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Noninformative priors for product of exponential means

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Abstract

In this paper, we develop the noninformative priors for the product of different powers of k means in the exponential distribution. We developed the first and second order matching priors. It turns out that the second order matching prior matches the alternative coverage probabilities, and is the highest posterior density matching prior. Also we revealed that the derived reference prior is the second order matching prior, and Jeffreys' prior and reference prior are the same. We showed that the proposed reference prior matches very well the target coverage probabilities in a frequentist sense through simulation study, and an example based on real data is given.

Keywords: Matching prior, nonlinear functions of exponential means, reference prior.

1. Introduction

The exponential distribution plays an important role in the field of reliability. The usefulness of the exponential distribution in reliability applications can be found in the early work of Davis (1952), Epstein and Sobel (1953), and others. Further justification, in the form of theoretical arguments to support the use of the exponential distribution as the failure law of complex equipment, is presented in the book by Barlow and Proschan (1975) and Lawless (2003).

The problem of making inference about the product of means has been studied by Southwood (1978) and Yfantis and Flatman (1991). A problem of making inference about the product of normal means can be recognized as the determination of an area of a rectangle based on measurements of length and width when the distribution of the length and width is normal. Also it can be viewed as the volume of a cuboid when the length, width, and height represent the means of three normal random variables. In environmental applications, such as exposure assessment and risk modeling, the estimation of product of normal means is desired (Southwood, 1978; Yfantis and Flatman, 1991).

In Bayesian view points, Berger and Bernardo (1989) studied the reference prior analysis for product of means of two normal populations with common known variance. Sun and

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Ye (1995) developed the two group reference prior for k normal populations with common known variance. In general, the variances are seldom known in real applications. So Sun and Ye (1999) derived the two group reference prior for k normal populations with unknown variances. It is very difficult or impossible to compute reference priors for other groups of ordering of parameters (Sun and Ye, 1995, 1999). Also for Poisson distributions, Kim (2006) and Raubenheimer (2012) derived the probability matching prior for the product of different powers of k Poisson rates.

Suppose that $X_{ij}, i = 1, \cdots, k, j = 1, \cdots, n_i$ are independent exponential random variables with mean λ_i for $i = 1, \cdots, k$. The parameter of interest is, $\theta = \prod_{i=1}^{k} \lambda_i^{c_i}$, nonlinear functions of k means. When $c_i = 1/k$, then θ is the geometric mean, and the estimation of θ arises in environmental applications and in economic applications (Kenneth *et al.*, 1998). For $c_i = 1$, the θ is the product of means. When $k = 2$, $c_1 = 1$ and $c_2 = -1$, θ is the ratio of two means. When $k = 4$, $c_1 = c_2 = 1/2$ and $c_3 = c_4 = -1/2$, θ is the ratio of two geometric means.

In this paper, we focus on the development of noninformative priors for θ. For the Bayesian inference of θ, we want to develop noninformative priors for this parameter. The Bayesian analysis using the noninformative or objective prior has been very popular and many authors have made an effort for developing the noninformative priors in various parameters of interest under many statistical models.

We consider two kinds of noninformative prior in this paper. One is a probability matching prior introduced by Welch and Peers (1963) which matches the posterior and frequentist probabilities of confidence intervals. Interest in such priors has been revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), DiCiccio and Stern (1994), Datta and Ghosh (1995, 1996), Mukerjee and Ghosh (1997). The other is the reference prior introduced by Bernardo (1979) which maximizes the Kullback-Leibler divergence between the prior and the posterior. Ghosh and Mukerjee (1992), and Berger and Bernardo (1989, 1992) give a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems (Kang *et al.*, 2013, 2014). Quite often reference priors satisfy the matching criterion described earlier.

The outline of the remaining sections is as follows. In Section 2, we develop the first order and the second order probability matching priors. We reveal that the second order matching prior is a highest posterior density (HPD) matching prior and matches the alternative coverage probabilities up to the second order. Also we derive the reference priors for the parameter. It turns out that the reference prior and Jeffreys' prior are the second order matching prior. Section 3 devotes to show that the propriety of the posterior distribution for the general prior including the reference prior and the matching prior. In Section 4, simulated frequentist coverage probabilities under the proposed prior and an example are given.

2. The noninformative priors

Let $x_{i1}, x_{i2}, \cdots, x_{in_i}$ denote observations from the exponential distribution with mean $\lambda_i, i = 1, \cdots, k$. Then likelihood function is given by

$$f(\mathbf{x}|\lambda_1, \cdots, \lambda_k) = \prod_{i=1}^{n_i} \lambda^{-n_i} \exp\left\{ -\sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{x_{ij}}{\lambda_i} \right\}, \tag{2.1}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_k)$ and $\mathbf{x}_i = (x_{i1}, \cdots, x_{in_i})$. We want to make a Bayesian inference about the product of different powers of k means based on noninformative prior or objective priors. Therefore we will develop the noninformative priors for $\prod_{i=1}^{k} \lambda_i^{c_i}$, $-\infty < c_i < \infty, i = 1, \cdots, k$.

2.1. The probability matching priors

For a prior π, let $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ denote the $(1-\alpha)$th posterior quantile of θ_1, that is,

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})|\mathbf{X}] = 1 - \alpha, \tag{2.2}$$

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_t)^T$ and θ_1 is the parameter of interest. We want to find priors π for which

$$P_{\theta}[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})] = 1 - \alpha + o(n^{-r}), \tag{2.3}$$

for some $r > 0$, as n goes to infinity. Priors π satisfying (2.3) are called matching priors. If $r = 1/2$, then π is referred to as a first order matching prior, while if $r = 1$, π is referred to as a second order matching prior.

Firstly, we develop the matching prior for the scale parameter. In order to find such matching priors π, let

$$\theta_1 = \prod_{i=1}^{k} \lambda_i^{c_i} \text{ and } \theta_i = \lambda_1^{-\frac{c_i/n_i}{c_1/n_1}} \lambda_i, i = 2, \cdots, k.$$

With this parametrization, the likelihood function of parameters $(\theta_1, \cdots, \theta_k)$ is given by

$$L(\theta_1, \cdots, \theta_k) \tag{2.4}$$

$$\propto \quad \theta_1^{-\frac{\sum_{i=1}^{k} c_i}{\sum_{i=1}^{k} c_i^2/n_i}} \prod_{i=2}^{k} \theta_i^{-n_i + \frac{c_i \sum_{j=1}^{k} c_j}{\sum_{j=1}^{k} c_j^2/n_j}}$$

$$\times \quad \exp\left\{ -\left[\frac{\theta_1}{\prod_{i=2}^{k} \theta_i^{c_i}} \right]^{-\frac{c_1/n_1}{\sum_{i=1}^{k} c_i^2/n_i}} S_1 - \sum_{i=2}^{k} \left[\frac{\theta_1}{\prod_{j=2}^{k} \theta_j^{c_j}} \right]^{-\frac{c_i/n_i}{\sum_{j=1}^{k} c_j^2/n_j}} \theta_i^{-1} S_i \right\}, \tag{2.5}$$

where $S_i = \sum_{j=1}^{n_i} x_{ij}, i = 1, \cdots, k$. Based on (2.4), the Fisher information matrix is given by

$$\mathbf{I}(\theta_1, \cdots, \theta_k) = \begin{pmatrix} [\sum_{i=1}^{k} c_i^2/n_i]^{-1}\theta_1^{-2} & 0 & \cdots & 0 \\ 0 & n_2\theta_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n_k\theta_k^{-2} \end{pmatrix}. \tag{2.6}$$

From the above Fisher information matrix $\mathbf{I}$, θ_1 is orthogonal to $\theta_2, \cdots, \theta_k$ in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of the first order probability matching prior is characterized by

$$\pi_m^{(1)}(\theta_1, \cdots, \theta_k) \propto \theta_1^{-1} d(\theta_2, \cdots, \theta_k), \tag{2.7}$$

where $d(\theta_2, \cdots, \theta_k) > 0$ is an arbitrary function differentiable in its argument.

The class of prior given in (2.7) can be narrowed down to the second order probability matching priors as given in Mukerjee and Ghosh (1997). A second order probability matching prior is of the form (2.7), and d must satisfy an additional differential equation (2.10) of Mukerjee and Ghosh (1997), namely

$$\frac{1}{6} d(\theta_2, \cdots, \theta_k) \frac{\partial}{\partial \theta_1} \{I_{11}^{-\frac{3}{2}} L_{1,1,1}\} + \sum_{s=2}^{k} \frac{\partial}{\partial \theta_s} \{I_{11}^{-\frac{1}{2}} L_{11s} I^{ss} d(\theta_2, \cdots, \theta_k)\} = 0, \tag{2.8}$$

where

$$\begin{aligned}
L_{1,1,1} &= E\left[\left(\frac{\partial \log L}{\partial \theta_1}\right)^3\right] = \sum_{j=1}^{k} \frac{2c_j^3/n_j^2}{[\sum_{i=1}^{k} c_i^2/n_i]^3} \theta_1^{-3}, \\
L_{11s} &= E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_s}\right] = -c_s \left[\sum_{j=1}^{k} \frac{c_j^3/n_j^2}{[\sum_{i=1}^{k} c_i^2/n_i]^3} + \frac{1}{\sum_{i=1}^{k} c_i^2/n_i}\right] \theta_1^{-2} \theta_s^{-1}, s = 2, \cdots, k, \\
I_{11} &= \left[\sum_{i=1}^{k} c_i^2/n_i\right]^{-1} \theta_1^{-2}, I^{ss} = \frac{\theta_s^2}{n_s}, s = 2, \cdots, k.
\end{aligned}$$

Then (2.8) simplifies to

$$\sum_{s=2}^{k} \frac{\partial}{\partial \theta_s} \left\{ \frac{c_s}{n_s} \left[\sum_{j=1}^{k} \frac{c_j^3/n_j^2}{[\sum_{i=1}^{k} c_i^2/n_i]^3} + \frac{1}{\sum_{i=1}^{k} c_i^2/n_i}\right] \theta_1^{-1} \theta_s d(\theta_2, \cdots, \theta_k) \right\} = 0. \tag{2.9}$$

A solution of (2.9) is of the form $d(\theta_2, \cdots, \theta_k) = \prod_{i=2}^{k} \theta_i^{-1}$. Thus the resulting second order probability matching prior is

$$\pi_m^{(2)}(\theta_1, \cdots, \theta_k) \propto \theta_1^{-1} \cdots \theta_k^{-1}. \tag{2.10}$$

Remark 2.1 There are alternative ways through which matching can be accomplished. Datta et al. (2000) provided a theorem which establishes the equivalence of second order matching priors and HPD matching priors (DiCiccio and Stern, 1994; Ghosh and Mukerjee, 1995) within the class of first order matching priors. The equivalence condition is that $I_{11}^{-3/2} L_{111}$ dose not depend on θ_1. Since

$$L_{111} = E\left[\frac{\partial^3 \log L}{\partial \theta_1^3}\right] = \left[\sum_{j=1}^{k} \frac{c_j^3/n_j^2}{[\sum_{i=1}^{k} c_i^2/n_i]^3} + \frac{3}{\sum_{i=1}^{k} c_i^2/n_i}\right] \theta_1^{-3},$$

$I_{11}^{-3/2}L_{111}$ does not depend on θ_1. Therefore the second order probability matching prior (2.10) is a HPD matching prior. Also

$$\begin{aligned} L_{11,1} &= E\left[\frac{\partial^2 \log L}{\partial\theta_1^2}\frac{\partial \log L}{\partial\theta_1}\right] = w_1\theta_1^{-3}, \\ L_{11,j} &= E\left[\frac{\partial^2 \log L}{\partial\theta_1^2}\frac{\partial \log L}{\partial\theta_2}\right] = w_j\theta_1^{-2}\theta_j^{-1}, j = 2, \cdots, k, \end{aligned}$$

where $w_j, j = 1, \cdots, k$, is a constant. And $d(\theta_2, \cdots, \theta_k) = \theta_1^{-1} \cdots \theta_k^{-1}$. Then

$$\sum_{s=2}^{k} \frac{\partial}{\partial\theta_s}\{L_{11s}I^{ss}I_{11}^{-1/2}d(\theta_2, \cdots, \theta_k)\} = 0, \sum_{s=2}^{k} \frac{\partial}{\partial\theta_s}\{L_{11,s}I^{ss}I_{11}^{-1/2}d(\theta_s)\} = 0,$$
$$\frac{\partial}{\partial\theta_1}\{I_{11}^{-3/2}L_{111}\} = 0, \frac{\partial}{\partial\theta_1}\{I_{11}^{-3/2}L_{11,1}\} = 0.$$

Therefore the second order matching prior (2.10) matches the alternative coverage probabilities (Mukerjee and Reid, 1999).

2.2. The reference priors

Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1992) have become very popular over the years for the development of noninformative priors. In this section, we derive the reference priors for different groups of ordering of $(\theta_1, \cdots, \theta_k)$. Then due to the orthogonality of the parameters, following Datta and Ghosh (1995b), choosing rectangular compacts for each $\theta_1, \cdots, \theta_{k-1}$ and θ_k when θ_1 is the parameter of interest, the reference priors are given by as follows.

For the likelihood (2.4), if θ_1 is the parameter of interest, then the reference prior distributions for group of ordering of $\{(\theta_1, \cdots, \theta_k)\}$ is

$$\pi_J(\theta_1, \cdots, \theta_k) \propto \theta_1^{-1} \cdots \theta_k^{-1}.$$

For groups of ordering of $\{\theta_1, \cdots, \theta_k\}$ and $\{\theta_1, (\theta_2, \cdots, \theta_k)\}$, the reference prior is

$$\pi_r(\theta_1, \cdots, \theta_k) \propto \theta_1^{-1} \cdots \theta_k^{-1}.$$

Remark 2.2 From the above reference priors, we know that the one-at-a-time reference prior π_r and Jeffreys' prior are the same. Also the one-at-a-time reference prior π_r and Jeffreys' prior π_J satisfy the second order matching criterion.

3. Implementation of the Bayesian procedure

We investigate the propriety of posteriors for a general class of priors which includes the reference priors and the matching priors. We consider the class of priors

$$\pi(\theta_1, \cdots, \theta_k) \propto \theta_1^{-a_1} \cdots \theta_k^{-a_k}, \tag{3.1}$$

where $a_i > 0, i = 1, \cdots, k$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of $(\theta_1, \cdots, \theta_k)$ under the prior (3.1) is proper if $n_1 + c_1(a_1 - 1) - \frac{\sum_{i=1}^k (a_i-1)c_i/n_i}{c_1/n_1} + a_1 - 1 > 0$ and $n_i + c_i(a_1 - 1) + a_i - 1, i = 2, \cdots, k$.

Proof: Note that the joint posterior for $\theta_1, \cdots, \theta_{k-1}$ and θ_k given $\mathbf{x}$ is given by

$$\begin{aligned}\pi(\theta_1, \cdots, \theta_k|\mathbf{x}) \quad \propto \quad & \theta_1^{-\frac{\sum_{i=1}^k c_i}{\sum_{i=1}^k c_i^2/n_i} - a_1} \prod_{i=2}^k \theta_i^{-n_i + \frac{c_i \sum_{j=1}^k c_j}{\sum_{j=1}^k c_j^2/n_j} - a_i} \\ \times \quad & \exp\left\{ - \left[\frac{\theta_1}{\prod_{i=2}^k \theta_i^{c_i}}\right]^{-\frac{c_1/n_1}{\sum_{i=1}^k c_i^2/n_i}} S_1 - \sum_{i=2}^k \left[\frac{\theta_1}{\prod_{j=2}^k \theta_j^{c_j}}\right]^{-\frac{c_i/n_i}{\sum_{j=1}^k c_j^2/n_j}} \frac{S_i}{\theta_i} \right\},\end{aligned} \tag{3.2}$$

where $S_i = \sum_{j=1}^{n_i} x_{ij}, i = 1, \cdots, k$. Let $\theta_1 = \prod_{i=1}^k \lambda_i^{c_i}$ and $\theta_i = \lambda_1^{-\frac{c_i/n_i}{c_1/n_1}} \lambda_i, i = 2, \cdots, k$.
Then we have

$$\begin{aligned}\pi(\lambda_1, \cdots, \lambda_k|\mathbf{x}) \quad \propto \quad & \lambda_1^{-(n_1 + c_1(a_1-1) - \frac{\sum_{i=1}^k (a_i-1)c_i/n_i}{c_1/n_1} + a_1)} \\ \times \quad & \lambda_2^{-(n_2 + c_2(a_1-1) + a_2)} \cdots \lambda_k^{-(n_k + c_k(a_1-1) + a_k)} \exp\left\{ - \sum_{i=1}^k \frac{S_i}{\lambda_i} \right\}.\end{aligned} \tag{3.3}$$

Thus

$$\int_0^\infty \cdots \int_0^\infty \pi(\lambda_1, \cdots, \lambda_k|\mathbf{x})) d\lambda_1 \cdots d\lambda_k < \infty \tag{3.4}$$

if $n_1 + c_1(a_1 - 1) - \frac{\sum_{i=1}^k (a_i-1)c_i/n_i}{c_1/n_1} + a_1 - 1 > 0$ and $n_i + c_i(a_1 - 1) + a_i - 1, i = 2, \cdots, k$. This completes the proof. $\square$

Theorem 3.2 Under the prior (3.1), the marginal posterior density of θ_1 is given by

$$\begin{aligned}& \pi(\theta_1|\mathbf{x}) \\ \propto \quad & \int_0^\infty \cdots \int_0^\infty \theta_1^{-\frac{\sum_{i=1}^k c_i}{\sum_{i=1}^k c_i^2/n_i} - a_1} \prod_{i=2}^k \theta_i^{-n_i + \frac{c_i \sum_{j=1}^k c_j}{\sum_{j=1}^k c_j^2/n_j} - a_i} \\ \times \quad & \exp\left\{ - \left[\frac{\theta_1}{\prod_{i=2}^k \theta_i^{c_i}}\right]^{-\frac{c_1/n_1}{\sum_{i=1}^k c_i^2/n_i}} S_1 - \sum_{i=2}^k \left[\frac{\theta_1}{\prod_{j=2}^k \theta_j^{c_j}}\right]^{-\frac{c_i/n_i}{\sum_{j=1}^k c_j^2/n_j}} \frac{S_i}{\theta_i} \right\} d\theta_2 \cdots d\theta_k.\end{aligned} \tag{3.5}$$

Note that actually the marginal density of θ_1 requires $k - 1$ dimensional integration. Therefore, we use the Markov Chain Monte Carlo numerical integration, and so it is easy to compute the marginal moments of θ_1. In Section 4, we investigate the frequentist coverage probabilities for Jeffreys' prior π_J and the reference prior π_r, respectively.

4. Numerical studies

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of product of means , that is $\theta_1 = \prod_{i=1}^k \lambda_i$, under the noninformative prior π given in (3.1) for several configurations k, $(\lambda_1, \cdots, \lambda_k)$ and $n_1, \cdots, n_k$. That is to say, the frequentist coverage of a $(1-\alpha)th$ posterior quantile should be close to $1-\alpha$. Since no closed form posteriors are available, the posterior quantiles are obtained via application of the Markov Chain Monte Carlo numerical integration. We provide below some of the implementational details.

For general priors, we can derive the conditional posteriors, and so we provide below some of the implementational details.

The joint posterior of $\lambda_1, \cdots, \lambda_k$ given $\mathbf{x}$ is

$$\begin{aligned}\pi(\lambda_1, \cdots, \lambda_k|\mathbf{x}) &\propto \lambda_1^{-\left(n_1+c_1(a_1-1)-\frac{\sum_{i=1}^k (a_i-1)c_i/n_i}{c_1/n_1}+a_1\right)} \\ &\times \lambda_2^{-(n_2+c_2(a_1-1)+a_2)} \cdots \lambda_k^{-(n_k+c_k(a_1-1)+a_k)} \exp\left\{-\sum_{i=1}^k \frac{S_i}{\lambda_i}\right\}.\end{aligned}$$

This leads to the full conditionals

$$(\lambda_1|\lambda_2, \cdots, \lambda_k, \mathbf{x}) \propto \lambda_1^{-\left(n_1+c_1(a_1-1)-\frac{\sum_{i=1}^k (a_i-1)c_i/n_i}{c_1/n_1}+a_1\right)} \exp\left\{-\frac{S_1}{\lambda_1}\right\}, \quad (4.1)$$

$$(\lambda_i|\lambda_{j\neq i}, \mathbf{x}) \propto \lambda_i^{-(n_i+c_i(a_1-1)+a_i)} \exp\left\{-\frac{S_i}{\lambda_i}\right\}, i = 2, \cdots, k, j = 1, \cdots, k. \quad (4.2)$$

Note that the conditionals of $\lambda_i, i = 1, \cdots, k$ have the inverted gamma distributions.

In each case we generate samples $20,000$ (discarding the first 10,000), compute the $\theta_1 = \prod_{i=1}^k \lambda_i$ each time, and find numerically the 5% and 95% posterior quantiles of $\prod_{i=1}^k \lambda_i$. The whole process is repeated $10,000$ times, and we find the proportion of times the true $\prod_{i=1}^k \lambda_i$ belong to this interval. This is the estimated frequentist coverage probability of the Bayesian credible interval. Table 4.1 gives numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the our prior.

Tables 4.1 indicates that the reference prior meets very well the target coverage probabilities. We also note that the matching prior provides good coverage in small sample size, and the results are less sensitive to the change of the values of $(\lambda_1, \cdots, \lambda_k)$ and k. Thus, we recommend to use the reference prior.

Example 4.1 This example is taken from Saraçoğlu1 *et al.* (2012). The following data sets show the breaking strengths of jute fiber at two different gauge lengths, and used by Xia *et al.* (2009).

Breaking strength of jute fiber of gauge length 10 mm:

```
693.73 704.66 323.83 778.17 123.06 637.66 383.43 151.48 108.94  50.16
671.49 183.16 257.44 727.23 291.27 101.15 376.42 163.40 141.38 700.74
262.90 353.24 422.11 43.93  590.48 212.13 303.90 506.60 530.55 177.25.
```

Table 4.1 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of $\theta_1 = \prod_{i=1}^{k} \lambda_i$

k	$\lambda_1, \cdots, \lambda_k$	n	π_r
2	1.0, 1.0	3,3	0.049 (0.949)
		3,5	0.053 (0.949)
		5,5	0.050 (0.952)
		5,10	0.051 (0.949)
		10,10	0.048 (0.952)
	1.0, 10.0	3,3	0.050 (0.948)
		3,5	0.052 (0.947)
		5,5	0.049 (0.952)
		5,10	0.049 (0.951)
		10,10	0.050 (0.953)
	1.0, 100.0	3,3	0.054 (0.952)
		3,5	0.048 (0.949)
		5,5	0.054 (0.952)
		5,10	0.045 (0.951)
		10,10	0.048 (0.948)
	1.0, 1000.0	3,3	0.050 (0.951)
		3,5	0.050 (0.952)
		5,5	0.053 (0.952)
		5,10	0.048 (0.950)
		10,10	0.049 (0.947)
3	1.0, 1.0, 1.0	3,3,3	0.046 (0.949)
		3,5,5	0.049 (0.951)
		5,5,5	0.051 (0.947)
		5,10,10	0.053 (0.953)
		10,10,10	0.051 (0.950)
	1.0, 2.0, 3.0	3,3,3	0.048 (0.954)
		3,5,5	0.049 (0.953)
		5,5,5	0.050 (0.952)
		5,10,10	0.049 (0.952)
		10,10,10	0.043 (0.948)
	1.0, 5.0, 10.0	3,3,3	0.047 (0.953)
		3,5,5	0.048 (0.949)
		5,5,5	0.048 (0.953)
		5,10,10	0.051 (0.951)
		10,10,10	0.052 (0.954)
	1.0, 10.0, 100.0	3,3,3	0.049 (0.952)
		3,5,5	0.053 (0.953)
		5,5,5	0.050 (0.952)
		5,10,10	0.048 (0.950)
		10,10,10	0.052 (0.952)
5	1.0, 1.0, 1.0, 1.0, 1.0	3,3,3,3,3	0.051 (0.947)
		3,3,5,5,5	0.048 (0.948)
		5,5,5,5,5	0.053 (0.950)
		5,5,10,10,10	0.053 (0.954)
		10,10,10,10,10	0.047 (0.952)
	1.0, 2.0, 3.0, 4.0, 5.0	3,3,3,3,3	0.048 (0.952)
		3,3,5,5,5	0.048 (0.946)
		5,5,5,5,5	0.048 (0.952)
		5,5,10,10,10	0.047 (0.950)
		10,10,10,10,10	0.049 (0.952)
	1.0, 3.0, 5.0, 7.0, 10.0	3,3,3,3,3	0.051 (0.950)
		3,3,5,5,5	0.049 (0.952)
		5,5,5,5,5	0.050 (0.950)
		5,5,10,10,10	0.048 (0.942)
		10,10,10,10,10	0.048 (0.953)
	1.0, 5.0, 10.0, 50.0, 100.0	3,3,3,3,3	0.052 (0.954)
		3,3,5,5,5	0.054 (0.952)
		5,5,5,5,5	0.052 (0.951)
		5,5,10,10,10	0.048 (0.947)
		10,10,10,10,10	0.050 (0.950)

Breaking strength of jute fiber of gauge length 20 mm:

```
 71.46 419.02 284.64 585.57 456.60 113.85 187.85 688.16 662.66  45.58
578.62 756.70 594.29 166.49  99.72 707.36 765.14 187.13 145.96 350.70
547.44 116.99 375.81 581.60 119.86  48.01 200.16  36.75 244.53  83.55.
```

For this data sets, the Kolmogorov-Smirnov distances between the empirical distribution functions and the fitted distribution functions have been used to check the goodness-of-fit. The Kolmogorov-Smirnov Z values are 0.958 and 0.727 and the associated p values are 0.317 and 0.666, respectively. Therefore one cannot reject the hypothesis that the data are coming from exponential distributions (Saraçcoğlu1 *et al.*, 2012).

For this data sets, the maximum likelihood estimate (MLE) and the corresponding 90% asymptotic confidence interval of $\theta_1 = \lambda_2/\lambda_1$ are given in Table 4.2. Also Bayes estimate and the 90% credible interval based on the matching prior given in Table 4.2. For the Bayesian credible interval, we consider 10 independent sequences with a sample of size 110,000 discarding the first 10,000.

The Bayes estimate based on the reference prior and the MLE give some different results. Also the confidence interval based on the MLE is slightly shorter than the credible interval of the reference prior. However we know that the reference prior meets very well the target coverage probabilities in results of our simulation.

Table 4.2 Estimate and confidence interval for $\theta_1 = \frac{\lambda_2}{\lambda_1}$

MLE	π_r
0.9317 (0.5360, 1.3274)	0.9635 (0.6071, 1.4281)

5. Concluding remarks

In the exponential models, we have found the matching prior and the reference prior for the product of different powers of means. We revealed that the reference prior and Jeffreys' prior satisfy a second order matching criterion, and the reference prior and Jeffreys' prior are the same. As illustrated in our numerical study, the reference prior meets very well target coverage probabilities even though small sample size. Thus we recommend the use of the reference prior for Bayesian inference of nonlinear function of means in the exponential distributions.

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