

OPTIMAL CONTROL FOR THE FOREST KINEMATIC MODEL

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ABSTRACT. This paper is concerned with the optimal control for the forest kinematic model. That is, we show the existence of the strong solution for the forest kinematic model and then show the existence of the optimal control.

1. Introduction

In this paper we consider the following optimal control problem

$$(P) \quad \text{minimize } \mathcal{J}(u)$$

with the cost functional $\mathcal{J}(u)$ of the form

$$\begin{aligned} \mathcal{J}(u) = \int_0^T \|y(u) - y_d\|_{L^2(I)}^2 dt + \int_0^T \|\rho(u) - \rho_d\|_{L^2(I)}^2 dt \\ + \gamma \|u\|_{H^1(0,T)}^2, \quad u \in H^1(0,T), \end{aligned}$$

where $y = y(u)$ and $\rho = \rho(u)$ are governed by the forest dynamical system

$$\begin{aligned} \frac{\partial y}{\partial t} &= d \frac{\partial^2 y}{\partial x^2} - \gamma(\rho)y - fy + g\rho & \text{in } I \times (0, T], \\ \frac{\partial \rho}{\partial t} &= fy - h\rho - u(t)\rho & \text{in } I \times (0, T], \\ \frac{\partial y}{\partial x}(0, t) &= \frac{\partial y}{\partial x}(L, t) = 0 & \text{on } (0, T], \\ y(x, 0) &= y_0(x), \quad \rho(x, 0) = \rho_0(x) & \text{in } I. \end{aligned} \tag{1.1}$$

Here, $I = (0, L)$ is a bounded interval in \mathbb{R} . $y = y(x, t)$ denotes tree density of young age class in I at time t and $\rho = \rho(x, t)$ is tree density of old age class in I at time t . $g > 0$ is fertility of the species. $h > 0$ and $f > 0$ denote death and aging rates. $\gamma(\rho)$ denotes a mortality rate function of the young trees with $\gamma(\rho) = a(\rho - b)^2 + c$ ($a, b, c > 0$). $u(t)$ denotes the control term.

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The modelling of forest age structure dynamics is one of the most important problems of mathematical ecology. The model (1.1) is introduced as base mathematical model of mono-species forest with two age classes ([1], [2]).

Many authors studied for the optimal control problem governed by the reaction diffusion model. In [8], the optimal control problem for the chemotaxis model was studied. Brandaõ et al.([6]) considered the optimal control problem for FitzHugh-Nagumo equation. In [3], [4] and [7], the optimal control problem for prey-predator reaction diffusion model was studied. In this paper, we show the existence of the strong solution of (1.1). We also show the existence of the optimal control.

The paper is organized as follows. Section 2 is a preliminary section. In Section 3, we show the existence of the strong solutions. Section 4 show the existence of the optimal control.

Notation. Let J be an interval in the real line \mathbf{R} . $L^p(J; \mathcal{H})$, $1 \leq p \leq \infty$, denotes the L^p space of measurable functions in J with values in a Hilbert space \mathcal{H} . $C(J; \mathcal{H})$ denotes the space of continuous functions in J with values in \mathcal{H} . $W^{1,2}(J; \mathcal{H}) = \{y; D^j y \in L^2(J; \mathcal{H}), j = 0, 1\}$, where D is the derivative in the sense of distributions. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by $a, b, c, d, f, g, h, m, l$ and I .

2. Preliminaries

First we recall a general existence result which we use in the sequel([5]). Consider the following abstract problem

$$\begin{aligned} \frac{dY}{dt} &= \mathcal{A}Y + F(t, Y(t)), \quad t \in [0, T], \\ Y(0) &= Y_0, \end{aligned} \quad (2.1)$$

where \mathcal{A} is a linear operator defined on a Banach space X , with the domain $D(\mathcal{A})$ and $F : [0, T] \times X \rightarrow X$ is a given function. If X is a Hilbert space endowed with the scalar product (\cdot, \cdot) , then the linear operator \mathcal{A} is called dissipative if $(\mathcal{A}Y, Y) \leq 0$, for all $Y \in D(\mathcal{A})$.

Theorem 2.1. ([5]) *Let X be a real Banach space, $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup of linear contractions $\{S(t), t \geq 0\}$ on X , and $F : [0, T] \times X \rightarrow X$ be a measurable function in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0, T]$.*

(i) *If $Y_0 \in X$, then problem (2.1) admits a unique mild solution, i.e. a function $Y \in C([0, T]; X)$ which verifies the equality*

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)F(s, Y(s))ds, \quad \forall t \in [0, T].$$

(ii) If X is a Hilbert space, \mathcal{A} is self adjoint and dissipative on X and $Y_0 \in D(\mathcal{A})$, then the mild solution is in fact a strong solution and $Y \in W^{1,2}(0, T; X) \cap L^2(0, T; D(\mathcal{A}))$.

3. Existence of the strong solution

In this section, we show the existence and uniqueness of a local strong solution for (1.1).

We rewrite (1.1) as an abstract problem (2.1) in the Hilbert spaces $\mathcal{H} = L^2(I) \times L^2(I)$. To this end, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$AY = \begin{pmatrix} d \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \rho \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in D(\mathcal{A}).$$

Here, $D(\mathcal{A}) = \left\{ Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in H^2(I) \times L^\infty(I), \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(L) = 0 \right\}$. Then \mathcal{A} is a self adjoint dissipative operator in \mathcal{H} .

Thus, (1.1) is formulated to the following abstract form

$$\begin{aligned} \frac{dY}{dt} + AY &= F(t, Y(t)), \quad 0 < t \leq T, \\ Y(0) &= Y_0 \end{aligned} \quad (3.1)$$

in the space \mathcal{H} . Here, $F(t, Y(t)) : [0, T] \times \mathcal{H} \rightarrow \mathcal{H}$ is the mapping

$$F(t, Y(t)) = \begin{pmatrix} f(t, y, \rho) \\ g(t, y, \rho) \end{pmatrix} = \begin{pmatrix} -\gamma(\rho)y - fy + g\rho \\ fy - h\rho - u(t)\rho \end{pmatrix} \quad (3.2)$$

and Y_0 is defined by $Y_0 = \begin{pmatrix} y_0 \\ \rho_0 \end{pmatrix}$. $\mathcal{K} = \left\{ \begin{pmatrix} y_0 \\ \rho_0 \end{pmatrix} \in D(\mathcal{A}); 0 \leq y_0 \text{ and } 0 \leq \rho_0 \right\}$ and $U_{ad} = \{u \in H^1(0, T); \|u\|_{H^1(0, T)} \leq m, 0 \leq u(t) \leq l\}$.

Now, we have the following result for the local strong solution to (1.1).

Theorem 3.1. For $Y_0 \in \mathcal{K}$ and $u \in U_{ad}$, (1.1) has a unique strong solution $Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in W^{1,2}(0, T_{y_0, \rho_0, u}; \mathcal{H})$ such that

$$\begin{aligned} 0 \leq y &\in L^\infty((0, T_{y_0, \rho_0, u}) \times I) \cap L^\infty(0, T_{y_0, \rho_0, u}; H^1(I)) \cap L^2(0, T_{y_0, \rho_0, u}; H^2(I)), \\ 0 \leq \rho &\in L^\infty((0, T_{y_0, \rho_0, u}) \times I) \cap L^\infty(0, T_{y_0, \rho_0, u}; L^2(I)). \end{aligned}$$

Here, $T_{y_0, \rho_0, u} > 0$ is determined by $\|y_0\|_{L^\infty(I)}$, $\|\rho_0\|_{L^\infty(I)}$ and $\|u\|_{H^1(0, T)}$. Moreover, the estimates

$$\left\| \frac{\partial y}{\partial t} \right\|_{L^2(0, T_{y_0, \rho_0, u}; L^2(I))} + \|y\|_{L^2(0, T_{y_0, \rho_0, u}; H^2(I))} + \|y\|_{H^1(I)} + \|y\|_{L^\infty((0, T_{y_0, \rho_0, u}) \times I)} \leq C \quad (3.3)$$

and

$$\left\| \frac{\partial \rho}{\partial t} \right\|_{L^2(0, T_{y_0, \rho_0, u}; L^2(I))} + \|\rho\|_{L^\infty((0, T_{y_0, \rho_0, u}) \times I)} + \|\rho\|_{L^2(I)} \leq C \quad (3.4)$$

hold, where C is also determined by $\|y_0\|_{L^\infty(I)}$, $\|\rho_0\|_{L^\infty(I)}$ and $\|u\|_{H^1(0, T)}$.

Proof. Since $F(t, Y(t))$ defined in (3.2) is not Lipschitz continuous with respect to Y uniformly for $t \in [0, T]$, we can not apply Theorem 2.1 for (3.1).

Step 1. We use a truncation procedure for $F(t, Y(t))$ ([1], [2], [7]). For a fixed positive integer $N > 0$, let us consider the following auxiliary problem:

$$\begin{aligned} \frac{dY_N}{dt} + AY_N &= F_N(t, Y_N(t)), \quad 0 < t \leq T, \\ Y_N(0) &= Y_0, \end{aligned} \quad (3.5)$$

where $Y_N = \begin{pmatrix} y_N \\ \rho_N \end{pmatrix}$ and $F_N(t, Y_N(t)) = \begin{pmatrix} f^N(t, y_N, \rho_N) \\ g^N(t, y_N, \rho_N) \end{pmatrix}$ with $f^N(t, y_N, \rho_N)$, $g^N(t, y_N, \rho_N)$ defined as follows: If y and ρ are greater than N or less than $-N$, then we replace y and ρ by N or $-N$. If $|y|, |\rho| \leq N$, then y and ρ in $f(t, y, \rho)$ remain unchanged. Similarly, y and ρ in $g(t, y, \rho)$ remain unchanged. Thus $F_N(t, Y_N(t)) = \begin{pmatrix} f^N(t, y_N, \rho_N) \\ g^N(t, y_N, \rho_N) \end{pmatrix}$ is well defined on $[0, T] \times \mathcal{H}$. Then, we can check that the function $F_N(t, Y_N(t))$ is Lipschitz continuous with respect to Y_N uniformly for $t \in [0, T]$. Thus, it follows from Theorem 2.1 that there exists a unique strong solution $Y_N \in W^{1,2}(0, T; \mathcal{H}) \cap L^2(0, T; D(\mathcal{A}))$.

Step 2. We show that $y_N \in L^\infty(0, T; H^1(I))$. Indeed, from first equation of (3.5), we deduce that

$$\begin{aligned} \int_0^t \int_I \left| \frac{\partial y_N}{\partial t} \right|^2 dx d\tau + d^2 \int_0^t \int_I \left| \frac{\partial^2 y_N}{\partial x^2} \right|^2 dx d\tau + d \int_I \left| \frac{\partial y_N}{\partial x} \right|^2 dx \\ = d \int_I \left| \frac{\partial y_0}{\partial x} \right|^2 dx + \int_0^t \int_I |f^N(\tau, y_N, \rho_N)|^2 dx d\tau. \end{aligned}$$

By the Lipschitz property of $f^N(t, y_N, \rho_N)$ we obtain that

$$d \int_I \left| \frac{\partial y_N}{\partial x} \right|^2 dx \leq d \int_I \left| \frac{\partial y_0}{\partial x} \right|^2 dx + C \int_0^T \int_I (|y_N|^2 + |\rho_N|^2) dx dt.$$

Since $y_N, \rho_N \in W^{1,2}(0, T; L^2(I))$ and $y_0 \in H^2(I)$, we have $y_N \in L^\infty(0, T; H^1(I))$. Similarly, multiplying the second equation of (3.5) by ρ_N , we obtain that

$$\begin{aligned} \int_I |\rho_N|^2 dx &\leq \int_I |\rho_0|^2 dx + C \int_0^T \int_I |g^N(t, y_N, \rho_N)|^2 dx dt \\ &\leq \int_I |\rho_0|^2 dx + C \int_0^T \int_I (|y_N|^2 + |\rho_N|^2) dx dt. \end{aligned}$$

Since $y_N, \rho_N \in W^{1,2}(0, T; L^2(I))$ and $\rho_0 \in L^\infty(I)$, we have $\rho_N \in L^\infty(0, T; L^2(I))$.

Step 3. Let us prove the boundedness of Y_N on $(0, T) \times I$. Put

$$M = \max\{\|f^N\|_{L^\infty((0,T) \times I)}, \|g^N\|_{L^\infty((0,T) \times I)}, \|y_0\|_{L^\infty(I)}, \|\rho_0\|_{L^\infty(I)}\}.$$

Note that function $\tilde{y}_N(t, x) = y_N(t, x) - Mt - \|y_0\|_{L^\infty(I)}$ satisfies the following problem

$$\begin{aligned} \frac{d\tilde{y}_N}{dt} &= d \frac{\partial^2 \tilde{y}_N}{\partial x^2} + f^N(t, y_N(t), \rho_N(t)) - M, \quad 0 < t \leq T, \\ \tilde{y}_N(0) &= y_0 - \|y_0\|_{L^\infty(I)}. \end{aligned} \quad (3.6)$$

Then the strong solution of (3.6) can be written as

$$\tilde{y}_N(t) = S(t)(y_0 - \|y_0\|_{L^\infty(I)}) + \int_0^t S(t-s)(f^N(s, y_N(s), \rho_N(s)) - M) ds,$$

where $\{S(t), t \geq 0\}$ is the C_0 -semigroup generated by the operator $B : D(B) \subset L^2(I) \rightarrow L^2(I)$.

$$By = d \frac{\partial^2 y}{\partial x^2}, \quad D(B) = \{y \in H^2(I); \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(L) = 0\}.$$

Since $y_0 - \|y_0\|_{L^\infty(I)} \leq 0$ and $f^N(t, y_N(t), \rho_N(t)) - M \leq 0$, it follows from the comparison principle of parabolic equation that $\tilde{y}_N(t, x) \leq 0$ for all $(t, x) \in (0, T) \times I$. In the same manner we can prove that $w_N(t, x) = y_N(t, x) + Mt + \|y_0\|_{L^\infty(I)}$ is nonnegative. Thus, we have

$$|y_N(t, x)| \leq Mt + \|y_0\|_{L^\infty(I)}, \quad (t, x) \in (0, T) \times I. \quad (3.7)$$

On the other hand, the function $\tilde{\rho}_N(t, x) = \rho_N(t, x) - Mt - \|\rho_0\|_{L^\infty(I)}$ satisfies the following problem

$$\begin{aligned} \frac{d\tilde{\rho}_N}{dt} &= g^N(t, y_N(t), \rho_N(t)) - M, \quad 0 < t \leq T, \\ \tilde{\rho}_N(0) &= \rho_0 - \|\rho_0\|_{L^\infty(I)}. \end{aligned}$$

Since $\rho_0 - \|\rho_0\|_{L^\infty(I)} \leq 0$ and $g^N(t, y_N(t), \rho_N(t)) - M \leq 0$, it follows that $\tilde{\rho}_N(t, x) \leq 0$ for all $(t, x) \in (0, T) \times I$. In the same manner we can prove that $z_N(t, x) = \rho_N(t, x) + Mt + \|\rho_0\|_{L^\infty(I)}$ is nonnegative. Therefore, we obtain

$$|\rho_N(t, x)| \leq Mt + \|\rho_0\|_{L^\infty(I)}, \quad (t, x) \in (0, T) \times I. \quad (3.8)$$

Thus we have proved that $y_N, \rho_N \in L^\infty((0, T) \times I)$, the upper bound being dependent only on N .

Step 4. To show the positiveness of y_N , we consider the following auxiliary problem:

$$\begin{aligned} \frac{d\bar{Y}_N}{dt} + A\bar{Y}_N &= \bar{F}_N(t, \bar{Y}_N(t)), \quad 0 < t \leq T, \\ \bar{Y}_N(0) &= Y_0, \end{aligned}$$

where

$$\bar{F}_N(t, \bar{Y}_N(t)) = \begin{pmatrix} -\gamma(\bar{\rho}_N)\bar{y}_N - f\bar{y}_N + g|\bar{\rho}_N| \\ f\bar{y}_N - h\bar{\rho}_N - u(t)\bar{\rho}_N \end{pmatrix}.$$

Let us verify first that $\bar{y}_N \geq 0$ by the truncation method([10]). Consider $H(\bar{y}_N)$ is $\mathcal{C}^{1,1}$ cutoff function for $-\infty < \bar{y}_N < \infty$ given by $H(\bar{y}_N) = \frac{\bar{y}_N^2}{2}$ for

$-\infty \leq \bar{y}_N < 0$ and $H(\bar{y}_N) = 0$ for $0 \leq \bar{y}_N < \infty$.

If we put

$$\psi(t) = \int_0^L H(\bar{y}_N(t))dx, \quad 0 \leq t \leq T,$$

then $\psi(t)$ is continuously differentiable function with the derivative

$$\begin{aligned} \frac{d}{dt}\psi(t) &= \int_0^L H'(\bar{y}_N(t))\bar{y}'_N(t)dx \\ &= \int_0^L H'(\bar{y}_N(t))\left(d\frac{\partial^2\bar{y}_N}{\partial x^2} - \gamma(\bar{\rho}_N)\bar{y}_N - f\bar{y}_N + g|\bar{\rho}_N|\right)dx \\ &= d\int_0^L H'(\bar{y}_N(t))\frac{\partial^2\bar{y}_N}{\partial x^2}dx - \int_0^L H'(\bar{y}_N(t))\gamma(\bar{\rho}_N)\bar{y}_Ndx \\ &\quad - f\int_0^L H'(\bar{y}_N(t))\bar{y}_Ndx + g\int_0^L H'(\bar{y}_N(t))|\bar{\rho}_N|dx \\ &= -a\int_0^L \left|\frac{\partial H'(\bar{y}_N(t))}{\partial x}\right|^2dx - c\int_0^L H'(\bar{y}_N(t))\bar{y}_Ndx \\ &\quad - f\int_0^L H'(\bar{y}_N(t))\bar{y}_Ndx + g\int_0^L H'(\bar{y}_N(t))|\bar{\rho}_N|dx. \end{aligned}$$

Since $H'(\bar{y}_N) \leq 0$, $H'(\bar{y}_N)\bar{y}_N \geq 0$, we obtain

$$\frac{d}{dt}\psi(t) \leq 0.$$

Therefore, $\psi(t) \leq \psi(0)$ for $0 \leq t \leq T$. Thus, $\psi(0) = 0$ implies $\psi(t) = 0$, that is, $\bar{y}_N(t) \geq 0$ for $0 \leq t \leq T$. Similarly, we obtain that $\bar{\rho}_N(t) \geq 0$ for $0 \leq t \leq T$. We conclude that $\bar{F}_N(t, \bar{Y}_N) = F_N(t, \bar{Y}_N)$. Thus we see that \bar{Y}_N is a solution of (3.5). By the uniqueness, we see that $\bar{Y}_N(t) = Y_N(t)$ for $0 \leq t \leq T$.

If we choose $N > 2\max\{\|y_0\|_{L^\infty(I)}, \|\rho_0\|_{L^\infty(I)}\}$, there exists $s \in (0, T)$ such that

$$Ms + \|y_0\|_{L^\infty(I)} \leq \frac{N}{2}, \quad Ms + \|\rho_0\|_{L^\infty(I)} \leq \frac{N}{2}.$$

From (3.7) and (3.8), we derive that $|y_N| \leq N, |\rho_N| \leq N$ for all $t \in (0, s)$. Thus $F_N(t, y_N, \rho_N) = F(t, y, \rho)$ for $(t, x) \in (0, s) \times I$, so $Y = \begin{pmatrix} y \\ \rho \end{pmatrix}$ is the local solution of (3.1) defined on $(0, s) \times I$. □

4. Existence of the optimal control

Let $S > 0$ be such that for each $u \in U_{ad}$, (3.1) has a unique strong solution $Y(u) \in W^{1,2}(0, S; \mathcal{H}) \cap L^2(0, S; D(\mathcal{A}))$ satisfying (3.3) and (3.4). Thus, the problem **(P)** is obviously formulated as follows:

$$(\bar{\mathbf{P}}) \quad \text{minimize } \mathcal{J}(u),$$

where

$$\mathcal{J}(u) = \int_0^S \|Y(u) - Y_d\|_{\mathcal{H}}^2 dt + \gamma \|u\|_{H^1(0,S)}^2, \quad u \in U_{ad}.$$

Here, $Y = \begin{pmatrix} y \\ \rho \end{pmatrix}$ and $Y_d = \begin{pmatrix} y_d \\ \rho_d \end{pmatrix}$ is a fixed element of $L^2(0, S; \mathcal{H})$ with $y_d, \rho_d \in L^2(0, S; L^2(I))$. γ is a positive constant. Then, we have the following result.

Theorem 4.1. *There exists an optimal control $u^* \in U_{ad}$ for $(\bar{\mathbf{P}})$ such that $\mathcal{J}(u^*) = \min_{u \in U_{ad}} \mathcal{J}(u)$.*

Proof. Let $\{u_n\} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} \mathcal{J}(u_n) = \min_{u \in U_{ad}} \mathcal{J}(u).$$

Since $\{u_n\}$ is bounded in $H^1(0, T)$, we can assume that $u_n \rightarrow u^*$ weakly in $H^1(0, T)$. By the compactness of $H^1(0, T) \hookrightarrow L^2(0, T)$, we see that

$$u_n \rightarrow u^* \text{ strongly in } L^2(0, T). \quad (4.1)$$

Let $Y_n = \begin{pmatrix} y_n \\ \rho_n \end{pmatrix}$ be the solution of (3.1) corresponding to u_n . That is, $Y_n = \begin{pmatrix} y_n \\ \rho_n \end{pmatrix}$ is the solution of the following equations

$$\begin{aligned} \frac{\partial y_n}{\partial t} &= d \frac{\partial^2 y_n}{\partial x^2} - \gamma(\rho_n)y_n - f y_n + g \rho_n && \text{in } I \times (0, S], \\ \frac{\partial \rho_n}{\partial t} &= f y_n - h \rho_n - u_n(t) \rho_n && \text{in } I \times (0, S], \\ \frac{\partial y_n}{\partial x}(0, t) &= \frac{\partial y_n}{\partial x}(L, t) = 0 && \text{on } (0, S], \\ y_n(x, 0) &= y_0(x), \quad \rho_n(x, 0) = \rho_0(x) && \text{in } I. \end{aligned} \quad (4.2)$$

Then, we see from (3.3) and (3.4) that

$$\left\| \frac{\partial y_n}{\partial t} \right\|_{L^2(0,S;L^2(I))}, \|y_n\|_{L^2(0,S;H^2(I))}, \|y_n\|_{H^1(I)}, \|y_n\|_{L^\infty((0,S) \times I)} \leq C, \quad (4.3)$$

$$\left\| \frac{\partial \rho_n}{\partial t} \right\|_{L^2(0,S;L^2(I))}, \|\rho_n\|_{L^\infty((0,S) \times I)}, \|\rho_n\|_{L^2(I)} \leq C. \quad (4.4)$$

Therefore, it is seen from (4.3) and (4.4) that there exists y^* such that

$$\begin{aligned} \frac{\partial y_n}{\partial t} &\rightharpoonup \frac{\partial y^*}{\partial t} \text{ weakly in } L^2(0, S; L^2(I)), \\ \frac{\partial^2 y_n}{\partial x^2} &\rightharpoonup \frac{\partial^2 y^*}{\partial x^2} \text{ weakly in } L^2(0, S; L^2(I)), \\ y_n &\rightarrow y^* \text{ weakly star in } L^\infty(0, S; H^1(I)), \\ y_n &\rightarrow y^* \text{ weakly star in } L^\infty((0, S) \times I), \\ \frac{\partial \rho_n}{\partial t} &\rightharpoonup \frac{\partial \rho^*}{\partial t} \text{ weakly in } L^2(0, S; L^2(I)) \\ \rho_n &\rightarrow \rho^* \text{ weakly star in } L^\infty(0, S; L^2(I)) \\ \rho_n &\rightarrow \rho^* \text{ weakly star in } L^\infty((0, S) \times I). \end{aligned}$$

Since $H^1(I)$ is compactly embedded in $L^2(I)$, we deduce that $\{y_n\}$ is relatively compact in $C([0, S]; L^2(I))$ ([9]). Therefore, we see that

$$y_n \rightarrow y^* \text{ strongly in } C([0, S]; L^2(I)). \tag{4.5}$$

Furthermore, multiplying the second equation in (4.2) by ρ_n we obtain

$$\int_I \rho_n^2 dx = \int_I \rho_0^2 dx + 2 \int_0^t \int_I (f y_n - h \rho_n - u_n(\tau) \rho_n) \rho_n dx d\tau. \tag{4.6}$$

From (4.6), we have

$$\left| \int_I \rho_n^2(t, x) dx - \int_I \rho_n^2(s, x) dx \right| \leq C|t - s|, \quad \forall t, s \in [0, S].$$

So, by Arzela-Ascoli Theorem, we can assume that

$$\rho_n \rightarrow \rho^* \text{ strongly in } L^2(I) \text{ uniformly with respect to } t \in [0, S]. \tag{4.7}$$

Now we show that $Y^* = (y^*, \rho^*)$ is the solution of (3.1) with respect to u^* . We first show that

$$\gamma(\rho_n) y_n \rightarrow \gamma(\rho^*) y^* \text{ strongly in } L^2(0, S; L^2(I)). \tag{4.8}$$

Indeed, since $\rho^*, \rho_n, y_n \in L^\infty((0, S) \times I)$, we have

$$\begin{aligned} &\int_0^S \int_I (\gamma(\rho_n) y_n - \gamma(\rho^*) y^*)^2 dx dt \\ &\leq C \int_0^S \int_I (\rho_n + \rho^* - 2b)^2 (\rho_n - \rho^*)^2 y_n^2 dx dt + C \int_0^S \int_I \gamma(\rho^*)^2 (y_n - y^*)^2 dx dt \\ &\leq C \left(\int_0^S \int_I (\rho_n - \rho^*)^2 dx dt + \int_0^S \int_I (y_n - y^*)^2 dx dt \right). \end{aligned}$$

From (4.5) and (4.7), we see that (4.8) is satisfied. Furthermore, since $y^* \in L^\infty((0, S) \times I)$, we have

$$\begin{aligned} & \int_0^S \int_I (u_n(t)y_n - u^*(t)y^*)^2 dx dt \\ & \leq C \left(\|u_n\|_{L^\infty(0,S)} \|y_n - y^*\|_{L^2(0,S;L^2(I))} + \|u_n(t) - u(t)\|_{L^2(0,S)} \|y^*\|_{L^\infty(0,S;L^\infty(I))} \right) \\ & \leq C \left(\|u_n\|_{H^1(0,S)} \|y_n - y^*\|_{L^2(0,S;L^2(I))} + \|u_n(t) - u(t)\|_{L^2(0,S)} \|y^*\|_{L^\infty(0,S;L^\infty(I))} \right). \end{aligned}$$

By using (4.1) and (4.5) we have

$$u_n y_n \rightarrow u^* y^* \text{ strongly in } L^2(0, S; L^2(I)).$$

Therefore, $Y^* = (y^*)$ is solution of (3.1) with respect to u^* . Since $Y_n - Y_d$ is strongly convergent to $Y^* - Y_d$ in $L^2(0, T; \mathcal{H})$, we have:

$$\min_{u \in U_{ad}} \mathcal{J}(u) \leq \mathcal{J}(u^*) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(u_n) = \min_{u \in U_{ad}} \mathcal{J}(u).$$

Hence, $\mathcal{J}(u^*) = \min_{u \in U_{ad}} \mathcal{J}(u)$. □

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