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Bayesian estimation of median household income for small areas with some longitudinal pattern

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Abstract

One of the main objectives of the U.S. Census Bureau is the proper estimation of median household income for small areas. These estimates have an important role in the formulation of various governmental decisions and policies. Since direct survey estimates are available annually for each state or county, it is desirable to exploit the longitudinal trend in income observations in the estimation procedure. In this study, we consider Fay-Herriot type small area models which include time-specific random effect to accommodate any unspecified time varying income pattern. Analysis is carried out in a hierarchical Bayesian framework using Markov chain Monte Carlo methodology. We have evaluated our estimates by comparing those with the corresponding census estimates of 1999 using some commonly used comparison measures. It turns out that among three types of time-specific random effects the small area model with a time series random walk component provides estimates which are superior to both direct estimates and the Census Bureau estimates.

Keywords: Gibbs sampler, hierarchical Bayesian, median household income, random walk, small areas.

1. Introduction

Sample survey methodologies are widely used for collecting relevant information about population of interest over time. Apart from providing population level estimates, surveys are also designed to estimate various features of subpopulations or domains. Domains maybe geographic areas like state, county, school district etc., or can even be identified by a particular socio-demographic characteristic like a specific age-sex group. Often the domain-specific sample size may be too small to yield direct estimates of adequate precision. This led to the development of small area estimation procedure (Rao, 2003).

Sometimes, observations on various characteristics of small areas are collected over time, and thus may possess a complicated underlying time-varying pattern. So it is desirable to exploit the longitudinal trend in observations of small areas in the estimation procedure. It is likely that models which exploit the time varying pattern in the observations may perform better than classical small area models which do not utilize this feature.

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The Small Area Income and Poverty Estimates (SAIPE) program of the U.S. Census Bureau was established for providing annual estimates of income and poverty statistics for all states across the U.S. The current methodology of the SAIPE program is based on combining state and county estimates of poverty and income obtained from the American Community Survey (ACS) with other indicators of poverty and income using the Fay-Herriot class of models (Fay and Herriot, 1979). The SAIPE program provides annual state and county level estimates of median household income. SAIPE regression model for estimating the median household income for 1999 use as covariates, the median adjusted gross income (AGI) derived from IRS tax returns and the median household income estimate for 1999 obtained from the 2000 Census. The response variable is direct estimate of median household income for 1999 obtained from the 2000 Current Population Survey (CPS).

In this paper, we consider the household income data for all U.S. states for the period 1995 through 1999 to estimate the true state specific median household income for 1999. We have viewed the state specific annual household median income values as longitudinal profiles. In our estimation procedure, we used the state wide CPS median household income values for only five years (1995-1999). We consider Bayesian Fay-Herriot type small area models which include time-specific random effect to accommodate any unspecified time varying income pattern during five years. Current works in Bayesian modeling is related to Goo and Kim (2013) and Ryu and Kim (2015).

The outline of the remaining sections is as follows. In section 2 we introduce hierarchical Bayesian Fay-Herriot type small area models with 3 types of time-specific random effects. Section 3 goes over a hierarchical Bayesian analysis we performed. In section 4, we provide the data analysis with regard to the median household income dataset. In section 5, we end with discussion. The appendix contains the expressions of the full conditional distributions for MCMC computations.

2. Model Specification

Let Y_{ij} and x_{ij} denote the CPS median household income and the IRS mean (or median) income recorded for i^{th} state at j^{th} year, respectively. The basic model can be expressed as

$$Y_{ij} = \mathbf{X}_{ij}'\boldsymbol{\beta} + b_i + \nu_j + e_{ij} \tag{2.1}$$

where $\theta_{ij} = \mathbf{X}_{ij}'\boldsymbol{\beta} + b_i + \nu_j$ is our target of inference.

In the model (2.1), $\mathbf{X}_{ij} = (1, x_{ij})'$ and $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ is the vector of regression coefficients. Here m is the number of small areas and t is the number of time points at which the response and covariates are measured. In our case, $m = 51$, for the 50 U.S states and the District of Columbia and $t = 5$ for the years 1995-1999. b_i is a area-specific random effect while ν_j represents a time-specific random effect. We assume $b_i \overset{iid}{\sim} N(0, \sigma_b^2)$. Moreover, we assume that $e_{ij} \sim N(0, \sigma_{ij}^2)$ where σ_{ij}^2 is the sampling standard deviations corresponding to the CPS direct median income estimates obtained by fitting a model to the estimates of sampling error covariance matrices of the CPS median household income estimates for several years (Bell, 1999). In the datasets of Census Bureau, sampling standard deviations are given for all the states at each of the time points. So σ_{ij}^2's are known. Furthermore, all three random components b_i, ν_j and σ_{ij}^2 are assumed to be mutually independent.

One of main objectives of this paper is to study the three types of time-specific random effects to accommodate any unspecified time varying income pattern in Fay-Herriot type model (2.1). Specifically, we consier the distributiond for ν_j as follows:

$$\begin{aligned}
&\text{model 1) } \nu_j \overset{iid}{\sim} N(0, \sigma_\nu^2);\\
&\text{model 2) } \nu_j \overset{ind}{\sim} N(0, \psi_j^2);\\
&\text{model 3) } \nu_j|\nu_{j-1} \overset{ind}{\sim} N(\rho\nu_{j-1}, \sigma_\nu^2), \text{ with } \nu_0 = 0.
\end{aligned}$$

Model 1 and 2 assume uncorrelated structure for time-specific random effects. But two models are different in the sense of one for homogeneity and the other for heterogeneity. Model 3 has the first-order autoregressive (AR(1)) structure. This is also called a first-order Markov process. Ghosh *et al.* (1996) assumed that $\nu_j|\nu_{j-1} \overset{ind}{\sim} N(\nu_{j-1}, \sigma_\nu^2)$. Alternatively, this can be written by $\nu_j = \nu_{j-1} + w_j$ where $w_j \overset{iid}{\sim} N(0, \sigma_\nu^2)$. This is the so-called random walk model and is similar to the dynamic equation used in dynamic linear models.

3. Hierarchical Bayesian inference

Let $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{it})'$ be the response and $\mathbf{X}_i = (X_{i1}, \ldots, X_{it})'$ be the covariate for the i^{th} state. For model 1, let $\mathbf{\Omega}_i^{(1)} = (\boldsymbol{\theta}_i, \boldsymbol{\beta}, b_i, \sigma_b^2, \sigma_\nu^2)$ be the parameter space corresponding to the i^{th} state where $\boldsymbol{\theta}_i = (\theta_{i1}, \ldots, \theta_{it})'$. Thus, the full parameter space will be given by $\mathbf{\Omega}^{(1)} = \mathbf{\Omega}_1^{(1)} \times \mathbf{\Omega}_2^{(1)} \times \ldots \times \mathbf{\Omega}_m^{(1)}$. For the i^{th} state, the likelihood corresponding to model 1 can be written as

$$\begin{aligned}
L(\mathbf{Y}_i, \mathbf{X}_i|\mathbf{\Omega}_i^{(1)}) &\propto L(\mathbf{Y}_i|\boldsymbol{\theta}_i)L(\boldsymbol{\theta}_i|\boldsymbol{\beta}, b_i, \sigma_\nu^2, \mathbf{X}_i)L(b_i|\sigma_b^2) \qquad (3.1)\\
&= \prod_{j=1}^{t} \left\{ L(Y_{ij}|\theta_{ij}, \sigma_{ij}^2)L(\theta_{ij}|X_{ij}'\boldsymbol{\beta} + b_i, \sigma_\nu^2) \right\} L(b_i|\sigma_b^2).
\end{aligned}$$

Here, $L(U|a, b)$ denotes a normal density with mean a and variance b while $L(b_i|\sigma_b^2)$ denotes a normal distribution with mean 0 and variances σ_b^2. To complete the Bayesian specification of model 1, we need to assign prior distribution to the unknown parameters. The full posterior of the parameters given the data is obtained in the usual way by combining the likelihood and the prior distribution as follows

$$\pi(\mathbf{\Omega}^{(1)}|\mathbf{Y}, \mathbf{X}) \propto \prod_{i=1}^{m} L(\mathbf{Y}_i, \mathbf{X}_i|\mathbf{\Omega}_i^{(1)})\pi(\boldsymbol{\beta})\pi(\sigma_b^2)\pi(\sigma_\nu^2). \qquad (3.2)$$

In model 2, we assumed the variance of ν_j as ψ_j^2 instead of $\sigma_\nu^2 (j = 1, \ldots, t)$. Let $\mathbf{\Omega}_i^{(2)} = (\boldsymbol{\theta}_i, \boldsymbol{\beta}, b_i, \sigma_b^2, \psi_1^2, \cdots, \psi_t^2)$. Then the likelihood for the i^{th} state corresponding to model 2 can be written as

$$L(\mathbf{Y}_i, \mathbf{X}_i|\Omega_i^{(2)}) \propto \prod_{j=1}^{t} \left\{ L(Y_{ij}|\theta_{ij}, \sigma_{ij}^2)L(\theta_{ij}|X_{ij}'\boldsymbol{\beta} + b_i, \psi_j^2) \right\} L(b_i|\sigma_b^2). \qquad (3.3)$$

Then the full posterior of the parameters given the data under the model 2 is given by

$$\pi(\mathbf{\Omega}^{(2)}|\mathbf{Y},\mathbf{X}) \propto \prod_{i=1}^{m} L(\mathbf{Y}_i,\mathbf{X}_i|\mathbf{\Omega}_i^{(2)})\pi(\boldsymbol{\beta})\pi(\sigma_b^2)\prod_{j=1}^{t}\pi(\psi_j^2). \tag{3.4}$$

In model 3, we assumed time specific random component ν_j as a random walk model. That is, $(\nu_j|\nu_{j-1},\sigma_\nu^2) \sim N(\rho\nu_{j-1},\sigma_\nu^2)$ with $\nu_0 = 0$. Alternatively, we can write, $\nu_j = \rho\nu_{j-1} + w_j$ where $w_j \overset{iid}{\sim} N(0,\sigma_\nu^2)$. Let $\mathbf{\Omega}_i^{(3)} = (\boldsymbol{\beta}, b_i, \nu_1, \cdots, \nu_t, \sigma_b^2, \sigma_\nu^2, \rho)$. Then the likelihood corresponding to model 3 can be written as

$$L(\mathbf{Y}_i,\mathbf{X}_i|\Omega_i^{(3)}) \propto \prod_{j=1}^{t}\left\{L(Y_{ij}|X_{ij}'\boldsymbol{\beta}+b_i+\nu_j,\sigma_{ij}^2)L(\nu_j|\rho\nu_{j-1},\sigma_\nu^2)\right\}L(b_i|\sigma_b^2). \tag{3.5}$$

The full posterior of the parameters given the data under the model 3 is obtained as follows

$$p(\mathbf{\Omega}^{(3)}|\mathbf{Y},\mathbf{X}) \propto \prod_{i=1}^{m} L(\mathbf{Y}_i,\mathbf{X}_i|\mathbf{\Omega}_i^{(3)})\pi(\boldsymbol{\beta})\pi(\sigma_b^2)\pi(\sigma_\nu^2)\pi(\rho). \tag{3.6}$$

We assume bivariate normal prior for the regression coefficients $\boldsymbol{\beta}$, proper conjugate gamma priors on the inverse of the variance components $\psi_j^2, \sigma_\nu^2, \sigma_b^2$. We choose small values (0.01) for the gamma shape and rate parameters. Thus, we have the following priors: $\boldsymbol{\beta} \sim N_2(\mathbf{b}_0,\mathbf{\Sigma})$, $(\sigma_\nu^2)^{-1} \sim G(c,d)$, $(\psi_j^2)^{-1} \sim G(c,d)$, $(\sigma_b^2)^{-1} \sim G(c,d)$, $\rho \sim U(-1,1)$. Here $X \sim G(a,b)$ denotes a gamma distribution with shape parameter a and rate parameter b having expression $f(x) \propto x^{a-1}exp(-bx), x \geq 0$.

Our target of inference is the true median household income of all states, $(\theta_{it}, i = 1,\ldots,m)$. Since the marginal posterior of θ_{ij} is analytically intractable, high-dimensional integration needs to be carried out in a theoretical framework. However, this task can be easily accomplished in MCMC computations by using Gibbs sampler to sample from the full conditionals of θ_{ij} and other relevant parameters. The full conditionals for three models are given in the Appendix. We have monitored the convergence of Gibbs sampler using trace plots, autocorrelation plots and Geweke test of stationarity. We run three independent chains each with a sample sizes of 10,000 and with a burn-in sample of another 5,000. This burn-in period is long enough to get random samples, which is based on the trace plots and Geweke test. The correlations are all nonsignificant. Also, Geweke test demonstrates stationarity of our sampler.

4. Data analysis

We applied the hierarchical Bayesian models to analyze the median household income dataset. The response variable Y_{ij} and covariates X_{ij} denote respectively the CPS median household income estimate and the corresponding IRS mean income estimate for the i^{th} state at the j^{th} year $(i = 1,\ldots,51; j = 1,\ldots,5)$. The state-specific mean income figures are obtained from IRS tax return data. The Census Bureau gets files of individual tax return data from the IRS for use in specifically approved projects such as SAIPE. For each state, the IRS mean income is the mean adjusted gross income (AGI) across all the tax returns in that state. Figure 4.1 shows the plots of the CPS median income against the IRS mean income for all states, while Figure 4.2 shows sample longitudinal median household income profiles for four states for the years 1995 through 1999. It is apparent that CPS median income may have quite a different underlying pattern with respect to IRS mean income, and the pattern of CPS median income is different depending on the area.

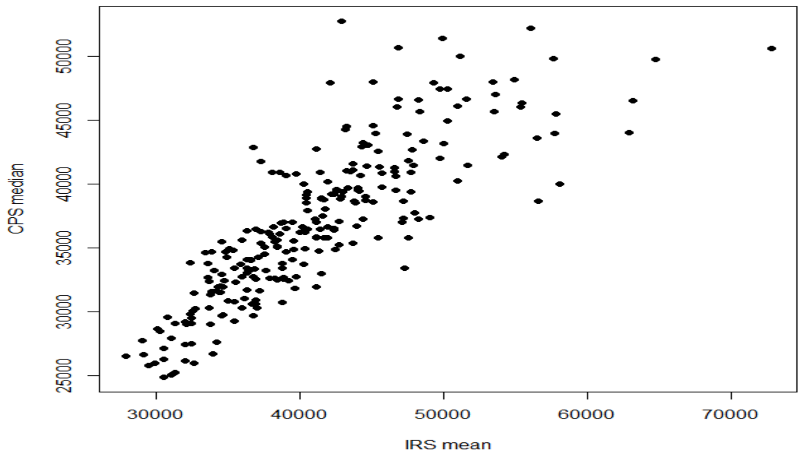

Figure 4.1 CPS median income against the IRS mean incomes for all the states for the years 1995-1999

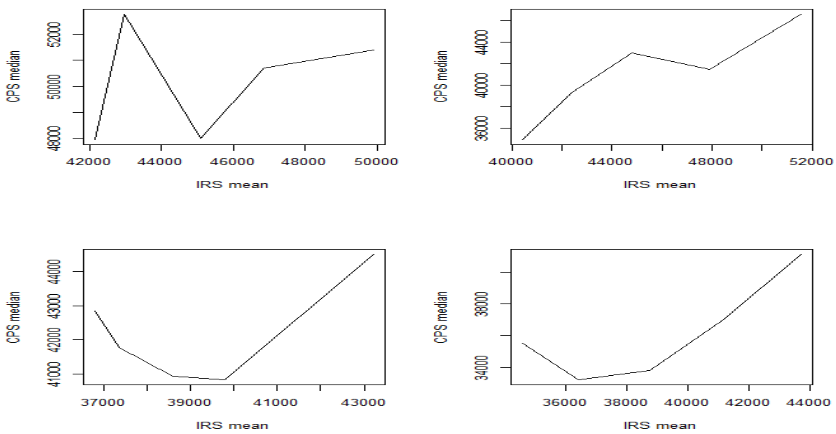

Figure 4.2 Longitudinal CPS median income profiles for 4 states plotted against IRS mean

Our dataset originally contained the median income of all the U.S states and the District of Columbia for the years 1995-2004. However, we only used the information for the five year period 1995-1999 since our target of inference are the state-specific median household incomes for 1999. We evaluated the performance of our estimates by comparing them to the corresponding census figures for 1999. This is because, in small area estimation problems, the census estimates are often treated as "gold standard" against which all other estimates are compared. In order to check the performance of our estimates, we plan to use four comparison measures. These were originally recommended by the panel on small area estimates of population and income set up by the Committee on National Statistics in July 1978 and are available in their July 1980 report. These are as follows.

- Average Relative Bias (ARB)= $(51)^{-1}\sum_{i=1}^{51}\frac{|c_i-e_i|}{c_i}$
- Average Squared Relative Bias (ASRB) = $(51)^{-1}\sum_{i=1}^{51}\frac{|c_i-e_i|^2}{{c_i}^2}$
- Average Absolute Bias (AAB) = $(51)^{-1}\sum_{i=1}^{51}|c_i-e_i|$
- Average Squared Diviation (ASD) = $(51)^{-1}\sum_{i=1}^{51}(c_i-e_i)^2$

Here c_i and e_i respectively denote the census and model based estimate of median household income for i^{th} state ($i=1,\ldots,51$). Clearly, lower values of these measures would imply a better model based estimate.

CPS median income ranged from \$24,879.68 to \$52,778.94 with a mean of \$36,868.49 and standard deviation of \$5954.94, while IRS mean annual income ranged from \$27,910 to \$72769.38 with a mean of \$41,133.45 and standard deviation of \$7196.56. The results of four comparison measures for the three types of small area models with time-specific random effects are shown in Table 4.1. Among three models, model 3 with $\rho=1$ or random walk model has the lowest values of four comparative measures. Overall, random walk model leads to some improvement in the performance of estimates over other models with different types of time-specific random component. As shown in Table 4.2, the percentage improvement of random walk model over the CPS is quite large, but it is slightly improved over SAIPE estimates. Also we report posterior mean, median and 95% HPD interval for the model parameters in Table 4.3.

Table 4.1 Results of four comparison measures

Estimate	ARB	ASRB	AAB	ASD
CPS	0.04154	0.00270	1,753.33	5,300,023
SAIPE	0.03260	0.00150	1,423.75	3,134,906
model 1	0.03751	0.00233	1,560.01	4,244,836
model 2	0.03265	0.00164	1,356.51	2,964,621
model 3	0.03130	0.00157	1,302.18	2,864,616
model 3 ($\rho=1$)	0.03125	0.00156	1,300.28	2,853,515

Table 4.2 Percentage improvements of estimates under model 3 over CPS and SAIPE estimates

Estimate	ARB	ASRB	AAB	ASD
CPS	24.6%	42.6%	25.7%	46.0%
SAIPE	4.2%	-4.7%	8.5%	8.6%

Table 4.3 Parameter estimates under the models

	β_0				β_1			
	Mean	Median	95% HPD		Mean	Median	95% HPD	
model 1	12266.4	12264.8	12223.2	12302.7	0.596	0.596	0.595	0.597
model 2	10735.4	10596.5	10358.6	11245.5	0.639	0.643	0.629	0.649
model 3	12230.6	12225.9	10814.9	13705.2	0.598	0.598	0.565	0.633
model 3 ($\rho=1$)	12228.7	12226.1	10723.4	13606.1	0.598	0.598	0.565	0.633

5. Discussion

This article has presented hierarchical Bayesian modeling with time specific random effects to accommodate any unspecified time varying income pattern for estimating median household income in U.S. states. A comparison of these estimates with those obtained from the 2000 decennial census reveals that Fay-Herriot type small area model utilizing longitudinal random walk effects performs the best among other types of longitudinal random effects. Also our random walk model could be an attractive alternative to the existing methodology of the Bureau of Census.

Appendix A: Full conditional distributions

Model 1

(i) $[\theta_{ij}|\boldsymbol{\beta}, b_i, \sigma_\nu^2] \sim N\Big(\Big(\frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_\nu^2}\Big)^{-1}\Big(\frac{Y_{ij}}{\sigma_{ij}^2} + \frac{\mathbf{X}'_{ij}\boldsymbol{\beta}+b_i}{\sigma_\nu^2}\Big), \Big(\frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_\nu^2}\Big)^{-1}\Big)$

(ii) $[b_i|\theta_{ij}, \boldsymbol{\beta}, \sigma_\nu^2, \sigma_b^2] \sim N\Big(\Big(\frac{t}{\sigma_\nu^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big(\frac{t(\theta_{ij}-\mathbf{X}'_{ij}\boldsymbol{\beta})}{\sigma_\nu^2}\Big), \Big(\frac{t}{\sigma_\nu^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big)$

(iii) $[\boldsymbol{\beta}|\mathbf{b}, \boldsymbol{\theta}, \sigma_\nu^2]$
$\sim N\Big(\Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\sigma_\nu^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big(\mathbf{b}'_0\boldsymbol{\Sigma}^{-1} + \frac{\sum_{i,j}(\theta_{ij}-b_i)\mathbf{X}'_{ij}}{\sigma_\nu^2}\Big), \Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\sigma_\nu^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big)$

(iv) $[\sigma_b^2|\mathbf{b}] \sim IG\Big(\frac{m}{2} + c, \frac{1}{2}\sum_i b_i^2 + d\Big)$

(v) $[\sigma_\nu^2|\boldsymbol{\theta}, \boldsymbol{\beta}, \mathbf{b}] \sim IG\Big(\frac{m}{2} + c, \frac{1}{2}\sum_{i,j}(\theta_{ij} - \mathbf{X}'_{ij}\boldsymbol{\beta} - b_i)^2 + d\Big)$

Model 2

(i) $[\theta_{ij}|\boldsymbol{\beta}, b_i, \psi_j^2] \sim N\Big(\Big(\frac{1}{\sigma_{ij}^2} + \frac{1}{\psi_j^2}\Big)^{-1}\Big(\frac{Y_{ij}}{\sigma_{ij}^2} + \frac{\mathbf{X}'_{ij}\boldsymbol{\beta}+b_i}{\psi_j^2}\Big), \Big(\frac{1}{\sigma_{ij}^2} + \frac{1}{\psi_j^2}\Big)^{-1}\Big)$

(ii) $[b_i|\theta_{ij}, \boldsymbol{\beta}, \psi_j^2, \sigma_b^2] \sim N\Big(\Big(\sum_{j=1}^t \frac{1}{\psi_j^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big(\sum_{j=1}^t \frac{(\theta_{ij}-\mathbf{X}'_{ij}\boldsymbol{\beta})}{\psi_j^2}\Big), \Big(\sum_{j=1}^t \frac{1}{\psi_j^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big)$

(iii) $[\boldsymbol{\beta}|\mathbf{b}, \boldsymbol{\theta}, \psi_j^2]$
$\sim N\Big(\Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\psi_j^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big(\mathbf{b}'_0\boldsymbol{\Sigma}^{-1} + \sum_{i,j}\frac{(\theta_{ij}-b_i)\mathbf{X}'_{ij}}{\psi_j^2}\Big), \Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\psi_j^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big)$

(iv) $[\sigma_b^2|\mathbf{b}] \sim IG\Big(\frac{m}{2} + c, \frac{1}{2}\sum_i b_i^2 + d\Big)$

(v) $[\psi_j^2|\boldsymbol{\theta}, \boldsymbol{\beta}, \mathbf{b}] \sim IG\Big(\frac{m}{2} + c, \frac{1}{2}\sum_j(\theta_{ij} - \mathbf{X}'_{ij}\boldsymbol{\beta} - b_i)^2 + d\Big)$

Model 3

(i) $[b_i|\nu_j, \boldsymbol{\beta}, \sigma_b^2] \sim N\Big(\Big(\sum_{j=1}^t \frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big(\sum_{j=1}^t \frac{(y_{ij}-\mathbf{X}'_{ij}\boldsymbol{\beta}-\nu_j)}{\sigma_{ij}^2}\Big), \Big(\sum_{j=1}^t \frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_{b^2}}\Big)^{-1}\Big)$

(ii) $[\boldsymbol{\beta}|\mathbf{b}, \boldsymbol{\nu}]$
$\sim N\Big(\Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\sigma_{ij}^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big(\mathbf{b}'_0\boldsymbol{\Sigma}^{-1} + \sum_{i,j}\frac{(Y_{ij}-b_i-\nu_j)\mathbf{X}'_{ij}}{\sigma_{ij}^2}\Big), \Big(\sum_{i,j} \frac{\mathbf{X}_{ij}\mathbf{X}'_{ij}}{\sigma_{ij}^2} + \boldsymbol{\Sigma}^{-1}\Big)^{-1}\Big)$

(iii) $[\nu_j|\mathbf{b}, \nu_{j-1}, \rho, \sigma_\nu^2]$
$\sim N\Big(\Big(\sum_{i=1}^m \frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_{\nu^2}}\Big)^{-1}\Big(\sum_{i=1}^m \frac{(y_{ij}-\mathbf{X}'_{ij}\boldsymbol{\beta}-b_i)}{\sigma_{ij}^2} + \rho\nu_{j-1}\Big), \Big(\sum_{i=1}^m \frac{1}{\sigma_{ij}^2} + \frac{1}{\sigma_{\nu^2}}\Big)^{-1}\Big)$

(iv) $[\sigma_b^2|\mathbf{b}] \sim IG\Big(\frac{m}{2} + c, \frac{1}{2}\sum_i b_i^2 + d\Big)$

(v) $[\sigma_\nu^2|\boldsymbol{\nu},\rho] \sim IG\Big(\frac{t}{2}+a, \frac{1}{2}\sum_j(\nu_j-\rho\nu_{j-1})^2+b\Big)$

(vi) $[\rho|\boldsymbol{\nu},\sigma_\nu^2] \sim N\Big(\frac{\sum_j \nu_j\nu_{j-1}}{\sum_j \nu_{j-1}^2}, \frac{\sigma_\nu^2}{\sum_j \nu_{j-1}^2}\Big)$, for $-1 \le \rho \le 1$

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