apply Taylor expansion at the origin to f. Then

$$f(z) = z + a_k z^k + \text{higher order terms}+,$$

where a_k is the first nonzero Taylor coefficient. We consider n-times function composition of f, $f^n := f \circ \cdots \circ f$ can be expressed by

$$f^n(z) = z + n a_k z^k + \text{higher order terms} + .$$

By Cauchy Integral formulae, for any holomorphic map φ from D to D,

$$\varphi^{(k)}(o) = \frac{k!}{2\pi i} \int_{|\zeta| = \varepsilon} \frac{\varphi(\zeta)}{\zeta^{k+1}} d\zeta,$$

where $\varphi^{(k)}(o)$ is the k-th derivative at the origin. Letting $\varphi = f^n$ and dividing by $k!$,

$$|na_k| = \left|\frac{f^{n(k)}(z)}{k!}\right| \leq \frac{1}{2\pi}\int_{|\zeta|=\varepsilon}\left|\frac{f^{n(k)}(\zeta)}{\zeta^{k+1}}\right||d\zeta|.$$

Since D is bounded, the right handed side is bounded independent of n. But left hand side is diverging. This is contradiction. □

The above proof is working for any dimension.

We call a domain D circular if for every point $z \in D$ and $\theta \in \mathbb{R}$, $e^{i\theta}z$ is contained in D. Applying Theorem 2.1, we obtain the Cartan's Linearization Theorem.

Theorem 2.2 (Cartan's Linearization Theorem)**.** *Let D_1, D_2 be bounded circular domains in $\mathbb{C}^n$ and the origin $o \in D_j$ $(j = 1, 2)$. If f is a bijective holomorphic map from D_1 to D_2 with $f(o) = o$, then f is a linear map.*

Proof. For a real number θ, an automorphism Π_θ is defined by $\Pi_\theta(z) = e^{i\theta}z$ of $D_j (j = 1, 2)$. We consider a map g defined by

$$g(z) = f^{-1} \circ \Pi_\theta^{-1} \circ f \circ \Pi_\theta(z).$$

Then g is contained in the $\mathrm{Aut}(D_1)$ and g preserves the origin. Moreover

$$dg(o) = df^{-1}(o)d\Pi_\theta^{-1}(o)df(o)d\Pi_\theta(o).$$

Since $d\Pi_\theta(o)$ is a diagonal matrix, we obtain that $dg(o)$ is the identity matrix. By Cartan Uniqueness Theorem, g is identically to the identity map. Hence we get commutative relation $f \circ \Pi_\theta = \Pi_\theta \circ f$. By Taylor series expansion, we easily get that f is linear. □

By the above theorem, the isotropy group $\mathrm{Aut}(B)_o$ of the unit ball B at the origin o is consists of all unitary maps $\mathcal{U}(n)$ and the isotropy group $\mathrm{Aut}(\Delta^2)_o$ of the bidisc Δ^2 at the origin o is consists of rotations $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{i\eta}z_2)$ and transition $(z_1, z_2) = (z_2, z_1)$. The full automorphism group is the unit disc in one dimensional complex space, is well-known by Schwarz's Lemma.

$$\mathrm{Aut}(\Delta) = \left\{e^{i\theta}\frac{z+a}{1+\bar{a}z} \mid \theta \in \mathbb{R}, |a| < 1\right\}$$

Using these facts, we can prove H. Poincaré's Theorem.

Proof. Let f be a bijective holomorphic map from B onto Δ^2. We can assume that $f(o) = (a_1, a_2)$. Define an automorphism by

$$\varphi(z_1, z_2) = \left(\frac{z_1 - a_1}{1 - \bar{a_1}z_1}, \frac{z_2 - a_2}{1 - \bar{a_2}z_2}\right).$$

Then $\varphi \circ f$ is biholomorphism from B onto Δ^2 satisfying $\varphi \circ f(o) = o$. We can obtain that the group isomorphism $\Psi : \mathrm{Aut}(B)_o \to \mathrm{Aut}(\Delta^2)_o$ is defined by $\Psi(g) = (\varphi \circ f) \circ g \circ (\varphi \circ f)^{-1}$ for all $g \in \mathrm{Aut}(B)_o$. Note that $\mathrm{Aut}(B)_o$ is non-abelian group but $\mathrm{Aut}(\Delta^2)_o$ is abelian. Hence, this is contradiction. The proof is done. □

3. Old Results of Automorphism Groups

In Section 2, we use the isotropy subgroup of given domain. Now, we find all automorphisms of the unit ball, the bidisc and the Thullen domain.

We now define an automorphism group of the unit ball. Given $a \in \mathbb{C}$ satisfying $|a| < 1$,

$$\varphi_a^1(z_1, z_2) = \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z_1} z_2 \right)$$

is the one-to-one onto holomorphic map of the unit ball. Obviously,

$$\varphi_a^2(z_1, z_2) = \left(\frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z_2} z_1, \frac{z_2 - a}{1 - \bar{a}z_2} \right)$$

is also an autormophism. Using these maps, for any point $(\alpha, \beta) \in B$, there are two automorphisms φ_a^1, φ_b^2 such that $\varphi_a^1 \circ \varphi_b^2(\alpha, \beta) = (0, 0)$. This means that the unit ball is homogeneous domain. Equivalently, the automorphism group $\mathrm{Aut}(B)$ transitively acts on the unit ball.

Theorem 3.1. *Let $B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit ball in $\mathbb{C}^2$. Then if f is an automorphism of B, then f is a finite composition of unitary map $U(z)$ and automorphisms φ_a^1, φ_b^2 of the unit ball.*

Proof. Let f be an automorphism of the unit ball and let $f(0, 0) = (\alpha, \beta) \in B$. We consider new automorphism $g = \varphi_\alpha^1 \circ \varphi_\beta^2 \circ f$. Then g preserves the origin. By Cartan's Linearization Theorem, g is linear. So, g preserve the boundary of the unit ball. It means that g is identically to the unitary map U. Therefore $f = \varphi_{-\alpha}^1 \circ \varphi_{-\beta}^2 \circ U$. □

Note that $\mathrm{Aut}(B^2)$ acts transitively on B^2. Equivalently, for any points p, q in the unit ball B, there is an automorphism $f \in \mathrm{Aut}(B^2)$ such that $f(p) = q$.

We will compute automorphism of the unit bids in $\mathbb{C}^2$. There are well-known two types of automorphisms of the bidisc.

$$L(z_1, z_2) = (z_2, z_1)$$

$$\varphi_{a,b}(z_1, z_2) = \left(e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a}z_1}, e^{i\theta_2} \frac{z_2 - b}{1 - \bar{b}z_2} \right),$$

where $|a|, |b| < 1$ and $\theta_1, \theta_2 \in \mathbb{R}$. The first map L is linear and the second map is the Möbius transformation.

Theorem 3.2. *Let $\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ be the bidisc in $\mathbb{C}^2$. Then if f is an automorphism of Δ^2, then f is a finite composition of transition map L and the Möbius transformation $\varphi_{a,b}$.*

Proof. Let f be an automorphism Δ^2 and let $(a,b) = f(0,0)$. We consider $\varphi_{a,b} \circ f$ satisfying

$$\varphi \circ f(0,0) = \varphi_{a,b}(a,b) = (0,0).$$

By Cartan's Linearization Theorem, we can easily get $\varphi \circ f = L$ or $\varphi \circ f$ is an identity map. Thus the proof is done. □

Let m be an integer with $m > 1$. The domain $E_m = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ is called the Thullen domain in $\mathbb{C}^2$. The automorphism of the Thullen domain can be described by the following form

$$(z_1,z_2) \mapsto \left(e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a}z_1}, e^{i\theta_2} \frac{\sqrt[2m]{1-|a|^2}}{\sqrt[m]{1-\bar{a}z_1}} z_2 \right)$$

where $|a| < 1$.

4. Recent Results of Automorphism Groups

The Kohn-Nirenberg domain Ω_{KN} defined by

$$\Omega_{KN} = \left\{ (z_1,z_2) \in \mathbb{C}^2 : \mathrm{Re}\, z_2 + |z_1 z_2|^2 + |z_1|^8 + \frac{15}{7}|z_1|^2 \mathrm{Re}\, z_1^6 < 0 \right\}$$

was constructed by Kohn and Nirenberg [9]. The Kohn-Nirenberg domain is a well-known example of a pseudoconvex domain not admitting a local holomorphic support function. In fact, the Kohn-Nirenberg domain cannot be realized as a convex set by any local holomorphic coordinate change at the origin. In [2], authors compute all automorphism of the Kohn-Nirenberg domain.

Theorem 4.1 (Byun and Cho). *The automorphism group of the Kohn-Nirenberg domain Ω_{KN} is generated by the map $(z_1,z_2) \mapsto (e^{i\frac{\pi}{3}} z_1, z_2)$. Therefore, it is compact and a cyclic group of order 6.*

In [4], J.E. Fornæss considered the germ of a domain near the origin in $\mathbb{C}^2$ such that $\Omega_t = \{(z_1,z_2) \mid \mathrm{Re}\, z_2 + |z_1 z_2|^2 + |z_1|^6 + t|z_1|^2 \mathrm{Re}\,(z_1)^4\}$ to study the holomorphic peak function that is smooth up to the boundary. J.E. Fornæss proved that for $1 < t < 9/5$ the domain Ω_t does not admit a holomorphic function on Ω_t that is C^1 up to the boundary and that peaks at the origin.

In [3], they compute all automorphism of the Fornæss domain. Moreover, they generalize domain $\Omega_{n,t}$ defined by

$$\mathrm{Re}\, z_2 + |z_1 z_2|^2 + |z_1|^{2n+2} + t|z_1|^2 \mathrm{Re}\, z_1^{2n} < 0$$

for positive integer n. These domains include the Kohn-Nirenberg domain and the Fornæss domain.

Theorem 4.2 (Byun and Cho). *The automorphism group of $\Omega_{n,t}$ is equal to the set*

$$\{{\Pi_n}^k \mid k = 1,2,\cdots,2n\},$$

where $\Pi_n(z_1,z_2) = (e^{i\frac{\pi}{n}} z_1, z_2)$ and ${\Pi_n}^k$ is the k-times function composition of Π_n. Therefore, it is compact and a cyclic group of order $2n$.

In [6], they introduce the general Kohn-Nirenberg domain in $\mathbb{C}^2$ definded by

$$\Omega_k = \{(z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Re} z_2 + |z_1 z_2|^2 + |z_1|^{2n} + k_1|z_1|^{2n2m}\operatorname{Re}(z_1^{2m}) + k_2|z_1|^{2n2m}\operatorname{Im}(z_1^{2m}) < 0\}$$

, where $k_i \in \mathbb{R}$, $n, m \in \mathbb{Z}^+$, and $nm1 \geq 0$. The main result of [6] is :

Theorem 4.3. *Let Ω_k be the above set. If $|k_1| + |k_2| < \frac{n^2}{n^2-m^2}$ and $k_1^2 + k_2^2 > 1$ then the automorphism group of Ω_k is equal to the set $\{{\Pi_k}^{\ell} : \ell = 1, 2, \cdots, 2m\}$, where $\Pi_k(z_1, z_2) = (e^{i\frac{\pi}{m}} z_1, z_2)$ and ${\Pi_k}^{\ell}$ is the ℓ-times function composition of Π_k. Therefore, it is compact and a cyclic group of order $2m$.*

The above three theorems is based on the following steps :

STEP I These domain Ω are unbounded domains of D'Angelo finite type. The origin is a boundary point of domain and an isolated point with respect to the D'Angelo type. Every automorphism f of domains should either preserves the origin or diverges to the infinite point.

STEP II Case $f(0) = 0$
Since domains are satisfied the Bell's Condition (R), every automorphism can be extended near the origin. Let H be the holomorphic tangent space at the origin to the boundary of domain. Using defining equation and power series, authors show that automorphism f preserves H. The set $\Omega \cap H$ is considered as the set in $\mathbb{C}$. The boundary of $\Omega \cap H$ consists of several straight lines intersecting only at the origin. Since f is an automorphism of $\Omega \cap H$ and $f(0) = 0$, the derivative of f at the origin is equal to 1 after linear map composition. This setting is similar to Cartan's Uniqueness Theorem except that the origin stays in the boundary. By reflection principle, f is an identity map of $\Omega \cap H$. Choose a point p in $\Omega \cap H$. The point p is an interior point of the domain. Again, we will use Cartan's Uniqueness Theorem. Denoted a derivative of f at p by A. Since sequence $f \circ \cdots \circ f$ is a normal family, A is an identity matrix. Hence f is an identity map of domain.

STEP III f diverges to the infinite point. It means that there is a sequence p_j in domain satisfying

$$\lim_{j\to\infty} p_j = 0, \quad \lim_{j\to\infty} |f(p_j)| = \infty.$$

After an inversion map $I(z_1, z_2) = (\frac{1}{z_1}, z_2)$ acting domain, the sequence $I \circ f(p_j)$ converges to the origin. The local germ of domain $I(\Omega)$ at the origin is strongly pseudoconvex boundary point. We apply scaling technique to sequences p_j and $I \circ f(p_j)$. The original domain converges finite type model domain $M_P = \{(z_1, z_2) \mid \operatorname{Re} z_2 + P(z_2) < 0\}$ and the second scaled limit domain $\big\{(z_1, z_2) \mid \operatorname{Re} z_2 + |z_2|^2 < 0\big\}$. This is impossible.

In [11], author constructs the bounded Kohn Nirenberg domain D in $\mathbb{C}^2$ defined by

$$\operatorname{Re} z_2 + \frac{1}{5}|z_2|^2 + |z_1 z_2|2 + |z_1|^8 + \frac{15}{7}|z_1|^2 \operatorname{Re}(z_2)^6 + 10|z_1|^{10} < 0.$$

The automorphism group $\operatorname{Aut}(D)$ is a compact group by the well-known results. But we do not full automorphisms of domain D.

Problem 4.4. Find all automorphisms of the bounded Kohn Nirenberg domain.

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