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# Robust Discretization of LTI Systems with Polytopic Uncertainties and Aperiodic Sampling

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**Abstract** – In the previous work, the authors studied the problem of robust discretization of linear time-invariant systems with polytopic uncertainties, where linear matrix inequality (LMI) conditions were developed to find an approximate discrete-time (DT) model of a continuous-time (CT) system with uncertainties in polytopic domain. The system matrices of obtained DT model preserved the polytopic structures of the original CT system. In this paper, we extend the previous approach to solve the problem of robust discretization of polytopic uncertain systems with aperiodic sampling. In contrast with the previous work, the sampling period is assumed to be unknown, time-varying, but contained within a known interval. The solution procedures are presented in terms of unidimensional optimizations subject to LMI constraints which are numerically tractable via LMI solvers. Finally, an example is given to show the validity of the proposed techniques.

**Keywords**: Discrete-time LTI systems, Polytopic uncertainty, Linear matrix inequality (LMI), Discretization, Sampled-data control

## 1. Introduction

Continuous-time (CT) systems controlled by digital controllers are referred to as sampled-data (SD) control systems, which are composed of CT systems to be controlled, discrete-time (DT) controllers controlling them, and the ideal sampler and zero-order holder to convert the CT signals into DT ones and vice versa [4]. When the digital controller is implemented on an actual CT plant, the control action through the zero-order holder appears as a piecewise constant signal in time, which is termed a SD controller. Significant research efforts on the SD control design have been made in the literature, and they can be divided into several categories. For instance, the so-called the direct DT design [17] is a design method based on the discretization of the CT system, where a DT controller is designed in DT domain directly. In the so-called lifting techniques [1, 4, 27], the SD controller design problem is transformed into an equivalent finite-dimensional discrete problem. The so-called jump system-based method [14, 25] is based on the representation of the system in the form of hybrid discrete/continuous model. The input delay approach [10, 11, 20] treats the SD systems as a CT system with uncertain but bounded time-varying delay in the control input.

Among the promising results, this paper focuses on the direct DT design method, in which the computation of an exact DT model of the original CT plant is required. While for LTI systems, the exact DT model is available in principle, this is not the case for nonlinear systems [15, 16] or uncertain LTI systems [3, 17]. Rather, an approximate DT model can be used in replacement of the exact DT model for the SD control design. A major drawback of the approximation technique is that they can suffer from degradation in performance or even lead to instability of the resulting SD control system when the approximation error is relatively large [18].

Especially for DT LTI systems with poyltopic uncertainties, substantial LMI-based results on robust control problems have been made up to date (e.g., [5-9, 13, 19, 28-38]), and most of them implicitly assumed that either exact or approximate polytopic DT model of the original CT plant is available. In order to apply the linear matrix inequality (LMI) methods for control design of DT systems, it is essential for the obtained approximate DT model to preserve the polytopic structure of the original CT system. A widely used simplest method is to take an approximation via the first-order Taylor series of the exact DT model under the assumption of fast sampling/fast hold [18]. This strategy usually works well under fast sampling, but the approximation error may become prohibitively large if the sampling period is relatively long. To alleviate this problem, in the previous work [18], we developed new LMI-based techniques to search for more exact approximation of the exact DT models of the original CT polytopic uncertain LTI systems, in which the discrepancy between the exact and the approximate DT models was minimized. To this end, we exploited higher-order truncated Taylor series of the exact DT model so that the truncation error of the approximate DT model can be reduced.

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Although the proposed method was successful in reducing the approximation error, there was still an unsolved problem: it can be applied only to the case that the sampling period is constant in time. To resolve this problem, in this paper, we investigate the robust discretization problem under aperiodic sampling. Specifically, it is assumed that the sampling period is time-varying and unknown, but lies within a known interval. Similarly to [18], this problem is tackled by minimizing the norm distances between the system matrices of the approximate and exact DT models. To obtain numerically tractable method to compute the approximation, the truncated Taylor series of the exact DT model is used similarly to [18]. The solution procedures are given in terms of unidimensional optimizations subject to LMIs, which can be readily tractable via convex optimizations [2]. To derive the LMI constraints, the so-called matrix-dilation technique [9], [22-24] is applied. A sufficient LMI condition to design a state feedback SD controller for the computed DT models is also studied briefly as one of applications of the proposed robust discretization strategy. Finally, an illustrative example is given to demonstrate the potential of the developed method.

## 2. Preliminaries

### 2.1 Notations

The adopted notation is as follows: $\mathbb{R}_+$ and $\mathbb{Z}_+$: sets of nonnegative real numbers and nonnegative integer, respectively; $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$: the *n*-dimensional Euclidean space and the set of all $m\times n$ real matrices, respectively; $A^T$ : transpose of matrix *A*; $A>0$ ($A<0$, $A\geq 0$, and $A\leq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix *A*; $A\otimes B$ : Kronecker's product of matrices *A* and *B*; $\mathrm{He}\{A\}$ : a shorthand notion for $A+A^T$; $I_n$: $n\times n$ identity matrix; $0_n$ and $0_{m\times n}$ : $n\times 1$ zero vector and $m\times n$ zero matrix, respectively $\mathcal{L}_h := [I_h \;\; 0_h] \in \mathbb{R}^{h\times(h+1)}$; $\mathcal{R}_h := [0_h \;\; I_h] \in \mathbb{R}^{h\times(h+1)}$ ; $*$ inside a matrix: transpose of its symmetric term; $\|\cdot\|$ : Euclidean vector norm for vectors or the matrix two-norm for matrices; $A_\perp$ : any matrices whose columns form bases of the right null-space of matrix $A$; $e_i^{[j]}$ : unit vector of dimension *j* with a 1 in the $i$-th component and 0 's elsewhere.

### 2.2 Problem formulation

Consider the CT LTI polytopic uncertain system

$$\dot{x}_c(t) = A_c(\alpha)x_c(t) + B_c(\alpha)u_c(t), \tag{1}$$

where $t\in\mathbb{R}_+ := \{x\in\mathbb{R} \, : \, x\geq 0\}$, $x_c(t)\in\mathbb{R}^n$ is the state, $u_c(t)\in\mathbb{R}^m$ is the control input, and matrices $A_c(\alpha)\in\mathbb{R}^{n\times n}$ and $B_c(\alpha)\in\mathbb{R}^{n\times n}$ are not precisely known but assumed to belong to the convex set

$$(A_c(\alpha), B_c(\alpha)) \in \left\{ (A,B) : (A,B) = \sum_{i=1}^{N} \alpha_i (A_{c,i}, B_{c,i}); \, \alpha\in\Delta_N \right\}.$$

where $\Delta_N$ is the unit simplex given by

$$\Delta_N := \left\{ \alpha\in\mathbb{R}^N : \sum_{i=1}^{N}\alpha_i = 1, \quad \alpha_i\geq 0, \quad i=1,2,\ldots,N \right\}.$$

It is assumed that the system is controlled by the SD controller

$$u_c(t) = u_c(t_k), \;\; \forall t\in[t_k, t_{k+1}), \;\; k\in\mathbb{Z}_+ := \{0,1,2,\ldots\}, \tag{2}$$

where $\{t_0, t_1, \ldots\}$ represents an unbounded monotonously increasing sequence of sampling instants with elements in $\mathbb{R}_+$, i.e., $\lim_{k\to\infty} t_k = \infty;\; t_0 = 0; t_k < t_{k+1};\; t_k\in\mathbb{R}_+;\; \forall k\in\mathbb{Z}_+$. We assume that the sampling interval, denoted by $\theta_k := t_{k+1} - t_k$, is time-varying and unknown but lies in a known compact set, $\theta_k\in[\theta_{\min}, \theta_{\max}]$, where $0<\theta_{\min}<\theta_{\max}<\infty$. The closed-loop SD control system composed of (1) and (2) is given by

$$\begin{aligned} &\dot{x}_c(t) = A_c(\alpha)x_c(t) + B_c(\alpha)u_c(t_k), \\ &\forall t\in[t_k, t_{k+1}), \quad k\in\mathbb{Z}_+, \end{aligned} \tag{3}$$

The state at time $t_{k+1}$ is

$$x_c(t_{k+1}) = e^{A_c(\alpha)\theta_k} x_c(t_k) + \int_0^{\theta_k} e^{A_c(\alpha)\tau} d\tau B_c(\alpha) u_c(t_k).$$

Introducing notation $A_d(\alpha,\theta_k) := e^{A_c(\alpha)\theta_k}$,

$$B_d(\alpha,\theta_k) := \int_0^{\theta_k} e^{A_c(\alpha)\tau} d\tau B_c(\alpha), \;\; x_d(k) := x_c(t_k),$$

$u_d(k) := u_c(t_k)$, system (3) can be converted to the uncertain DT linear time-varying (LTV) system

$$x_d(k+1) = A_d(\alpha,\theta_k)x_d(k) + B_d(\alpha,\theta_k)u_d(k), \tag{4}$$

where $k\in\mathbb{Z}_+$. DT LTV system (4) can be viewed as the exact discretization of (3) in the sense that $\|x_c(t_k) - x_d(k)\| = 0, \;\; \forall k\in\mathbb{Z}_+, \;\; \alpha\in\Delta_N$ is satisfied with $x_c(0) = x_d(0)$ and any control input sequence $\{u_d(0), u_d(1), \ldots\}$. Note that (4) is the exact DT model of the SD control system (3) (or CT system (1)). As indicated in [18], due to the nonlinear and infinite dimensional nature of $A_d(\alpha,\theta_k)$ and $B_d(\alpha,\theta_k)$ with respect to the uncertain parameters and sampling period $\theta_k$, it may be difficult to find their exact representations that preserve the polytopic structures of $A_c(\alpha)$ and $B_c(\alpha)$. To simplify the problem, let us consider uniform sampling period $\theta = \theta_0 = \theta_1 = \cdots$. In this case, most researches addressing the robust control of DT polytopic uncertain LTI systems approximate $A_d(\alpha,\theta_k)$ and $B_d(\alpha,\theta_k)$ to their first-order power series with the assumption that the sampling period $\theta$ is sufficiently small. However, when $\theta$ is relatively large, the approximations become inaccurate. To alleviate this problem, the concept of the robust discretization was suggested in the previous work [18]. Roughly speaking, the robust discretization problem is finding approximations $G(\alpha)$ and $H(\alpha)$ of matrices $A_d(\alpha,\theta_k)$ and $B_d(\alpha,\theta_k)$, respectively, such that both $G(\alpha)$ and $H(\alpha)$ preserve the polytopic structures of $A_c(\alpha)$ and $B_c(\alpha)$. In other words, it is required that the approximations can be expressed as convex combinations of given vertices. Specifically, a simplified robust discretization problem addressed in [18] can be expressed as finding matrices $G_i, H_i, i \in \{1,2,\ldots,N\}$ that solve the optimizations

$$\min_{G_i, i\in\{1,2,\ldots,N\}} \max_{\alpha\in\Delta_N} \| A_d^{[h]}(\alpha,\theta_k) - G(\alpha) \|,$$
$$\min_{H_i, i\in\{1,2,\ldots,N\}} \max_{\alpha\in\Delta_N} \| B_d^{[h]}(\alpha,\theta_k) - H(\alpha) \|,$$

where

$$G(\alpha) := \sum_{i=1}^{N} \alpha_i G_i, \quad H(\alpha) := \sum_{i=1}^{N} \alpha_i H_i$$

and

$$A_d^{[h]}(\alpha,\theta_k) := \sum_{i=0}^{h} \frac{\theta_k^i}{i!} A_c(\alpha)^i,$$
$$B_d^{[h]}(\alpha,\theta_k) := \sum_{i=1}^{h} \frac{\theta_k^i}{i!} A_c(\alpha)^{i-1} B_c(\alpha)$$

are the $h$-order Taylor series approximations of matrices $A_d(\alpha, \theta_k)$ and $B_d(\alpha, \theta_k)$, respectively. As mentioned in the introduction, the research in [18] only considered the case of the uniform sampling period. If the sampling period is time-varying within a known bound, the problem becomes more complicated. In this paper, we cope with the robust discretization problem under aperiodic sampling. The robust discretization problem considered in [18] is modified as follows.

**Problem (Robust discretization under aperiodic sampling)**. Let integer $h \geq 1$ be given. Compute matrices $G_{ij}, H_{ij}, (i,j) \in \{1,2,\ldots,N\}\times\{1,2\}$ that solve the following optimizations:

$$\min_{G_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \max_{\substack{\alpha\in\Delta_N,\\ \theta_k\in[\theta_{\min},\theta_{\max}]}} \| A_d^{[h]}(\alpha,\theta_k) - G(\alpha,\theta_k) \|, \tag{5}$$
$$\min_{H_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \max_{\substack{\alpha\in\Delta_N,\\ \theta_k\in[\theta_{\min},\theta_{\max}]}} \| B_d^{[h]}(\alpha,\theta_k) - H(\alpha,\theta_k) \|, \tag{6}$$

where

$$G(\alpha,\theta_k) := \sum_{i=1}^{N}\sum_{j=1}^{2} \alpha_i \beta_j(\theta_k) G_{ij}, \tag{7}$$

$$H(\alpha,\theta_k) := \sum_{i=1}^{N}\sum_{j=1}^{2} \alpha_i \beta_j(\theta_k) H_{ij}, \tag{8}$$
$$\beta_1(\theta_k) := \frac{\theta_{\max}-\theta_k}{\theta_{\max}-\theta_{\min}}, \quad \beta_2(\theta_k) := \frac{\theta_k-\theta_{\min}}{\theta_{\max}-\theta_{\min}}.$$

Note that $G(\alpha,\theta_k)$ and $H(\alpha,\theta_k)$ depend on $\theta_k$ and have poyltopic structures with respect to $\theta_k$.

## 3. Main Result

In this section, LMI solutions to the robust discretization with aperiodic sampling are presented. As in [18], optimizations (5) and (6) can be rewritten by

$$\min_{G_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \gamma_A \quad \text{subject to} \tag{9}$$
$$(A_d^{[h]}(\alpha,\theta_k) - G(\alpha,\theta_k))^T (A_d^{[h]}(\alpha,\theta_k) - G(\alpha,\theta_k)) \leq \gamma_A I_n,$$
$$\forall(\alpha,\theta_k) \in \Delta_N \times [\theta_{\min},\theta_{\max}]$$

$$\min_{H_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \gamma_B \quad \text{subject to} \tag{10}$$
$$(B_d^{[h]}(\alpha,\theta_k) - H(\alpha,\theta_k))^T (B_d^{[h]}(\alpha,\theta_k) - H(\alpha,\theta_k)) \leq \gamma_B I_n,$$
$$\forall(\alpha,\theta_k) \in \Delta_N \times [\theta_{\min},\theta_{\max}]$$

Alternative expressions are

$$\min_{G_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \gamma_A \quad \text{subject to} \tag{11}$$
$$(A_d^{[h]}(\alpha,\theta_k) - G(\alpha,\theta_k))(A_d^{[h]}(\alpha,\theta_k) - G(\alpha,\theta_k))^T \leq \gamma_A I_n,$$
$$\forall(\alpha,\theta_k) \in \Delta_N \times [\theta_{\min},\theta_{\max}]$$

$$\min_{H_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\}} \gamma_B \quad \text{subject to} \tag{12}$$
$$(B_d^{[h]}(\alpha,\theta_k) - H(\alpha,\theta_k))(B_d^{[h]}(\alpha,\theta_k) - H(\alpha,\theta_k))^T \leq \gamma_B I_n,$$
$$\forall(\alpha,\theta_k) \in \Delta_N \times [\theta_{\min},\theta_{\max}]$$

which are equivalent to (9) and (10), respectively. We will use expressions (11) and (12) rather than (9) and (10) since (11) and (12) are more suitable to be converted into LMI conditions. The following results can be viewed as the main results of this paper. They establish sufficient LMI conditions that ensure constraints (11) and (12).

**Theorem 1**: Let $h \geq 1$ be given. If there exist matrices $G_{ij} \in \mathbb{R}^{n\times n}$, $(i,j) \in \{1,2,\ldots,N\}\times\{1,2\}$, $M \in \mathbb{R}^{n(h+1)\times nh}$ and a scalar $\gamma_A \geq 0$ such that

$$\begin{bmatrix} \begin{bmatrix} -\gamma_A e_1^{[h+1]} e_1^{[h+1]T} \otimes I_n \\ +\mathrm{He}\{M(\mathcal{L}_h \otimes \theta_j A_{c,i}^T - \mathcal{R}_h \otimes I_n)\} \end{bmatrix} & * \\ \begin{bmatrix} (I_n - G_{ij}^T) & \frac{1}{1!} I_n & \cdots & \frac{1}{h!} I_n \end{bmatrix} & -I_n \end{bmatrix} \leq 0, \tag{13}$$
$$\forall(i,j) \in \{1,2,\ldots,N\}\times\{1,2\},$$

where $(\theta_1, \theta_2) = (\theta_{\min}, \theta_{\max})$, then constraint in (11) is satisfied.

**Proof**. First of all, multiplying (13) by $\alpha_i \beta_j(\theta_k)$ and summing for $(i,j) \in \{1,2,\ldots,N\} \times \{1,2\}$, we obtain

$$\begin{bmatrix} \begin{bmatrix} -\gamma_A e_1^{[h+1]} e_1^{[h+1]T} \otimes I_n \\ +\mathrm{He}\{M(\mathcal{L}_h \otimes \theta A_c(\alpha)^T - \mathcal{R}_h \otimes I_n)\} \end{bmatrix} & * \\ \begin{bmatrix} I_n - G(\alpha,\theta_k)^T & \frac{1}{1!} I_n & \cdots & \frac{1}{h!} I_n \end{bmatrix} & -I_n \end{bmatrix} \le 0, \tag{14}$$
$$\forall (\alpha,\theta) \in \Delta_N \times [\theta_{\min}, \theta_{\max}],$$

where $G(\alpha,\theta_k)$ and $H(\alpha,\theta_k)$ are defined in (8). Applying the Schur complement to the above inequalities yields

$$\begin{aligned} & -\gamma_A e_1^{[h+1]} e_1^{[h+1]T} \otimes I_n \\ & + \begin{bmatrix} I_n - G(\alpha,\theta_k)^T & \frac{1}{1!} I_n & \cdots & \frac{1}{h!} I_n \end{bmatrix}^T \\ & \times \begin{bmatrix} I_n - G(\alpha,\theta_k)^T & \frac{1}{1!} I_n & \cdots & \frac{1}{h!} I_n \end{bmatrix} \\ & + \mathrm{He}\{M(\mathcal{L}_h \otimes \theta A_c(\alpha)^T - \mathcal{R}_h \otimes I_n)\} \\ & \le 0, \quad \forall (\alpha,\theta) \in \Delta_N \times [\theta_{\min}, \theta_{\max}] \end{aligned}$$

Pre- and post-multiplying the last inequality by $\Pi^{[h]T}$ and its transpose, where

$$\Pi^{[h]} := \begin{bmatrix} I_n \\ \theta A_c(\alpha)^T \\ \vdots \\ \theta^h (A_c(\alpha)^h)^T \end{bmatrix}$$

and using relation $(\mathcal{L}_h \otimes \theta A_c(\alpha)^T - \mathcal{R}_h \otimes I_n)\Pi^{[h]} = 0_{nh \times n}$, we can obtain the constraint in (11). This completes the proof. □

Similarly to Theorem 1, an LMI condition that ensures constraint (12) can be obtained.

**Theorem 2**: Let $h \ge 1$ be given. If there exist matrices $H_{ij} \in \mathbb{R}^{n \times m}, (i,j) \in \{1,2,\ldots,N\} \times \{1,2\}, M \in \mathbb{R}^{n(h+1) \times nh}$ and a scalar $\gamma_B \ge 0$ such that

$$\begin{bmatrix} \begin{bmatrix} -\gamma_B e_1^{[h+1]} e_1^{[h+1]T} \otimes I_n \\ +\mathrm{He}\{M(\mathcal{L}_h \otimes \theta_j A_{c,i}^T - \mathcal{R}_h \otimes I_n)\} \end{bmatrix} & * \\ \begin{bmatrix} \left(\frac{\theta_j}{1!} B_{c,i}^T - H_{ij}^T\right) & \frac{\theta_j}{2!} B_{c,i}^T & \cdots & \frac{\theta_j}{(h+1)!} B_{c,i}^T \end{bmatrix} & -I_m \end{bmatrix} \le 0,$$
$$\forall (i,j) \in \{1,2,\ldots,N\} \times \{1,2\} \tag{15}$$

where $(\theta_1, \theta_2) = (\theta_{\min}, \theta_{\max})$, then constraint (12) is satisfied.

**Proof**. The proof is straightforwardly extended from the proof of Theorem 1 so omitted for brevity. □

In this regard, the optimizations in (11) and (12) can be replaced by the following optimizations subject to LMI constraints:

$$\min_{G_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\},M} \gamma_A \quad \text{subject to LMIs in (13)} \tag{16}$$

$$\min_{H_{ij},(i,j)\in\{1,2,\ldots,N\}\times\{1,2\},M} \gamma_B \quad \text{subject to LMIs in (15)} \tag{17}$$

**Remark**. Optimizations (16) and (17) are single-parameter minimization problems subject to LMI constraints, and hence, can be solved by means of a sequence of LMI optimizations, i.e. a line search or a bisection process over $\gamma_A$ and $\gamma_B$, respectively, or solved by the eigenvalue problem (EVP) [2], which is convex optimization, and hence, can be directly treated with LMI solvers [12, 21, 26].

## 4. Application

Although the proposed strategy provides only approximate solutions to the robust discretization problem with aperiodic sampling, it may be at least more precise than the first-order Taylor series approximation. Moreover, the proposed technique would be effective from the practical point of view since as stated in [18], once a discretized model of a CT system is obtained, then it can be stored in database and used repeatedly for various SD control design purposes through existing LMI-based DT control design techniques (e.g., [5-7, 9] to name a few) in the literature. For instance, let us assume that matrices $G_{ij} \in \mathbb{R}^{n \times n}$, $H_{ij} \in \mathbb{R}^{n \times n}$, $(i,j) \in \{1, 2, \ldots, N\} \times \{1, 2\}$ are solutions to optimizations (16) and (17), respectively. Instead of considering exact discretization (4) of the original CT system (3), consider the following DT system which is an approximate discretization of (3) under aperiodic sampling:

$$\xi(k+1) = G(\alpha,\theta_k)\xi(k) + H(\alpha,\theta_k)\pi(k), \tag{18}$$

where $\xi(k) \in \mathbb{R}^n$ is the state and $\pi(k) \in \mathbb{R}^m$ is the control input. Note that DT system (18) can be viewed as an approximate DT model of the exact DT model (4). In addition, let us consider the following state-feedback control law:

$$\pi(k) = F\xi(k).$$

The closed-loop system is

$$\xi(k+1) = (G(\alpha,\theta_k) + H(\alpha,\theta_k)F)\xi(k). \tag{19}$$

Based on the LMI design approach developed in [6], we can readily establish the following LMI-based state-feedback design condition.

**Proposition 1**. If there exist matrices $P_{ij} = P_{ij}^T \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$, and $K \in \mathbb{R}^{m\times n}$ such that LMIs

$$\begin{bmatrix} -P_{ij} & * \\ G_{ij}S + H_{ij}K & P_{il} - S - S^T \end{bmatrix} < 0, \qquad (20)$$
$$\forall i \in \{1,2,\ldots,N\}, \quad \forall (j,l) \in \{1,2\}^2$$

hold, then state-feedback gain given by $F = KS^{-1}$ stabilizes closed-loop system (19) for all $\alpha \in \Delta_N$ and for all time-varying sampling period $\theta_k \in [\theta_{\min}, \theta_{\max}]$.

**Proof**. Multiplying (20) by $\alpha_i \beta_j(\theta_k)\beta_l(\theta_{k+1})$ and summing for $(i,j,l) \in \{1,2,\ldots,N\} \times \{1,2\}^2$, we obtain

$$\begin{bmatrix} -P(\alpha,\theta_k) & * \\ G(\alpha,\theta_k)S + H(\alpha,\theta_k)K & P(\alpha,\theta_{k+1}) - S - S^T \end{bmatrix} < 0,$$
$$\forall (\alpha,\theta_k,\theta_{k+1}) \in \Delta_N \times [\theta_{\min},\theta_{\max}]^2,$$

where

$$P(\alpha,\theta_k) := \sum_{i=1}^{N}\sum_{j=1}^{2} \alpha_i \beta_j(\theta_k) P_{ij}$$

and $G(\alpha,\theta_k)$ and $H(\alpha,\theta_k)$ are defined in (8). Next, by pre- and post-multiplying the last inequality by

$$\begin{bmatrix} S^{-1} & 0_{n\times n} \\ 0_{n\times n} & S^{-1} \end{bmatrix}^T$$

and its transpose, and by applying the extended Schur complement in [6], it follows that

$$(G(\alpha,\theta_k) + H(\alpha,\theta_k)F)^T X(\alpha,\theta_{k+1})$$
$$\times (G(\alpha,\theta_k) + H(\alpha,\theta_k)F) - X(\alpha,\theta_k) < 0,$$
$$\forall (\alpha,\theta_k,\theta_{k+1}) \in \Delta_N \times [\theta_{\min},\theta_{\max}]^2$$

where $F = KS^{-1}$ and $X(\alpha,\theta_k) = S^{-T}P(\alpha,\theta_k)S^{-1}$. By means of the Lyapunov theory, one concludes that (19) is asymptotically stable for all $\alpha \in \Delta_N$ and for all time-varying sampling period $\theta_k \in [\theta_{\min},\theta_{\max}]$. This completes the proof. □

On the other hand, let us consider the SD state-feedback controller

$$u_c(t) = u_c(t_k) = Fx_c(t_k), \quad \forall t \in [t_k, t_{k+1}), \quad k \in Z_+, \qquad (21)$$

for system (3). The closed-loop SD control system is

$$\dot{x}_c(t) = (A_c(\alpha) + B_c(\alpha)F)x_c(t_k), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}_+. \qquad (22)$$

If $G(\alpha,\theta_k) = A_d(\alpha,\theta_k)$ and $H(\alpha,\theta_k) = B_d(\alpha,\theta_k)$ for all $\alpha \in \Delta_N$ and $\theta_k \in [\theta_{\min},\theta_{\max}]$, then one can expect that $\| x_c(t_k) - \xi(k) \| = 0$, $\forall k \in \mathbb{Z}_+$, $\alpha \in \Delta_N$ is satisfied with $x_c(0) = \xi(0)$ and any control input sequence $\{\pi(0), \pi(1), \ldots\}$. Although the idealistic case may not occur in reality, we can still expect that if $G(\alpha,\theta_k) \cong A_d(\alpha,\theta_k)$ and $H(\alpha,\theta_k) \cong B_d(\alpha,\theta_k)$, then the solution $x_c(t)$ to (22) closely matches the solution $\xi(k)$ to (19) at every sampling instants $\{t_0, t_1, \ldots\}$. In this respect, the proposed robust discretization under aperiodic sampling can be viewed as a practically useful and simple approach to deal with various SD control problems.

All numerical examples in the sequel were treated with the help of MATLAB R2012b running on a PC with Intel Core i7-3770 3.4GHz CPU, 32GB RAM. The LMI problems were solved with SeDuMi 1.3 [26] and Yalmip [21].

**Example 1.** Let us consider the linearized model of the inverted pendulum system taken from [3]. Its state-space realization is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\bar{m}\bar{g}/\bar{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (\bar{M}+\bar{m})\bar{g}/(\bar{M}\bar{l}) & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1/\bar{M} \\ 0 \\ 1/(\bar{M}\bar{l}) \end{bmatrix},$$
$$x_c(t) = \begin{bmatrix} x_{c,1}(t) \\ x_{c,2}(t) \\ x_{c,3}(t) \\ x_{c,4}(t) \end{bmatrix},$$

$x_{c,1}(t)$ is the position of the cart, $x_{c,2}(t) = \dot{x}_{c,1}(t)$, $x_{c,3}(t)$ is the angle of the pendulum from the vertical, $x_{c,4}(t) = \dot{x}_{c,3}(t)$, $\bar{m}$ is the mass of the pendulum, $\bar{M}$ is the mass of the cart, $\bar{l}$ is the length of the pendulum, and $u_c(t)$ is the horizontal force applied to the cart. We assume $(\bar{M}, \bar{l}) = (8\text{kg}, 1\text{m})$ and $\bar{m} \in [1\text{kg}, 3\text{kg}]$. Then, the system can be described by (1) with two vertices. By applying Theorems 1 and 2 with $(\theta_{\min}, \theta_{\max}, h) = (0.01\text{s}, 0.1\text{s}, 7)$, we obtain the approximate DT system (18) with $(\gamma_A, \gamma_B) =$ $(0.1112, 4.8251\times 10^{-8})$ and

$$G_{11} = \begin{bmatrix} 1 & 0.0297 & -0.0044 & 0.0031 \\ 0 & 0.9998 & -0.1024 & -0.0030 \\ 0 & -0.0016 & 1.0195 & 0.0089 \\ 0 & 0 & 0.4258 & 1.0083 \end{bmatrix},$$
$$G_{12} = \begin{bmatrix} 1.0001 & 0.0741 & -0.0084 & 0.0001 \\ 0 & 0.9975 & -0.1603 & 0.0044 \\ -0.0001 & 0.0048 & 1.0417 & 0.0911 \\ -0.0001 & 0.0042 & 0.9338 & 1.0284 \end{bmatrix},$$

$$G_{21}=\begin{bmatrix}1 & 0.0295 & -0.0047 & 0.0036\\ 0 & 0.9998 & -0.1149 & -0.0047\\ 0 & -0.0008 & 1.0201 & 0.0076\\ 0 & 0.0006 & 0.4388 & 1.0091\end{bmatrix},$$

$$G_{22}=\begin{bmatrix}0.9999 & 0.0756 & -0.0126 & -0.0046\\ 0 & 1.0004 & -0.2841 & -0.0155\\ 0.0002 & -0.0020 & 1.0463 & 0.0979\\ 0 & 0.0002 & 1.0589 & 1.0455\end{bmatrix},$$

$$H_{11}=\begin{bmatrix}0\\ 0.0013\\ 0\\ -0.0013\end{bmatrix},\quad H_{12}=\begin{bmatrix}0.0005\\ 0.0125\\ -0.0005\\ -0.0127\end{bmatrix},$$

$$H_{21}=\begin{bmatrix}0\\ 0.0013\\ 0\\ -0.0013\end{bmatrix},\quad H_{22}=\begin{bmatrix}0.0005\\ 0.0126\\ -0.0005\\ -0.0127\end{bmatrix}.$$

By using Proposition 1, the state-feedback gain is calculated as follows:

$$F=[37.8 \quad 79.5 \quad 1148.3 \quad 210.5].$$

The simulation results with $x_0=[5 \quad -3 \quad 2 \quad -3]^T$ and $\alpha=[0.5,0.5]^T$ are depicted in Figs. 1(a)-(d), where $x_c(t)$ (solid line) is the solution to the SD closed-loop system (22) and $\xi(k)$ at each sampling instant (dot) is the solution to the DT closed-loop system (19). In other words, the dotted lines in Figs. 1(a)-(d) can be viewed as the state trajectories $\xi(k)$ of the approximately discretized model of the original CT system (1) and the solid lines indicate the state trajectories $x_c(t)$ of the CT plant (1). The closeness of the two trajectories implies that the robust discretization approach proposed in this paper is an exact approximation of the exact discritization of the CT plant (1). From the figure, we confirm that the trajectory of $\xi(k)$ closely matches the trajectory of $x_c(t)$ at sampling instants $\{t_0,t_1,\ldots\}$.

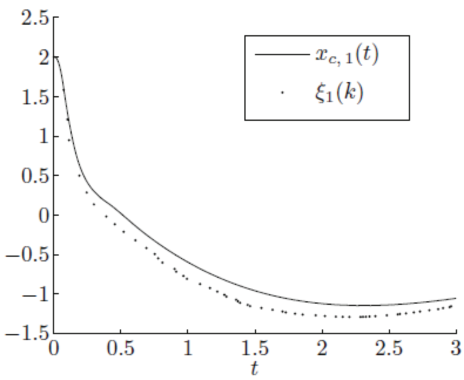


**Fig. 1.** The solid line is the solution to the SD closed-loop system (22) $x_{c,1}(t)$ and the dotted line is the solution to the DT closed-loop system (19) $\xi_1(k)$ at each sampling instant.

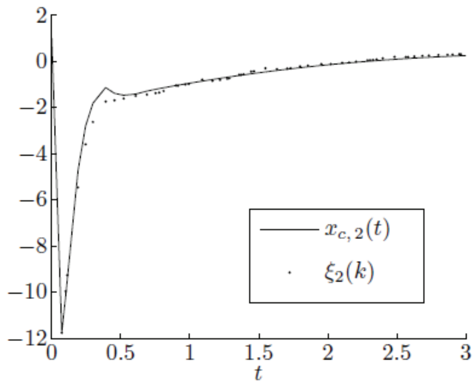


**Fig. 2.** The solid line is the solution to the SD closed-loop system (22) $x_{c,2}(t)$ and the dotted line is the solution to the DT closed-loop system (19) $\xi_2(k)$ at each sampling instant.

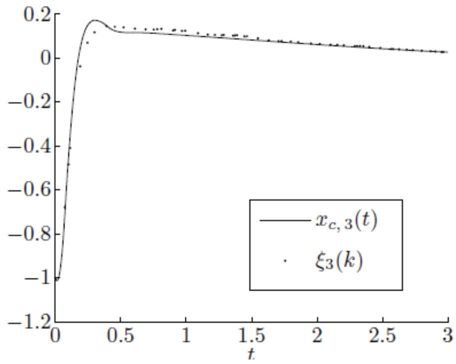


**Fig. 3.** The solid line is the solution to the SD closed-loop system (22) $x_{c,3}(t)$ and the dotted line is the solution to the DT closed-loop system (19) $\xi_3(k)$ at each sampling instant.

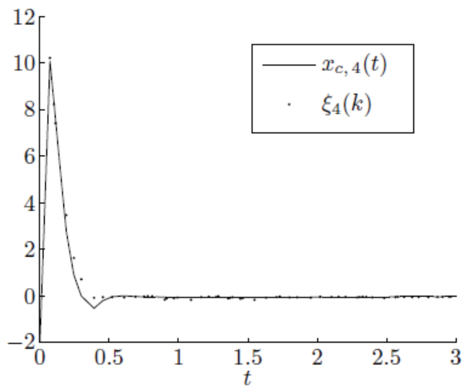


**Fig. 4.** The solid line is the solution to the SD closed-loop system (22) $x_{c,4}(t)$ and the dotted line is the solution to the DT closed-loop system (19) $\xi_4(k)$ at each sampling instant.

## 5. Conclusion

In this paper, our previous work on the robust discretization problem has been extended to deal with the same problem with aperiodic sampling. LMI conditions to compute approximate DT models of the original CT plants have been developed. Finally, an example has been given to illustrate the developed method.

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