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Default Bayesian testing equality of scale parameters in several inverse Gaussian distributions

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Abstract

This paper deals with the problem of testing about the equality of the scale parameters in several inverse Gaussian distributions. We propose default Bayesian testing procedures for the equality of the shape parameters under the reference priors. The reference prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. Therefore we propose the default Bayesian testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. Simulation study and an example are provided.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, inverse Gaussian distribution, reference prior, scale parameter.

1. Introduction

The probability density function of the inverse Gaussian distribution with mean parameter μ and the scale parameter λ is defined by

$$f(x|\lambda,\mu) = \sqrt{\frac{\lambda}{2\pi}} x^{-\frac{3}{2}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\},\ x>0,\ \mu>0,\ \lambda>0. \qquad (1.1)$$

Because of the versatility and flexibility in modelling right-skewed data, the inverse Gaussian distribution has potentially useful applications in a wide variety of fields such as biology, economics, reliability theory and life testing as discussed in Chhikara and Folks (1989) and Seshadri (1999).

The present paper focuses on testing the equality of the scale parameters in several inverse Gaussian distributions. This equality test is important because the well-known analysis of reciprocals F-test is based on the assumption of equality of scale parameters. That is,

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when the equality of the scale parameters is not guaranteed, this F-test may give a wrong conclusion.

Without the homogeneity of the scale parameter assumption, Tian (2006) developed a generalized p-value based test procedure for the equality of several inverse Gaussian means. Krishnamoorthy and Tian (2008) further used generalized p-value based test to make an inference on the difference and ratio of two independent inverse Gaussian means. Subsequently, a parametric bootstrap approach for testing equality of several inverse Gaussian means under heterogeneity was applied by Ma and Tian (2009). When the means of several inverse Gaussian distributions are the same, Ye *et al.* (2010) suggested the generalized inference-based tests associated with the large-sample theory, to perform statistical inference for the common mean without assuming the scalar parameters are known and equal.

For testing the equality of the scale parameters, there are several relevant proposals available in the literature. Chang *et al.* (2012) developed a test using the generalized p-value. Sadooghi-Alvandi and Malekzadeh (2013) proposed an exact likelihood-ratio test, and showed that the proposed test is more powerful than the test proposed by Chang *et al.* (2012) by simulation study. Liu and He (2013) developed the generalized test variables for testing the equality of inverse Gaussian scale parameters based on generalized likelihood ratio. But there is no result for this test based on Bayesian hypothesis testing procedure.

In Bayesian model selection or testing, the posterior probabilities of models or hypotheses are computed using the Bayes factor. And then the model which has the highest posterior probability is selected. In this sense, the Bayes factor plays an important role in Bayesian model selection.

The use of objective priors like Jeffreys' prior or reference prior of Berger and Bernardo (1989, 1992) has been very popular in Bayesian inference. The Bayesian inference using objective priors has many advantages over the Bayesian inference based on subjective priors. At least, the inference based on objective priors does not need to study about the influences of hyper parameter. But the objective priors or noninformative priors such as Jeffreys' or reference prior are typically improper.

It is well known that the Bayes factor under the improper prior contains arbitrary constants and these constants affect the values of the Bayes factor. And this fact causes a serious problem in Bayesian model selection. To overcome this problem, Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the idea of imaginary training sample to choose the arbitrary constants. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a fraction of the likelihood. An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001). The last two approaches of intrinsic Bayes factor and fractional Bayes factor have shown to be quite useful in many statistical inference (Kang *et al.*, (2013, 2014)).

In this paper, we propose the objective Bayesian hypothesis testing procedures for the equality of the scale parameters in several inverse Gaussian distributions based on the Bayes factors. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses $H_1,\ H_2,\cdots,\ H_q$ are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$.

Consider the noninformative prior π_i^N for the prior distribution of θ_i which is improper. Then the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants of the scales of π_i^N and π_j^N.

The idea of Berger and Perricchi (1996) is to use part of the data as a training sample $\mathbf{x}(l)$ which satisfies

$$0 < m_i^N(\mathbf{x}(l)) < \infty,\ i = 1, \cdots, q, \tag{2.3}$$

and denote $\mathbf{x}(-l)$ as the remainder of the data.

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$ with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively. It is clear that the arbitrary constants are cancelling out in (2.4).

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. A minimal training sample satisfies (2.3) with the property that no subset of the minimal training sample is proper. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^{N} \times \frac{1}{L} \sum_{l=1}^{L} B_{ij}^{N}(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^{N} \times ME[B_{ij}^{N}(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} of (2.5) and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^{F} = B_{ji}^{N} \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^{N} \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. See O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995).

3. Bayesian hypothesis testing procedures

Let $x_{ij}, i = 1, \cdots, k, j = 1, \cdots, n_i$ denote observations from inverse Gaussian distribution with the scale parameter λ_i and the mean parameter μ_i. Then likelihood function is given by

$$\begin{aligned} & f(\mathbf{x}|\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k) \\ = \ & (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-\frac{3}{2}} \right) \left(\prod_{i=1}^{k} \lambda_i^{\frac{n_i}{2}} \right) \exp\{ - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_i (x_{ij} - \mu_i)^2}{2\mu_i^2 x_{ij}} \}, \end{aligned} \tag{3.1}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_k)$ and $\mathbf{x}_i = (x_{i1}, \cdots, x_{in_i})$ and $n = n_1 + \cdots + n_k$. We are interested in testing the hypotheses $H_1 : \lambda_1 = \cdots = \lambda_k$ versus $H_2 : \lambda_1 \neq \cdots \neq \lambda_k$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \lambda_1 = \cdots = \lambda_k \equiv \lambda$ is

$$L_1(\lambda, \mu_1, \cdots, \mu_k|\mathbf{x}) = (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-\frac{3}{2}} \right) \lambda^{\frac{n}{2}} \exp\{ - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda (x_{ij} - \mu_i)^2}{2\mu_i^2 x_{ij}} \}. \tag{3.2}$$

And under the hypothesis H_1, the reference prior for $(\lambda, \mu_1, \cdots, \mu_k)$ can be obtained directly from the Fisher information by Chhikara and Folks (1989) and is given as follows.

$$\pi_1^N(\lambda, \mu_1, \cdots, \mu_k) \propto \lambda^{-1}\mu_1^{-\frac{3}{2}} \cdots \mu_k^{-\frac{3}{2}}. \tag{3.3}$$

Then from the likelihood (3.2) and the reference prior (3.3), the element $m_1^b(\mathbf{x})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty L_1^b(\lambda, \mu_1, \cdots, \mu_k|\mathbf{x})\pi_1^N(\lambda, \mu_1, \cdots, \mu_k) d\lambda d\mu_1 \cdots d\mu_k \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}}\right) \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}}\right) \lambda^{\frac{bn}{2}-1} \\ &\times \exp\left\{-\sum_{i=1}^k \sum_{j=1}^{n_i} \frac{b\lambda(x_{ij}-\mu_i)^2}{2\mu_i^2 x_{ij}}\right\} d\lambda d\mu_1 \cdots d\mu_k \\ &= (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}}\right) \Gamma\left[\frac{bn}{2}\right] \\ &\times \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}}\right) \left[\sum_{i=1}^k \frac{b}{2}\left(s_i + \frac{n_i(\bar{x}_i-\mu_i)^2}{\mu_i^2 \bar{x}_i}\right)\right]^{-\frac{bn}{2}} d\mu_1 \cdots d\mu_k, \end{aligned} \tag{3.4}$$

where $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $s_i = \sum_{j=1}^{n_i}(\frac{1}{x_{ij}} - \frac{1}{\bar{x}_i})$. For the hypothesis H_2, the reference prior for $(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k)$ is

$$\pi^N(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k) \propto \prod_{i=1}^k \lambda_i^{-1}\mu_i^{-\frac{3}{2}}. \tag{3.5}$$

This reference prior can be obtained directly from the Fisher information by Chhikara and Folks (1989). The likelihood function under the hypothesis H_2 is given by equation (3.1). Thus from the likelihood (3.1) and the reference prior (3.5), the element $m_2^b(\mathbf{x})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}) &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty L_2^b(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k|\mathbf{x}) \\ &\times \pi_2^N(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k) d\lambda_1 \cdots d\lambda_k d\mu_1 \cdots d\mu_k \\ &= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}}\right) \left(\prod_{i=1}^k \lambda_i^{\frac{bn_i}{2}-1} \mu_i^{-\frac{3}{2}}\right) \\ &\times \exp\{-\sum_{i=1}^k \sum_{j=1}^{n_i} \frac{\lambda_i(x_{ij}-\mu_i)^2}{2\mu_i^2 x_{ij}}\} d\eta_1 \cdots d\eta_k d\mu_1 \cdots d\mu_k \qquad (3.6) \\ &= (2\pi)^{-\frac{bn}{2}} \left(\prod_{i=1}^k \prod_{j=1}^{n_i} x_{ij}^{-\frac{3b}{2}}\right) \prod_{i=1}^k \Gamma\left[\frac{bn_i}{2}\right] \int_0^\infty \mu_i^{-\frac{3}{2}} \left[\frac{b}{2}\left(s_i + \frac{n_i(\bar{x}_i-\mu_i)^2}{\mu_i^2 \bar{x}_i}\right)\right]^{-\frac{bn_i}{2}} d\mu_i. \end{aligned}$$

Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{\prod_{i=1}^k \Gamma[\frac{n_i}{2}] S_{2i}(\mathbf{x})}{\Gamma[\frac{n}{2}] S_1(\mathbf{x})}, \tag{3.7}$$

where

$$S_1(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}} \right) \left[\sum_{i=1}^k \left(s_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i^2 \bar{x}_i} \right) \right]^{-\frac{n}{2}} d\mu_1 \cdots d\mu_k$$

and

$$S_{2i}(\mathbf{x}) = \int_0^\infty \mu_i^{-\frac{3}{2}} \left[\left(s_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i^2 \bar{x}_i} \right) \right]^{-\frac{n_i}{2}} d\mu_i.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{\Gamma[\frac{bn}{2}] S_1(\mathbf{x}; b)}{\prod_{i=1}^k \Gamma[\frac{bn_i}{2}] S_{2i}(\mathbf{x}; b)}, \tag{3.8}$$

where

$$S_1(\mathbf{x}; b) = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}} \right) \left[\sum_{i=1}^k \left(s_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i^2 \bar{x}_i} \right) \right]^{-\frac{bn}{2}} d\mu_1 \cdots d\mu_k$$

and

$$S_{2i}(\mathbf{x}; b) = \int_0^\infty \mu_i^{-\frac{3}{2}} \left[\left(s_i + \frac{n_i(\bar{x}_i - \mu_i)^2}{\mu_i^2 \bar{x}_i} \right) \right]^{-\frac{bn_i}{2}} d\mu_i.$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{\Gamma[\frac{bn}{2}]}{\Gamma[\frac{n}{2}]} \frac{S_1(\mathbf{x}; b)}{S_1(\mathbf{x})} \frac{\prod_{i=1}^k \Gamma[\frac{n_i}{2}] S_{2i}(\mathbf{x})}{\prod_{i=1}^k \Gamma[\frac{bn_i}{2}] S_{2i}(\mathbf{x}; b)}. \tag{3.9}$$

Note that the calculations of the FBF of H_2 versus H_1 require actually two dimensional integration.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is already computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal densities of $(X_{1i_1}, X_{1i_2}, \cdots, X_{ki_1}, X_{ki_2})$ are finite for all $(i_1 < i_2) \in \{1, 2, \cdots, n_i\}, i = 1, \cdots, k$ under each hypothesis. Thus we conclude that any training sample of size 2 from each population is a minimal training sample.

The marginal density $m_1^N(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})$ under H_1 is given by

$$\begin{aligned}
& m_1^N(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2}) \\
= & \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2} | \lambda, \mu_1, \cdots, \mu_k) \\
\times & \pi_1^N(\lambda, \mu_1, \cdots, \mu_k) d\lambda d\mu_1 \cdots d\mu_k \\
= & (2\pi)^{-k} \left(\prod_{i=1}^k \prod_{j=1}^2 x_{ii_j}^{-\frac{3}{2}} \right) \Gamma[k] \\
\times & \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}} \right) \left[\sum_{i=1}^k \frac{1}{2} \left(w_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i^2 (x_{ii_1} + x_{ii_2})} \right) \right]^{-k} d\mu_1 \cdots d\mu_k,
\end{aligned}$$

where $w_i = (x_{ii_1} + x_{ii_2})/2$. And the marginal density $m_2^N(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})$ under H_2 is given by

$$\begin{aligned}
& m_2^N(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2}) \\
= & \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2} | \lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k) \\
\times & \pi_2^N(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k) d\lambda_1 \cdots d\lambda_k d\mu_1 \cdots d\mu_k \\
= & (2\pi)^{-k} \left(\prod_{i=1}^k \prod_{j=1}^2 x_{ii_j}^{-\frac{3}{2}} \right) \int_0^\infty \mu_i^{-\frac{3}{2}} \left[\frac{1}{2} \left(w_i + \frac{n_i (x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i^2 (x_{ii_1} + x_{ii_2})} \right) \right]^{-1} d\mu_i.
\end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$\begin{aligned}
& B_{21}^{AI} \qquad (3.10) \\
= & \frac{\prod_{i=1}^k \Gamma[\frac{n_i}{2}] S_{2i}(\mathbf{x})}{\Gamma[\frac{n}{2}] S_1(\mathbf{x})} \left[\frac{1}{L} \sum_{(i_1<i_2)\in\{1,\cdots,n_1\}}^{n_1} \cdots \sum_{(i_1<i_2)\in\{1,\cdots,n_k\}}^{n_k} \frac{T_1(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})}{T_2(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})} \right],
\end{aligned}$$

where $L = \prod_{i=1}^k n_i(n_i - 1)/2$,

$$\begin{aligned}
& T_1(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2}) \\
= & \Gamma[k] \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k \mu_i^{-\frac{3}{2}} \right) \left[\sum_{i=1}^k \frac{1}{2} \left(w_i + \frac{(x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i^2 (x_{ii_1} + x_{ii_2})} \right) \right]^{-k} d\mu_1 \cdots d\mu_k,
\end{aligned}$$

and

$$\begin{aligned}
& T_2(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2}) \\
= & \prod_{i=1}^k \int_0^\infty \mu_i^{-\frac{3}{2}} \left[\frac{1}{2} \left(w_i + \frac{n_i (x_{ii_1} + x_{ii_2} - 2\mu_i)^2}{\mu_i^2 (x_{ii_1} + x_{ii_2})} \right) \right]^{-1} d\mu_i.
\end{aligned}$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{\prod_{i=1}^k \Gamma[\frac{n_i}{2}] S_{2i}(\mathbf{x})}{\Gamma[\frac{n}{2}] S_1(\mathbf{x})} ME \left[\frac{T_1(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})}{T_2(x_{1i_1}, x_{1i_2}, \cdots, x_{ki_1}, x_{ki_2})} \right]. \qquad (3.11)$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require actually two dimensional integration.

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of $(\lambda_1, \mu_1), \cdots, (\lambda_k, \mu_k)$ and $(n_1, \cdots, n_k)$. In particular, for fixed $(\lambda_i, \mu_i), i = 1, \cdots, k$, we take 200 independent random samples of $\mathbf{X}_i$ with sample sizes n_i from the inverse Gaussian distributions with (λ_i, μ_i), respectively. We want to test the hypotheses $H_1 : \lambda_1 = \cdots = \lambda_k$ versus $H_2 : \lambda_1 \neq \cdots \neq \lambda_k$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. For this simulation, FORTRAN equipped with IMSL subroutine is used. To compute the integration in the Bayes Factors, the IMSL subroutine GQRUL/DGQRUL is used.

Tables 4.1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. In Table 4.1, $P^F(\cdot)$, $P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF and MIBF, respectively. From Table 4.1, the FBF, the AIBF and the MIBF accept the hypothesis H_1 when the values of λ_2 and λ_3 are close to values of λ_1, whereas reject the hypothesis H_1 when the values of λ_2 and λ_3 are far from values of λ_1. Also the AIBF and the MIBF give a similar behavior for all sample sizes. However the FBF favors the hypothesis H_2 than the AIBF and the MIBF.

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

μ_1, μ_2, μ_3	$\lambda_1, \lambda_2, \lambda_3$	n_1, n_2, n_3	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0, 1.0, 1.0	1.0, 1.0, 1.0	5, 5, 5	0.598 (0.169)	0.725 (0.184)	0.745 (0.172)
		5, 10, 10	0.722 (0.164)	0.818 (0.153)	0.832 (0.147)
		10, 10, 10	0.726 (0.168)	0.844 (0.147)	0.857 (0.141)
	1.0, 1.5, 2.0	5, 5, 5	0.568 (0.181)	0.697 (0.198)	0.719 (0.185)
		5, 10, 10	0.664 (0.218)	0.755 (0.222)	0.773 (0.212)
		10, 10, 10	0.624 (0.246)	0.750 (0.239)	0.767 (0.230)
	1.0, 2.0, 3.0	5, 5, 5	0.514 (0.193)	0.637 (0.220)	0.662 (0.214)
		5, 10, 10	0.571 (0.254)	0.661 (0.268)	0.683 (0.261)
		10, 10, 10	0.522 (0.272)	0.649 (0.293)	0.668 (0.289)
	1.0, 3.0, 5.0	5, 5, 5	0.459 (0.221)	0.574 (0.262)	0.602 (0.256)
		5, 10, 10	0.441 (0.300)	0.511 (0.331)	0.531 (0.329)
		10, 10, 10	0.308 (0.291)	0.405 (0.341)	0.424 (0.343)
	1.0, 5.0, 10.0	5, 5, 5	0.304 (0.234)	0.380 (0.287)	0.406 (0.288)
		5, 10, 10	0.267 (0.299)	0.305 (0.332)	0.321 (0.335)
		10, 10, 10	0.104 (0.169)	0.152 (0.223)	0.165 (0.232)
1.0, 3.0, 5.0	1.0, 1.0, 1.0	5, 5, 5	0.613 (0.163)	0.740 (0.178)	0.766 (0.163)
		5, 10, 10	0.721 (0.171)	0.819 (0.158)	0.838 (0.145)
		10, 10, 10	0.731 (0.154)	0.851 (0.129)	0.867 (0.120)
	1.0, 1.5, 2.0	5, 5, 5	0.594 (0.173)	0.713 (0.194)	0.740 (0.176)
		5, 10, 10	0.649 (0.221)	0.739 (0.225)	0.767 (0.209)
		10, 10, 10	0.671 (0.219)	0.789 (0.210)	0.811 (0.196)
	1.0, 2.0, 3.0	5, 5, 5	0.518 (0.205)	0.630 (0.241)	0.664 (0.228)
		5, 10, 10	0.585 (0.241)	0.674 (0.258)	0.707 (0.247)
		10, 10, 10	0.513 (0.288)	0.627 (0.303)	0.656 (0.293)
	1.0, 3.0, 5.0	5, 5, 5	0.439 (0.226)	0.533 (0.276)	0.576 (0.266)
		5, 10, 10	0.481 (0.289)	0.550 (0.316)	0.584 (0.307)
		10, 10, 10	0.329 (0.280)	0.424 (0.326)	0.455 (0.329)
	1.0, 5.0, 10.0	5, 5, 5	0.284 (0.230)	0.341 (0.284)	0.386 (0.286)
		5, 10, 10	0.260 (0.273)	0.300 (0.310)	0.332 (0.316)
		10, 10, 10	0.110 (0.190)	0.149 (0.239)	0.165 (0.249)
1.0, 5.0, 10.0	1.0, 1.0, 1.0	5, 5, 5	0.624 (0.157)	0.748 (0.174)	0.775 (0.157)
		5, 10, 10	0.700 (0.185)	0.796 (0.184)	0.817 (0.170)
		10, 10, 10	0.751 (0.149)	0.864 (0.124)	0.880 (0.113)
	1.0, 1.5, 2.0	5, 5, 5	0.556 (0.189)	0.673 (0.218)	0.705 (0.205)
		5, 10, 10	0.687 (0.194)	0.779 (0.193)	0.805 (0.178)
		10, 10, 10	0.659 (0.231)	0.774 (0.228)	0.797 (0.215)
	1.0, 2.0, 3.0	5, 5, 5	0.523 (0.197)	0.632 (0.233)	0.672 (0.219)
		5, 10, 10	0.617 (0.240)	0.698 (0.255)	0.728 (0.241)
		10, 10, 10	0.528 (0.255)	0.649 (0.273)	0.680 (0.265)
	1.0, 3.0, 5.0	5, 5, 5	0.441 (0.226)	0.532 (0.275)	0.575 (0.266)
		5, 10, 10	0.472 (0.287)	0.537 (0.315)	0.578 (0.307)
		10, 10, 10	0.346 (0.281)	0.438 (0.321)	0.474 (0.321)
	1.0, 5.0, 10.0	5, 5, 5	0.327 (0.222)	0.384 (0.277)	0.435 (0.273)
		5, 10, 10	0.297 (0.289)	0.335 (0.323)	0.367 (0.329)
		10, 10, 10	0.144 (0.207)	0.192 (0.259)	0.217 (0.273)

Example 4.1 This example is taken from Liu and He (2013). This data set can be obtained from http://lib.stat.cmu.edu/DASL/Large Datafiles/Crash.data, originally provided by the National Transportation Safety Administration. Tian (2006) used a part of this data set to test equality of means of inverse Gaussian distributions under possible heterogeneity. Here, we also used the data to test homogeneity of the scale parameters of inverse Gaussian distributions. According to Tian's (2006) description, we consider the problem of comparing the crash injuries between three car makes-Dodge, Honda, and Hyundai. There are eight, seven, and five observations for these three car makes, respectively. For illustrating the proposed method, we only take one of the injury variable, that is, left femur load as an example. Mudholkar and Tian (2002) used the entropy goodness-of-fit test to show that the variable can be reasonably described by inverse Gaussian distribution.

The summary statistics of this data set are: $n_1 = 8, n_2 = 7, n_3 = 5$, $\bar{x}_1 = 8.578$, $\bar{x}_2 = 8.053$, $\bar{x}_3 = 15.968$, $s_1 = 0.203, s_2 = 0.150, s_3 = 0.082$. For this data set, the p-values by the generalized likelihood ratio test of Liu and He (2013) and the approximate χ^2 test are 0.903 and 0.932, respectively. Therefore, there is no obvious evidence to reject the common scale parameter.

We want to test the hypotheses $H_1 : \lambda_1 = \lambda_2 = \lambda_3$ versus $H_2 : \lambda_1 \neq \lambda_2 \neq \lambda_3$. The values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.2. From the results of Table 4.2, the posterior probabilities based on various Bayes factors give the same answer, and all Bayes factor select the hypothesis H_1. And the AIBF and the MIBF slightly seem to favor the simple hypothesis.

Table 4.2 Bayes factor and posterior probabilities of $H_1 : \lambda_1 = \lambda_2 = \lambda_3$

B_{21}^F	$P^F(H_1\|\mathbf{x})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x})$
0.212	0.825	0.092	0.915	0.087	0.920

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the equality of the scale parameters in several inverse Gaussian distributions under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. However the FBF favors the hypothesis H_2 than the AIBF and the MIBF. From our simulation and example, we recommend the use of the FBF than the AIBF and MIBF for practical application in view of its simplicity and ease of implementation.

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