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Estimation of Hurst Parameter in Longitudinal Data with Long Memory

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Abstract

This paper considers the problem of estimation of the Hurst parameter $H \in (1/2, 1)$ from longitudinal data with the error term of a fractional Brownian motion with Hurst parameter H that gives the amount of the long memory of its increment. We provide a new estimator of Hurst parameter H using a two scale sampling method based on Aït-Sahalia and Jacod (2009). Asymptotic behaviors (consistent and central limit theorem) of the proposed estimator will be investigated. For the proof of a central limit theorem, we use recent results on necessary and sufficient conditions for multi-dimensional vectors of multiple stochastic integrals to converges in distribution to multivariate normal distribution studied by Nourdin *et al.* (2010), Nualart and Ortiz-Latorre (2008), and Peccati and Tudor (2005).

Keywords: Malliavin calculus, multiple stochastic integrals, central limit theorem, Hurst parameter, longitudinal data, Hurst parameter, fractional Brownian motion

1. Introduction

A fractional Brownian motion $\{B^H, t \geq 0\}$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process with the covariance function

$$\mathbb{E}\left[B^H(t)B^H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0.$$

The Hurst parameter $H \in (0, 1)$ characterizes self-similar behavior of the process. This parameter gives the long-range dependence property of its increments and decides the regularity of the sample paths. Hence the problem of properly estimating Hurst parameter H is imperative. Many methods for estimating H of $\{B^H, t \geq 0\}$ have been proposed to solve this problem, such as wavelets, k-variations, variograms, maximum likelihood method and spectral methods, some of which can be found in the book by Beran (1994).

In this paper, we consider the problem of estimation of Hurst parameter H in the following longitudinal data, allowing intercept function varying over u and other coefficient β_1 being constant:

$$Y_i(t) = (\beta_0 + u_i) + \beta_1 x_i(t) + B_i^H(t), \quad i = 1, \ldots, d \;\text{ and }\; t \in [0, T], \tag{1.1}$$

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where B_i^H, $i = 1, \ldots, d$, are mutually independent fractional Brownian motions with Hurst parameter H and $x_i(t)$ is a non-random function. This paper investigates asymptotic behaviors (consistent and central limit theorem) of an estimator of the Hurst parameter H by introducing a method based on the ratio of two realized power variation with different sampling frequencies.

If the data $\{Y_i(t)\}$, $i = 1, \ldots, d$, have the long memory property for each series, then we may use the model (1.1) for a statistical application. In practice, the fractional Brownian motion B_H of the Hurst parameter $H \in (0,1)$ usually depends on scale parameter σ such that

$$\mathbb{E}\left[B_{H,\sigma}(t)^2\right] = \sigma^2 t^{2H}, \quad t \geq 0 \text{ and } \sigma > 0.$$

The process $B_{H,\sigma}(t)$ denote a path of a fractional Brownian motion with parameter $(H, \sigma) \in (0,1) \times (0, \infty)$. Suppose that we observe $\{Y_i(t)\}$ at times $j\Delta_n$, $j = 1, \ldots, [T/\Delta_n]$ and at cross section $i = 1, \ldots, d$. Assume that all series in the longitudinal data have the same Hurst parameter H and σ. For practical purpose, we have to estimate Hurst parameter H first, and then a realization, obtained by the data Y_i, of the estimator $\hat{H}_{ols}(n,d)$ proposed in this paper is plugged into H in the model (1.1). The model (1.1) becomes

$$Y_i(t) = (\beta_0 + u_i) + \beta_1 x_i(t) + \epsilon_i(t), \quad i = 1, \ldots, d \text{ and } t \in [0, T], \tag{1.2}$$

where the error term $\epsilon_i(t)$ is a fractional Brownian motion with $\epsilon_i(t+h) - \epsilon_i(t) \sim \mathcal{N}(0, \sigma^2 h^{2\hat{H}_{ols}(n,d)})$. After that, we may use the usual longitudinal data analysis in order to estimate the linear regression model (1.2).

The main tool for the proof of the central limit theorem is the Malliavin calculus and the result in Nourdin *et al.* (2010), which is a collection of some of the results contained in the paper by Peccati and Tudor (2005) and Nualart and Ortiz-Latorre (2008).

2. Preliminaries

In this section, we briefly review some basic facts about Malliavin calculus for Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that $\mathbb{H}$ is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. Let $X = \{X(h), h \in \mathbb{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_{\mathbb{H}}$. If $X = B^H$, then

$$\mathbb{E}\left[B^H(t)B^H(s)\right] = \langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathbb{H}} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For every $q \geq 1$, let $\mathcal{H}_q$ be the q^{th} Wiener chaos of X, that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_q(X(h)) : h \in \mathbb{H}, \|h\|_{\mathbb{H}} = 1\}$, where H_q is the q^{th} Hermite polynomial. We define a linear isometric mapping $I_q : \mathbb{H}^{\odot q} \to \mathcal{H}_q$ by $I_q(h^{\otimes q}) = H_q(X(h))$, where $\mathbb{H}^{\odot n}$ is the symmetric tensor product. The following duality formula holds

$$\mathbb{E}\left[F I_q(h)\right] = \mathbb{E}\left[\langle D^q F, h \rangle_{\mathbb{H}^{\otimes q}}\right], \tag{2.1}$$

for any element $h \in \mathbb{H}^{\odot q}$ and any random variable $F \in \mathbb{D}^{q,2}$. Here $\mathbb{D}^{q,2}$ is the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{q,2}^2 = \mathbb{E}\left[F^2\right] + \sum_{k=1}^{q} \mathbb{E}\left[\left\|D^k F\right\|_{\mathbb{H}^{\otimes k}}^2\right],$$

where D^k is the iterative Malliavin derivative. The linear isometric mapping I_q satisfies $I_q(f) = I_q(\tilde{f})$ and

$$\mathbb{E}\left[I_p(f)I_q(g)\right] = \begin{cases} 0, & \text{if } p \neq q, \\ p!\langle \tilde{f}, \tilde{g}\rangle_{\mathbb{H}}, & \text{if } p = q, \end{cases} \tag{2.2}$$

where $\tilde{f}$ denotes the symmetrization of f.

If $f \in \mathbb{H}^{\odot p}$, the Malliavin derivative of the multiple stochastic integrals is given by

$$D_z I_q(f_q) = qI_{q-1}\left(f_q(\,\cdot\,, z)\right) \quad \text{for } z \in [0,1]^2. \tag{2.3}$$

Let $\{e_l, l \geq 1\}$ be a complete orthonormal system in $\mathbb{H}$.

If $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$, the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, is the element of $\mathbb{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{l_1,\ldots,l_r=1}^{\infty} \langle f, e_{l_1} \otimes \cdots \otimes e_{l_r}\rangle_{\mathbb{H}^{\otimes r}} \otimes \langle g, e_{l_1} \otimes \cdots \otimes e_{l_r}\rangle_{\mathbb{H}^{\otimes r}}. \tag{2.4}$$

Notice that the tensor product $f \otimes g$ and the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, are not necessarily symmetric even though f and g are symmetric. We will denote their symmetrizations by $f\tilde{\otimes}g$ and $f\tilde{\otimes}_r g$, respectively. The following formula for the product of the multiple stochastic integrals will be frequently used to prove the main result in this paper.

Proposition 1. *Let $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$ be two symmetric functions. Then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2.5}$$

The main tool for the proof of our result is the Malliavin calculus and the following theorem in Nourdin *et al.* (2010), which is a collection of some of the results contained in the paper by Peccati and Tudor (2005) and Nualart and Ortiz-Latorre (2008).

Theorem 1. (Nourdin *et al.*, 2010) *Fix $d \geq 2$ and let $\Sigma = (\kappa_{i,j})_{i,j=1,\ldots,d}$ be a $d \times d$ positive definite matrix. Fix integers $1 \leq q_1 \leq \cdots \leq q_d$. For any $n \geq 1$ and $i = 1, \ldots, d$, $f_i^{(n)} \in \mathbb{H}^{\odot q_i}$. Assume that*

$$F_n = \left(I_{q_1}\left(f_1^{(n)}\right), \ldots, I_{q_d}\left(f_d^{(n)}\right)\right), \quad n \geq 1,$$

is such that

$$\lim_{n\to\infty} \mathbb{E}\left[I_{q_i}\left(f_i^{(n)}\right) I_{q_j}\left(f_j^{(n)}\right)\right] = \kappa_{i,j}, \quad 1 \leq i, j \leq d.$$

Then the followings are equivalent:

(i) For every $1 \leq i \leq d$, the sequence $\{I_{q_i}(f_i^{(n)}), n \geq 1\}$ converges to a normal distribution $\mathbb{N}(0, \kappa_{i,i})$.

(ii) For every $1 \leq i \leq d$, $\lim_{n\to\infty} \mathbb{E}[(I_{q_i}(f_i^{(n)}))^4] = 3\kappa_{i,i}^2$.

(iii) For every $1 \leq i \leq d$ and every $1 \leq r \leq q_i - 1$, $\lim_{n\to\infty} \|f_i^{(n)} \otimes_r f_i^{(n)}\|_{\mathbb{H}^{\otimes 2(q_i - r)}} = 0$.

(iv) The random vector F_n converges in distribution to a d-dimensional Gaussian vector $\mathcal{N}_d(0, \Sigma)$.

3. Main Results

By two scale sampling method based on Aït-Sahalia and Jacod (2009), we propose a statistic defined by

$$U_n^{(i)} = \frac{\sum_{l=1}^{[T/k\Delta_n]} \left|\Delta_{l,k}^n Y_i\right|^2}{\sum_{l=1}^{[T/\Delta_n]} \left|\Delta_l^n Y_i\right|^2}, \tag{3.1}$$

where $\Delta_l^n Y_i = Y_i(l\Delta_n) - Y_i((l-1)\Delta_n)$ and $\Delta_{l,k}^n Y_i = Y_i(lk\Delta_n) - Y_i((l-1)k\Delta_n)$ for determined positive integer k. Assume that

$$(\mathbf{A}) \text{ for all } i = 1, \ldots, d,\ |x_i(t) - x_i(s)| \le c|t-s| \text{ for a constant } c > 0.$$

We compute a realized quadratic variation of Y_i. First we write

$$\begin{aligned}
&\Delta_n^{1-2H} \sum_{l=1}^{[T/\Delta_n]} \left|\Delta_l^n Y_i\right|^2 \\
&= \Delta_n^{1-2H} \sum_{l=1}^{[T/\Delta_n]} \left[\beta_1^2 \{x_i(l\Delta_n) - x_i((l-1)\Delta_n)\}^2 + 2\beta_1 \{x_i(l\Delta_n) - x_i((l-1)\Delta_n)\} \Delta_l^n B_i^H + \left(\Delta_l^n B_i^H\right)^2\right] \\
&:= A_{1,n}(T) + A_{2,n}(T) + A_{3,n}(T).
\end{aligned} \tag{3.2}$$

By assumption **(A)**, the first term in (3.2) becomes, as $n \to 0$,

$$\left|A_{1,n}(T)\right| \le c\Delta_n^{2(1-H)} \to 0.$$

For every $\varepsilon > 0$, we obtain, as $n \to \infty$,

$$\mathbb{P}\left(\left|A_{2,n}(T)\right| > \varepsilon\right) \le \frac{c\Delta_n^{2-2H} \sum_{l=1}^{[T/\Delta_n]} \sqrt{\mathbb{E}\left[\left(\Delta_l^n B_i^H\right)^2\right]}}{\varepsilon} \le \frac{c\Delta_n^{1-H}}{\varepsilon} \to 0. \tag{3.3}$$

As for the term $A_{3,n}(T)$, we compute the expectation and variance of $A_{3,n}(T)$. Obviously,

$$\mathbb{E}\left[A_{3,n}(T)\right] = \Delta_n \left[\frac{T}{\Delta_n}\right], \tag{3.4}$$

and

$$\begin{aligned}
\mathbb{V}\text{ar}\left(A_{3,n}(T)\right) &= \Delta_n^{2-4H} \sum_{l=1}^{[T/\Delta_n]} \mathbb{V}\text{ar}\left(\left(\Delta_l^n B_i^H\right)^2\right) + \Delta_n^{2-4H} \sum_{l,l'=1}^{[T/\Delta_n]} \mathbb{C}\text{ov}\left(\left(\Delta_l^n B_i^H\right)^2, \left(\Delta_{l'}^n B_i^H\right)^2\right) \\
&= 3\Delta_n^2 \left[\frac{T}{\Delta_n}\right] + \Delta_n^2 \sum_{l=1}^{[T/\Delta_n]-1} \left(\left[\frac{T}{\Delta_n}\right] - l\right)\left((l+1)^{2H} + (l-1)^{2H} - 2l^{2H}\right) \\
&= 3\Delta_n^2 \left[\frac{T}{\Delta_n}\right] + \Delta_n^2 \left[\frac{T}{\Delta_n}\right]\left[\left(\left[\frac{T}{\Delta_n}\right]\right)^{2H} - 1 - \left(\left[\frac{T}{\Delta_n}\right] - 1\right)^{2H}\right].
\end{aligned} \tag{3.5}$$

Hence we obtain, from (3.4) and (3.5), that for every $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}\left(\left|A_{3,n}(T) - T\right| > \varepsilon\right) \le \frac{2}{\varepsilon}\mathbb{V}\mathrm{ar}\left(A_{3,n}(T)\right) + \mathbb{P}\left(\left|\Delta_n\left[\frac{T}{\Delta_n}\right] - T\right| > \varepsilon\right) \to 0. \tag{3.6}$$

Also it is clear that

$$\Delta_n^{1-2H}\sum_{l=1}^{[T/k\Delta_n]}\left|\Delta_{l,k}^n Y_i\right|^2 \xrightarrow{p} k^{2H-1}T, \tag{3.7}$$

where notation $\xrightarrow{p}$ denotes the convergence in probability. From (3.6) and (3.7), it follows that for $i = 1, \ldots, d$,

$$U_n^{(i)} \xrightarrow{p} k^{2H-1}, \quad i.e, \ \log\left(U_n^{(i)}\right) \xrightarrow{p} (2H-1)\log k. \tag{3.8}$$

In this section, we consider an estimator of H obtained by applying the least square method to the approximate relationship, given in (3.8),

$$\log\left(U_n^{(i)}\right) \approx (2H-1)\log k. \tag{3.9}$$

Then the ordinary least square estimator $\hat{H}_{ols}$ is given by

$$\hat{H}_{ols}(n,d) = \frac{\sum_{i=1}^d \log\left(U_n^{(i)}\right) + d\log k}{d\log k^2}. \tag{3.10}$$

Now we prove the consistency and central limit theorem of the estimator $\hat{H}_{ols}(n)$ given in (3.9). Let us set

$$\rho_H(l) = \frac{1}{2}\left(|l+1|^{2H} + |l-1|^{2H} - 2|l|^{2H}\right).$$

Note that

$$|\rho_H(l)| = H(2H-1)|l|^{2H-2} + o\left(|l|^{2H-2}\right) \quad \text{as } |l| \to \infty. \tag{3.11}$$

Theorem 2. *Under the assumption* **(A)**, *we have, as* $n \to \infty$,

(i) $\hat{H}_{ols}(n) \xrightarrow{p} H$.

(ii) If $H < 3/4$,

$$\frac{1}{\sqrt{\Delta_n}}\left(\hat{H}_{ols}(n,d) - H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2}{d\left(T\log k^2\right)^2}\right),$$

where σ^2 *is given by*

$$\sigma^2 = 2T(k+1)\sum_{j\in\mathbb{Z}}\rho_H(j)^2 - 2k^{-2H}\sum_{l\in\mathbb{Z}}\sum_{j=1}^{k}\left(\sum_{r=1}^{k}\rho_H(lk+r-j)\right)^2.$$

Proof: From (3.9), it is obvious that (i) holds. As for the proof of (ii), using the multiplication formula of multiple stochastic integral in (2.5) yields

$$(k\Delta_n)^{1-2H}\sum_{l=1}^{[T/k\Delta_n]}\left|\Delta_{l,k}^n B^H\right|^2 = I_2(f_{n,k}) + (k\Delta_n)\,k\left[\frac{T}{k\Delta_n}\right],$$
$$\Delta_n^{1-2H}\sum_{l=1}^{[T/\Delta_n]}\left|\Delta_l^n B^H\right|^2 = I_2(g_n) + \Delta_n\left[\frac{T}{\Delta_n}\right],$$

where the kernels $f_{n,k}$ and g_n are given by

$$f_{n,k} = (k\Delta_n)^{1-2H}\sum_{l=1}^{[T/k\Delta_n]}\mathbf{1}_{[(l-1)k\Delta_n,lk\Delta_n]}^{\otimes 2} \quad \text{and} \quad g_n = \Delta_n^{1-2H}\sum_{l=1}^{[T/\Delta_n]}\mathbf{1}_{[(l-1)\Delta_n,l\Delta_n]}^{\otimes 2}.$$

We will prove that for $H < 3/4$

$$\Delta_n^{-\frac{1}{2}}A_{i,n}(T) \xrightarrow{p} 0, \quad i = 1,2. \tag{3.12}$$

Obviously, as $n \to 0$,

$$\Delta_n^{-\frac{1}{2}}|A_{1,n}(T)| \le c\Delta_n^{\frac{3}{2}-2H} \to 0 \quad \text{for } H < \frac{3}{4}. \tag{3.13}$$

Let us set

$$W_n = \sum_{l=1}^{[T/\Delta_n]} 2\beta_1\,(x_i(l\Delta_n) - x_i((l-1)\Delta_n))\,\Delta_l^n B_i^H.$$

We note that

$$\begin{aligned}\mathbb{V}\text{ar}(W_n) &= 4\beta_1^2\Delta_n^{2H}\sum_{l=1}^{[T/\Delta_n]}(x_i(l\Delta_n) - x_i((l-1)\Delta_n))^2\\ &+ 2\beta_1^2\Delta_n^{2H}\sum_{l,l'=1}^{[T/\Delta_n]}(x_i(l\Delta_n) - x_i((l-1)\Delta_n))\,(x_i\,(l'\Delta_n) - x_i\,((l'-1)\,\Delta_n))\\ &\times\left(\left|l-l'+1\right|^{2H} + \left|l-l'-1\right|^{2H} - 2\left|l-l'\right|^{2H}\right) := \tilde{\sigma}_n^2.\end{aligned} \tag{3.14}$$

This (3.14) implies that W_n is the centered Gaussian random variable with the variance $\tilde{\sigma}_n^2$. For sufficiently large n, we estimate, from (3.11),

$$\begin{aligned}\tilde{\sigma}_n^2 &\le c\Delta_n^{1+2H} + \Delta_n^{2+2H}\sum_{|j|<[T/\Delta_n]}\left(\left[\frac{T}{\Delta_n}\right] - |j|\right)\left|\left|j'+1\right|^{2H} + \left|j'-1\right|^{2H} - 2|j|^{2H}\right|\\ &\le c_{H,T}\left\{\Delta_n^{1+2H} + \Delta_n^{2+2H}\sum_{j=1}^{[T/\Delta_n]}\Delta_n^{-1}\,j^{2H-2}\right\}\\ &\le c_{H,T}\left\{\Delta_n^{1+2H} + \Delta_n^{2+2H}\left(\Delta_n^{-1} + \Delta_n^{-2H}\right)\right\}\\ &\le c_{H,T}\Delta_n^2.\end{aligned} \tag{3.15}$$

For every $\varepsilon > 0$, we obtain, from (3.15), that, as $n \to \infty$,

$$\begin{aligned}\mathbb{P}\left(\Delta_n^{-\frac{1}{2}}|A_{2,n}(T)| > \varepsilon\right) &= \mathbb{P}\left(\Delta_n^{\frac{1}{2}-2H}|W_n| > \varepsilon\right)\\ &\le c_{H,T}\frac{\sqrt{2}\Delta_n^{\frac{1}{2}-2H}\tilde{\sigma}_n}{\sqrt{\pi}\varepsilon}\\ &\le c_{H,T}\frac{\Delta_n^{\frac{3}{2}-2H}}{\varepsilon} \to 0 \quad \text{for } H < \frac{3}{4}. \end{aligned} \tag{3.16}$$

From (3.13) and (3.16), it is suffice to consider only the terms $\sum_{l=1}^{[T/k\Delta_n]} |\Delta_{l,k}^n B^H|^2$ and $\sum_{l=1}^{[T/\Delta_n]} |\Delta_l^n B^H|^2$ in order to prove (ii). First we will show that a sequence of the random vector

$$\left(\Delta_n^{-\frac{1}{2}} I_2(f_{n,k}), \Delta_n^{-\frac{1}{2}} I_2(g_n)\right)$$

converges in distribution to a two-dimensional Gaussian random variable with mean 0 and covariance Σ. By Theorem 1, we need to show that

$$\begin{aligned}&\text{(i)}\ \ 2\lim_{n\to\infty}\Delta_n^{-1}\|f_{n,k}\|^2_{\mathbb{H}^{\otimes 2}} = \sigma_{11}^2, \quad 2\lim_{n\to\infty}\Delta_n^{-1}\|g_n\|^2_{\mathbb{H}^{\otimes 2}} = \sigma_{22}^2 \quad \text{and} \quad 2\lim_{n\to\infty}\Delta_n^{-1}\langle f_{n,k}, g_n\rangle_{\mathbb{H}^{\otimes 2}} = \sigma_{12}^2.\\ &\text{(ii)}\ \ \lim_{n\to\infty}\Delta_n^{-2}\|f_{n,k}\otimes_1 f_{n,k}\|^2_{\mathbb{H}^{\otimes 2}} = 0 \quad \text{and} \quad \lim_{n\to\infty}\Delta_n^{-2}\|g_n\otimes_1 g_n\|^2_{\mathbb{H}^{\otimes 2}} = 0.\end{aligned} \tag{3.17}$$

The first limit in (i) can be computed as

$$\begin{aligned}2\Delta_n^{-1}\left\|f_{n,k}\right\|^2_{\mathbb{H}^{\otimes 2}} &= 2\Delta_n^{-1}(k\Delta_n)^{2-4H}\sum_{l,l'=1}^{[T/k\Delta_n]}\left\langle \mathbf{1}^{\otimes 2}_{[(l-1)k\Delta_n, lk\Delta_n]}, \mathbf{1}^{\otimes 2}_{[(l'-1)k\Delta_n, l'k\Delta_n]}\right\rangle_{\mathbb{H}^{\otimes 2}}\\ &= 2\Delta_n^{-1}(k\Delta_n)^2\sum_{l,l'=1}^{[T/k\Delta_n]}\rho_H(l-l')^2\\ &= 2k^2\sum_{|j|<[T/k\Delta_n]}\Delta_n\left(\left[\frac{T}{k\Delta_n}\right]-|j|\right)\rho_H(j)^2.\end{aligned}$$

Hence, since $\sum_{j\in\mathbb{Z}}\rho_H(j)^2 < \infty$, we obtain, by dominated convergence theorem, that if $H < 3/4$,

$$2\lim_{n\to\infty}\Delta_n^{-1}\left\|f_{n,k}\right\|^2_{\mathcal{H}^{\otimes 2}} = 2kT\sum_{j\in\mathbb{Z}}\rho_H(j)^2. \tag{3.18}$$

When $k = 1$, we also have

$$2\lim_{n\to\infty}\Delta_n^{-1}\|g_n\|^2_{\mathcal{H}^{\otimes 2}} = 2T\sum_{j\in\mathbb{Z}}\rho_H(j)^2. \tag{3.19}$$

Using a similar argument as for the first term in (i) yields, as $n \to \infty$,

$$\begin{aligned}
\Delta_n^{-1}\langle f_{n,k}, g_n\rangle_{\mathbb{H}^{\otimes 2}} &= k^{1-2H}\Delta_n^{1-4H}\sum_{l=1}^{[T/k\Delta_n]}\sum_{l'=1}^{[T/\Delta_n]}\left\langle \mathbf{1}^{\otimes 2}_{[(l-1)k\Delta_n, lk\Delta_n]}, \mathbf{1}^{\otimes 2}_{[(l'-1)\Delta_n, l'\Delta_n]}\right\rangle_{\mathbb{H}^{\otimes 2}} \\
&= k^{1-2H}\Delta_n \sum_{l=1}^{[T/k\Delta_n]}\sum_{l'=1}^{[T/k\Delta_n]}\sum_{j=1}^{k}\left(\left\langle \mathbf{1}_{[(l-1)k, lk]}, \mathbf{1}_{[(l'-1)k+j-1,(l'-1)k+j]}\right\rangle_{\mathbb{H}}\right)^2 \\
&= k^{1-2H}\Delta_n \sum_{l=1}^{[T/k\Delta_n]}\sum_{l'=1}^{[T/k\Delta_n]}\sum_{j=1}^{k}\left(\sum_{r=1}^{k}\rho_H((l-l')k+r-j)\right)^2 \\
&= k^{1-2H}\Delta_n \sum_{|l|<[T/k\Delta_n]}\sum_{j=1}^{k}\left(\left[\frac{T}{k\Delta_n}\right]-|l|\right)\left(\sum_{r=1}^{k}\rho_H(lk+r-j)\right)^2 \\
&\to k^{-2H}\sum_{l\in\mathbb{Z}}\sum_{j=1}^{k}\left(\sum_{r=1}^{k}\rho_H(lk+r-j)\right)^2 . \qquad (3.20)
\end{aligned}$$

As for the first term in (ii), we compute

$$\begin{aligned}
f_{n,k}\otimes_1 f_{n,k} &= (k\Delta_n)^{2-4H}\sum_{l,l'=1}^{[T/k\Delta_n]}\langle \mathbf{1}_{[(l-1)k\Delta_n, lk\Delta_n]}, \mathbf{1}_{[(l'-1)k\Delta_n, l'k\Delta_n]}\rangle_{\mathbb{H}} \times \mathbf{1}_{[(l-1)k\Delta_n, lk\Delta_n]}\tilde{\otimes}\mathbf{1}_{[(l'-1)k\Delta_n, l'k\Delta_n]} \\
&= (k\Delta_n)^{2-2H}\sum_{l,l'=1}^{[T/k\Delta_n]}\rho_H(l-l')\mathbf{1}_{[(l-1)k\Delta_n, lk\Delta_n]}\tilde{\otimes}\mathbf{1}_{[(l'-1)k\Delta_n, l'k\Delta_n]}. \qquad (3.21)
\end{aligned}$$

Let us set $\rho_{n,H}(j) = |\rho_{n,H}(j)|\mathbf{1}_{\{|j|\le[T/k\Delta_n]\}}$. By using the arguments in the quadratic variation of the fractional Brownian motion studied by Nourdin (2013), we obtain, from (3.21),

$$\begin{aligned}
\Delta_n^{-2}\left\|f_{n,k}\otimes_1 f_{n,k}\right\|^2_{\mathbb{H}^{\otimes 2}} &= k^4\Delta_n^2\sum_{l,l',j,j'=1}^{[T/k\Delta_n]}\rho_H(l-l')\rho_H(j-j')\rho_H(l-j)\rho_H(l'-j') \\
&\le k^4\Delta_n^2\sum_{l,j'=1}^{[T/k\Delta_n]}\sum_{j,l'\in\mathbb{Z}}\rho_{n,H}(l-l')\rho_{n,H}(j-j')\rho_{n,H}(l-j)\rho_{n,H}(l'-j') \\
&\le k^4\Delta_n\sum_{l\in\mathbb{Z}}(\rho_{n,H}*\rho_{n,H})(l)^2 \le k^4\Delta_n\left(\sum_{|l|\le[T/k\Delta_n]}|\rho_H(l)|^{\frac{4}{3}}\right)^3 .
\end{aligned}$$

Hence we have, from (3.11), that as $n \to \infty$,

$$k^4\Delta_n\left(\sum_{|l|\le[T/k\Delta_n]}|\rho_H(l)|^{\frac{4}{3}}\right)^3 \le c\Delta_n\left(1+\left[\frac{T}{k\Delta_n}\right]^{\frac{8H-5}{3}}\right)^3 \to 0,$$

which implies that

$$\lim_{n\to\infty}\Delta_n^{-2}\left\|f_{n,k}\otimes_1 f_{n,k}\right\|^2_{\mathbb{H}^{\otimes 2}} = 0. \qquad (3.22)$$

The same arguments as for the proof of (3.22) yield that

$$\lim_{n\to\infty} \Delta_n^{-2} \|g_n \otimes_1 g_n\|^2_{\mathbb{H}^{\otimes 2}} = 0. \tag{3.23}$$

By combining the above results (3.18), (3.19), (3.20), (3.22) and (3.23), we obtain

$$\left(\Delta_n^{-\frac{1}{2}} I_2(f_{n,k}), \Delta_n^{-\frac{1}{2}} I_2(g_n)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma), \tag{3.24}$$

where the matrix $\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix}$ is given by

$$\sigma_{11}^2 = 2kT \sum_{j\in\mathbb{Z}} \rho_H(j)^2, \qquad \sigma_{22}^2 = 2T \sum_{j\in\mathbb{Z}} \rho_H(j)^2, \qquad \sigma_{12}^2 = k^{-2H} \sum_{l\in\mathbb{Z}} \sum_{j=1}^{k} \left(\sum_{r=1}^{k} \rho_H(lk + r - j) \right)^2 .$$

For $i = 1, \ldots, d$, we write

$$\frac{1}{\sqrt{\Delta_n}} \left(U_n^{(i)} - k^{2H-1} \right) = \frac{k^{2H-1}}{I_2(g_n) + \Delta_n^{2-2H}[T/\Delta_n]} \times \frac{I_2(f_{n,k}) - I_2(g_n)}{\sqrt{\Delta_n}}. \tag{3.25}$$

Obviously, from (3.12) and (3.22), it follows that

$$\frac{k^{2H-1}}{I_2(g_n) + \Delta_n^{2-2H}[T/\Delta_n]} \xrightarrow{p} \frac{k^{2H-1}}{T}. \tag{3.26}$$

By (3.25) together with (3.24) and (3.26), we have

$$\frac{1}{\sqrt{\Delta_n}} \left(U_n^{(i)} - k^{2H-1} \right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{k^{4H-2}\sigma^2}{T^2}\right), \tag{3.27}$$

where $\sigma^2 = \sigma_{11}^2 + \sigma_{22}^2 - 2\sigma_{12}^2$. By applying the delta-method with $f(x) = \log x$ to (3.27),

$$\frac{\sum_{i=1}^{d} \left(\log \left(U_n^{(i)} \right) - (2H - 1) \log k \right)}{\sqrt{\Delta_n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{d\sigma^2}{T^2}\right). \tag{3.28}$$

From (3.28), we get

$$\begin{aligned} \frac{1}{\sqrt{\Delta_n}} \left(\hat{H}_{ols}(n, d) - H \right) &= \frac{\sum_{i=1}^{d} \left(\log \left(U_n^{(i)} \right) - (2H - 1) \log k \right)}{\sqrt{\Delta_n} d \log k^2} \\ &\xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2}{d\,(T \log k^2)^2}\right). \end{aligned}$$

Hence the proof of theorem is completed. □

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