

# A Tuning Algorithm for LQ-PID Controllers using the Combined Time - and Frequency-Domain Control Method

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**Abstract** – This paper proposes a new method for tuning a linear quadratic - proportional integral derivative controller for second order systems to simultaneously meet the time and frequency domain design specifications. The suitable loop-shape of the controlled system and the desired step response are considered as specifications in the time and frequency domains, respectively. The weighting factors, Q and R of the LQ controller are determined by the algebraic Riccati equation with respect to the limiting behavior and target function matching. Numerical examples show the effectiveness of the proposed LQ-PID tuning method

**Keywords:** LQ-PID Control, Combined method, Loop-Shaping, Target function

## 1. Introduction

Linear optimal control theory has been extensively studied since Bellman's dynamic programming [1] and Pontryagin et al.'s minimum principle [2] were published early in the 1960s. In linear optimal control, the linear quadratic regulator (LQR) problem is to find the optimal feedback control law to minimize the quadratic performance criterion with respect to the states and inputs via weighting matrices Q and R [3-5]. Thus the central issue remains of how to relate the weighting matrices in the quadratic performance criterion to classical specifications in the time and frequency domains for the control system design [6-9].

Various analytic methods have been developed for frequency domain specifications such as disturbance attenuation, noise rejection, and good command following for the LQR [10, 11] and linear quadratic Gaussian / linear transfer recovery (LQG/LTR) [12, 13], based on the loop-shaping method. These existing methods have difficulty directly matching the quadratic performance indices with the filter dynamics to the frequency loop shape [3, 4].

On the other hand, eigenvalue positioning methods were proposed for the time domain approach to select the appropriate weighting matrices, Q and R, to locate the closed loop poles inside the desired region or match them with the specific eigenvalues through the ARE [4]. Existing pole positioning approaches are limited in their analysis of time domain performances such as overshoot, rising and settling time [4]. LQ-proportional integral (PID) control methods have been also developed separately in time and frequency domains [14, 15].

We propose a new optimal control method for LQ-PID design to simultaneously satisfy the design specifications in the time- and frequency-domain without any filter or pole-zero cancelation [16]. For frequency-domain performances, we improved the existing limiting behaviour method of the loop transfer function to more exactly interpret the loop shape with respect to all design factors, based on Kalman's equality [17, 18]. This loop shaping makes it possible to consider the magnitudes of the LQ loop system independently in low and high frequency ranges for good command following, disturbance rejection, and attenuation of noise effect [19]. Time domain specifications such as percent overshoot, rise time, and settling time are related to the design factor by target function matching regardless of the pole-zero cancellation [16]. Overshoot and the Routh- Hurwitz stability criterion are formulated by the coefficient analysis with respect to the normalized third-order closed-loop system [20]. The other time-related specifications are addressed by the time-regulation property of the target function [21]. In the proposed method, the LQ-PID control parameters are determined by optimizing to minimize the cost function, which is subject to constraints to meet the combined time - and frequency-domain specifications for classical performances. We choose a combination of integral absolute error (IAE) and 2-norm of the control gains for the cost function of the optimal problem [22].

The primary contributions of the proposed method include as follows:

- (i) The proposed method provides the direct matching manner to relate the weighting factor, Q, to the loop shape of the loop transfer function in frequency domain.
- (ii) The proposed target function matching method improves the existing time domain approach of LQ-PID control by considering the specific time domain performances of transient response such as overshoot,

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rising time, settling time.

- (iii) Both of time and frequency domain specifications are combined by the proposed optimization problem in order to be simultaneously satisfied.

The paper is organized as follows. In Section 2, we introduce the LQ-PID formulation to design the optimal PID controller by transforming the PID control into the LQ approach. The design methodologies for the proposed controller are discussed as the optimization problem with constraints subject to the time- and frequency-domain specifications in Section 3. Numerical examples are given in Section 4 to show the effectiveness of the proposed design method, compared to other design methods. Finally, Section 6 contains some concluding remarks and remaining challenges.

## 2. LQ-PID Control

Briefly, we introduce the LQ-PID control method proposed by Suh and Yang [14], in which the optimal linear feedback control law is matched to the conventional PID control for the second order system by augmenting the integral of the output variable as a new state.

Consider the following second order model:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = c\omega_n^2 u(t), \quad (1)$$

where  $y(t)$ ,  $u(t)$ ,  $\zeta$  and  $\omega_n$  are the output variable, the control variable, the damping ratio and the natural frequency. The initial conditions of  $y(t)$  and  $dy(t)/dt$  are specified.

The new augmented state variables are:

$$x_p(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \int_0^t y(\tau) d\tau \\ y(t) \\ \frac{dy(t)}{dt} \end{bmatrix}, \quad (2)$$

The state-space representation of the augmented system becomes:

$$\frac{dx(t)}{dt} = Ax(t) + B\hat{u}(t-L), \quad (3)$$

$$y(t) = Cx(t) \quad (4)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} = \begin{bmatrix} 0 & I_p \\ 0 & A_p \end{bmatrix}, \quad (5)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ c\omega_n^2 \end{bmatrix} = \begin{bmatrix} 0 \\ B_p \end{bmatrix}, \quad C = [0 \ 1 \ 0] = [0 \ C_p]. \quad (6)$$

If we apply the different input gain according to and as

$$\begin{cases} \hat{u}(t) = -Ge^{A_c t} e^{A(L-t)} x(t), & 0 \leq t < L \\ \hat{u}(t) = -Ge^{A_c t} x(t), & L \leq t \end{cases},$$

where  $A_c$  is the closed loop system matrix.

The time delay system can be represented as the system without the time delay as where [23].

The quadratic performance criterion is considered for the LQR formulation:

$$J = \frac{1}{2} \int_0^\infty (x(t)^T Qx(t) + \rho u(t)^T u(t)) dt, \quad (7)$$

with the assumption that the weighting factor,  $Q$ , is positive semi-definite and symmetric  $Q = Q^T \geq 0$ , and  $\rho$  is a positive value.

The linear feedback control law to regulate the state,  $x(t)$ , is obtained as follows:

$$u(t) = -Gx(t), \text{ and} \quad (8)$$

$$G = \rho^{-1} B^T K, \quad (9)$$

where the optimal gain matrix,  $K = K^T$ , is obtained by solving the ARE:

$$KA + A^T K + Q - \frac{1}{\rho} KBB^T K = 0. \quad (10)$$

Let the components of  $K$  and  $G$  be:

$$K = \begin{bmatrix} K_{00} & K_{10} & K_{20} \\ K_{10} & K_{11} & K_{21} \\ K_{20} & K_{21} & K_{22} \end{bmatrix}, \text{ and} \quad (11)$$

$$G = [g_0 \ g_1 \ g_2] = \frac{c\omega_n^2}{\rho} [K_{20} \ K_{21} \ K_{22}], \quad (12)$$

which we will hereafter refer to as the optimal Kalman gains of the LQ-PID.

The optimal control law of Eq. (9) is represented as the following PID control formula:

$$u(t) = -\frac{c\omega_n^2}{\rho} \left[ K_{20} \int_0^t y(\tau) d\tau + K_{21} y(t) + K_{22} \frac{dy(t)}{dt} \right]. \quad (13)$$

The state feedback system with the control input,  $u(t)$ , of Eq. (8) is obtained by substituting Eq. (12) into Eq. (3):



$$|S(j\omega)| \leq 1, \text{ for all } \omega \text{ [3, 4, 17, 19].}$$

The sensitivity function is defined in the usual way,

$$S(s) = (1 + g_{LQ}(s))^{-1}, \quad (21)$$

where the loop transfer function  $g_{LQ}(s)$  of LQR is obtained as follows:

$$\begin{aligned} g_{LQ}(s) &= G(sI - A)^{-1} B \\ &= [g_0 \ g_1 \ g_2] \begin{bmatrix} \frac{1}{s} & \frac{1}{s} & C_p (sI - A_p)^{-1} \\ 0 & (sI - A_p)^{-1} & \end{bmatrix} \begin{bmatrix} 0 \\ B_p \end{bmatrix}. \quad (22) \\ &= \frac{(g_2 s^2 + g_1 s + g_0) c \omega_n^2}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)} \end{aligned}$$

In order to associate the frequency loop shaping procedure with the weighting factors,  $Q$  and  $\rho$ , we use Kalman's identity for the single input single output case:

$$\rho(1 + g_{LQ}(-j\omega))(1 + g_{LQ}(j\omega)) = \rho + g_{oL}(-j\omega)g_{oL}(j\omega), \quad (23)$$

where  $g_{oL}(j\omega) = N(j\omega I - A)^{-1} B$  with a partitioned matrix,  $N$ , of the weighting matrix,  $Q$ . Eq. (23) can be represented as:

$$\begin{aligned} \rho(1 + g_{LQ}(-j\omega))(1 + g_{LQ}(j\omega)) \\ = \rho + N((-j\omega - A)^{-1} B)N((j\omega - A)^{-1} B). \end{aligned} \quad (24)$$

Deriving the magnitude of the complex function in Eq. (24), we have:

$$|1 + g_{LQ}(j\omega)| = \sqrt{1 + \frac{1}{\rho} |N(j\omega I - A)^{-1} B|^2}. \quad (25)$$

Eq. (25) can be approximated as Eq. (26) only for low and high frequencies since  $|g_{LQ}(j\omega)| \gg 1$  and  $|g_{LQ}(j\omega)| \ll 1$ , respectively [15].

$$|g_{LQ}(j\omega)| \approx \frac{1}{\sqrt{\rho}} |N(j\omega I - A)^{-1} B|. \quad (26)$$

The magnitudes of the loop transfer function,  $g_{LQ}(s)$ , are approximated via the limiting behaviours of  $g_{LQ}(j\omega)$  at low and high frequencies [19], respectively, as:

$$\lim_{\omega \rightarrow 0} |g_{LQ}(j\omega)| \approx \frac{|c|}{\sqrt{\rho}} \sqrt{\frac{n_0^2}{\omega^2} + n_1^2}. \quad (27)$$

and

$$\lim_{\omega \rightarrow \infty} |g_{LQ}(j\omega)| \approx \frac{|n_2 c \omega_n^2|}{\sqrt{\rho}} \sqrt{\frac{1}{\omega^2 + (2\zeta \omega_n)^2}}. \quad (28)$$

Let each asymptotic line of  $|g_{LQ}(j\omega)|$  at the low and high frequency be  $l_L(\omega)$  and  $l_H(\omega)$ , respectively.

We consider a design procedure that involves explicitly shaping the magnitude of the loop transfer function,  $g_{LQ}(s)$ , for frequency domain performances.

In classical loop shaping, better performances such as good command following and disturbance rejection are obtained when the magnitude of the loop transfer function,  $g_{LQ}(s)$ , is larger for low frequencies at  $|g_{LQ}(j\omega)| \gg 1$  [19]. Fortunately, robust stability, noise attenuation, and control energy reduction are valid primarily in high frequencies at  $|g_{LQ}(j\omega)| \ll 1$ . The shape of  $|g_{LQ}(j\omega)|$  then should not invade either the low and high frequency barriers,  $Q$  and  $\beta(\omega)$ , respectively, as described in Fig. 2.

For this adequate loop shaping, we can develop a loop shaping technique in which the design parameter,  $N$ , plays a key role in maintaining  $|g_{LQ}(j\omega)|$  beyond the required boundaries. Specifically, this means that we match the weighting matrix,  $Q$ , of the LQ problem directly with the performances of the frequency domain to satisfy the boundary conditions via the approximation of limiting behavior.

Let  $\Omega_r$  be a boundary frequency of the command following barrier,  $\alpha(\omega)$ , and scalar  $m_r$  be a magnitude of  $\alpha(\omega)$ , as shown in Fig. 2. Set the barrier,  $\alpha(\omega)$ , as:

$$\alpha(\omega) = \begin{cases} m_r, & \omega \leq \Omega_r \\ 0, & \text{else} \end{cases}. \quad (29)$$

Since  $l_L(\omega)$  should exist on the upper side of the barrier,  $\alpha(\omega)$ , the asymptotic line of  $|g_{LQ}(j\omega)|$  at the low frequency must satisfy:

$$l_L(\omega) = \frac{|c|}{\sqrt{\rho}} \sqrt{\frac{n_0^2}{\omega^2} + n_1^2} \geq \alpha(\omega), \omega \leq \Omega_r. \quad (30)$$

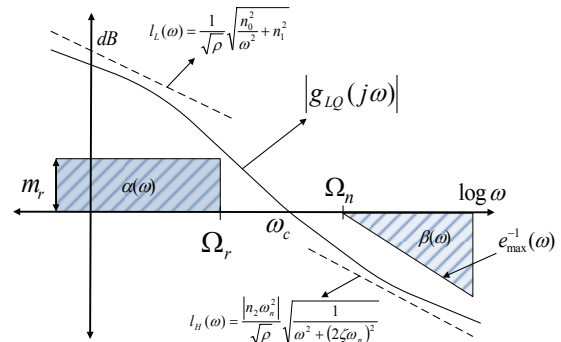


Fig. 2. Typical shape of  $|g_{LQ}(j\omega)|$  with barriers for the requirements

Solving Eq. (30) for the range of the parameters,  $n_0$  and  $n_1$ , gives:

$$\frac{n_0^2}{\Omega_r^2} + n_1^2 \geq \frac{\rho m_r^2}{c^2}. \quad (31)$$

Let  $\Omega_r$  be a boundary frequency of the sensor noise, and  $e_{\max}^{-1}(\omega)$  be the inverse of the maximum value of the modeling error in Fig. 2.

The barrier,  $\beta(\omega)$ , can then be expressed as:

$$\beta(\omega) = \begin{cases} 0, & \omega < \Omega_n \\ e_{\max}^{-1}(\omega), & \omega \geq \Omega_n \end{cases}. \quad (32)$$

The inequality constraint of  $l_H(\omega)$  for high frequencies is derived from Eq. (28) in the same manner:

$$l_H(\omega) = \frac{|n_2 c \omega_n^2|}{\sqrt{\rho}} \sqrt{\frac{1}{\omega^2 + (2\zeta\omega_n)^2}} \leq \beta(\omega). \quad (33)$$

Hence this constraint can be rearranged with the respect to the element  $n_2$  of the partial matrix,  $N$  of  $Q$  as

$$|n_2| \leq \frac{e_{\max}^{-1}(\omega)}{|c \omega_n^2|} \sqrt{\rho} \sqrt{\Omega_n^2 + (2\zeta\omega_n)^2}. \quad (34)$$

Finally, we can yield the constraints of Eqs. (31) and (34) with respect to the weighting factors,  $Q$  and  $\rho$ , of the LQ problem to satisfy frequency domain performances, adjusting the loop shaping of  $|g_{LQ}(j\omega)|$ .

### 3.1.2 Time domain performances

In the time domain, the desired transient performances are typically characterized by overshoot, and rise, and settling times. Designs for controllers that improve these transient performances of a second order closed loop system in the time domain are well explained in current literatures; however higher order systems are generally analyzed via the dominant pole concept as the reduced second order system is hard to formulate [24].

In this paper, we consider the special formulation of the third order closed loop system as the target function which is referred to [14] as:

$$F(s) = \frac{\omega_i^3}{\left(s + \frac{1}{r}\omega_i\right)(s^2 + p\omega_i s + r\omega_i^2)}. \quad (35)$$

The shape of the unit step response of Eq. (35) is determined by parameters  $p$ ,  $r$ , and  $\omega_i$ . Furthermore, it is found that the overshoot is maintained constantly, regardless of the variations in natural frequency,  $\omega_i$ , when

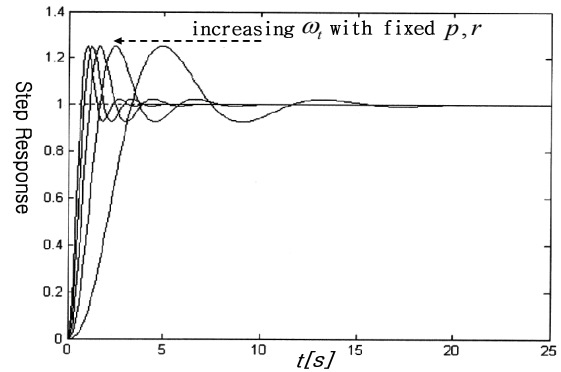


Fig. 3. Step response of the third order system according to variations in  $\omega_i$

parameters  $p$  and  $r$  are fixed. This is verified from the formulation of the output,  $y(t)$  [25].

$$y(t) = \frac{A^2 + B^2}{C} \sin\left(\frac{1}{2}t\omega_i\lambda + \theta\right) e^{-\frac{1}{2}pt\omega_i} - \frac{r^3}{(1 - pr + r^3)} e^{-\frac{\omega_i}{r}t} + 1, \quad (36)$$

where

$$\begin{aligned} A &= (pr - 1)(-p^2 + 4r), \\ B &= -(2r^2 + p - p^2r)\sqrt{-p^2 + 4r}, \\ C &= (1 - pr + r^3)(-p^2 + 4r), \\ \lambda &= \sqrt{-p^2 + 4r}, \text{ and} \\ \theta &= \begin{cases} \tan^{-1} \frac{B}{A} \pm \pi, & A < 0 \\ \tan^{-1} \frac{B}{A}, & A \geq 0 \end{cases}. \end{aligned} \quad (37)$$

Since the time,  $t$ , and the natural frequency,  $\omega_i$ , are united, we can modify the time,  $t$ , to a new time,  $\tau$ , using  $\omega_i$  as the time ratio, as follows:

$$y(\tau) = \frac{A^2 + B^2}{C} \sin\left(\frac{1}{2}\tau\lambda + \theta\right) e^{-\frac{1}{2}p\tau} - \frac{r^3}{(1 - pr + r^3)} e^{-\frac{\tau}{r}} + 1, \quad (38)$$

where  $\tau = t * \omega_i$ .

The output  $y(\tau)$  is represented as the response of the normalized third-order closed-loop system (NTCS), described as Eq. (35) with  $\omega_i = 1$ . Hence the time ratio of the same step response  $y(\tau)$  can be regulated independently on variations of the natural frequency,  $\omega_i$ , as shown in Fig. 3.

We consider two steps to formulate the constraints of

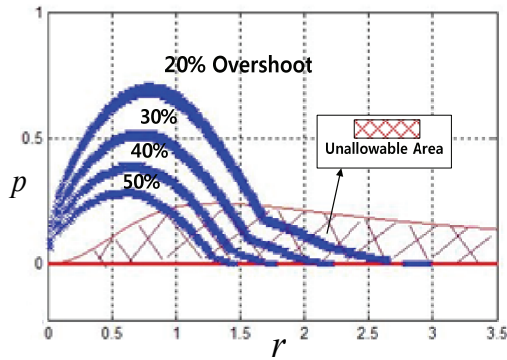


Fig. 4. Coefficient plane for overshoot in NTCS

time domain specifications. First, we determine the constraints with respect to  $p$ ,  $r$ , and  $\omega_i$  to satisfy the performance requirement in the time domain. For the overshoot requirement, the constraint function is illustrated as follows:

$$f_{OS}(r, p) \leq \Gamma, \quad \forall \tau, \quad (39)$$

where  $f_{OS}(r, p)$  represents the maximum step response function consistent with the variables,  $p$  and  $r$ .

Fig. 4 shows that the contours of maximum percentage overshoot with respect to the step reference can be plotted in the coefficient plane of  $p$  and  $r$ . If the Routh-Hurwitz stability criterion is applied to the normalized system transfer function of Eq. (35), Eq. (40) is required for simple absolute stability with  $h = 1$ :

$$\left(\frac{1}{r} + p\right) \left(\frac{p}{r} + r\right) > h. \quad (40)$$

A gain margin of 0.33, which is allowable in practice, corresponds to the line of Eq. (40) for  $h = 1.5$  [18]. The region between the two contours at  $h = 1$  and  $h = 1.5$  is the unallowable design area, which is distinct by oblique in the coefficient plane of Fig. 4.

Hence, except in terms of area, the overshoot coefficient is settled according to the boundary of the required overshoot percentage. The natural frequency,  $\omega_i$ , can be addressed as the design parameter, which plays a role in regulating the rise and settling time under the overshoot determined by  $p$  and  $r$ .

$$\omega_i \geq \max(\omega_{ST}, \omega_{RT}), \quad \text{fixed } p, r, \quad (41)$$

where  $\omega_{ST}$  and  $\omega_{RT}$  depend on the values of the natural frequency,  $\omega_i$ , to satisfy the required settling and rising time, respectively. In other words, the system parameters  $p$  and  $r$  are determined to meet the given overshoot requirement, and  $\omega_i$  is considered as a factor used to separately satisfy the given settling time and rising time requirements.

Secondly, the system parameters  $p$ ,  $r$ , and  $\omega_i$  should

be formulated as a function with respect to the optimal Kalman gains of the LQ-PID feedback system via pole matching to equalize the characteristic equation of the target function Eq. (16) of NTCS.

$$\begin{aligned} & s^3 + \left(2\zeta\omega_n + \frac{c^2\omega_n^4}{\rho}K_{22}\right)s^2 \\ & + \left(\omega_n^2 + \frac{c^2\omega_n^4}{\rho}K_{21}\right)s + \frac{c^2\omega_n^4}{\rho}K_{20}. \quad (42) \\ & = \left(s + \frac{1}{r}\omega_i\right)(s^2 + p\omega_i s + r\omega_i^2) \end{aligned}$$

Given the above Eq. (42), the following three equations are derived.

$$\begin{aligned} \omega_i^3 &= \frac{c^2\omega_n^4}{\rho}K_{20} \\ \frac{p\omega_i^2}{r} + r\omega_i^2 &= \omega_n^2 + \frac{c^2\omega_n^4}{\rho}K_{21}, \quad (43) \\ \frac{\omega_i}{r} + p\omega_i &= 2\zeta\omega_n + \frac{c^2\omega_n^4}{\rho}K_{22} \end{aligned}$$

Using Eq. (43), weighting matrices,  $Q$ , and  $\rho$ , can be expressed in terms of the variables of the target transfer function via the ARE to satisfy the time domain performances, as follows:

$$\begin{aligned} n_0^2 - \frac{c^2\omega_n^4}{\rho}K_{20}^2 &= 0 \\ 2K_{10} - 2\omega_n^2K_{21} + n_1^2 - \frac{c^2\omega_n^4}{\rho}K_{21}^2 &= 0 \\ 2K_{21} - 4\zeta\omega_nK_{22} + n_2^2 - \frac{c^2\omega_n^4}{\rho}K_{22}^2 &= 0 \\ K_{10} - 2\zeta\omega_nK_{20} - \frac{c^2\omega_n^4}{\rho}K_{22}K_{20} + n_2n_0 &= 0 \end{aligned} \quad (44)$$

By substituting Eq. (43) to Eq. (44), the components  $[n_0 \ n_1 \ n_2]$  of weighting matrix,  $Q$  are expressed as Eq. (45) in terms with the variables,  $p$ ,  $r$ , and  $\omega_i$  for time domain performances, given input weighting,  $\rho$ , and system parameters,  $\omega_n$ ,  $\zeta$  and  $c$  of the plant

$$\begin{aligned} n_0^2 &= \frac{\rho\omega_i^6}{\omega_n^4} \\ n_1^2 &= f_1(r, p, \omega_i), \\ n_2^2 &= f_2(r, p, \omega_i) \end{aligned} \quad (45)$$

where  $f_1$  and  $f_2$  are the functions to meet time domain specifications in terms of the system variables. Eq. (39)

and (41) then can be represented by the function of the optimal Kalman gains through Eqs. (42), (43), and (44) as follows:

$$f_{OS}(r, p) = f_{OS}(\rho, n_0, n_1, n_2) \leq \Gamma, \quad \forall \tau \quad (46)$$

$$\omega_i = f_{WT}(\rho, n_0, n_1, n_2) \geq \max(\omega_{ST}, \omega_{RT}), \quad \text{fixed } p, r \quad (47)$$

Therefore we can determine the feasible constraints Eqs. (46) and (47) in terms of positive weighting matrices,  $Q$  and  $\rho$ , of the LQ problem to meet transient step response requirements such as overshoot, rise, and settling time in time domain.

### 3.2 Formulation of an optimization problem

In the above sections, we derived that the constraint conditions Eqs. (31), (34), (46), and (47) to satisfy the combined time- and frequency-domain requirements. The conditions are represented as inequality functions with respect to weighting matrices,  $Q$  and  $\rho$ , of LQ problem, obtained via the loop shaping method and the pole matching method by the ARE. In this subsection, we formulate the optimal problem to determine optimal values of the design parameter  $N$  to minimize the object function subject to the constraints combined by time- and frequency-domain performances. We choose the weighted combination of two objectives, the 2-norm of PID gain and IAE. This objective of the optimal problem can vary depending on the control purpose of the designer in practice.

The first objective term is represented as Eq. (48) in order to avoid a high gain problem when we take  $\rho \ll 1$  for the cheap control as shown by Eq. (27) [26].

$$f_1(n_0, n_1, n_2) = \left\| \frac{\omega_n^2}{\rho} [K_{20} \ K_{21} \ K_{22}] \right\|_2. \quad (48)$$

The second objective term is the IAE index as Eq. (50) for the time response to consider the error arising from the difference between a nominal plant and a closed loop target function Eq. (35):

$$f_2(n_0, n_1, n_2) = \int_0^\infty |e(t)| dt. \quad (49)$$

These two objectives are combined by weighting vector,  $\gamma$ , as follows:

$$f(n_0, n_1, n_2) = [\gamma_1 \ \gamma_2] \begin{bmatrix} f_1(n_0, n_1, n_2) \\ f_2(n_0, n_1, n_2) \end{bmatrix}, \quad (50)$$

where the weighting vector can be chosen according to the importance of each objective.

In order to combine the time- and frequency-domain performances, the optimization problem is formulated with respect to the object functions in Eq. (50) and the

constraints in Eqs. (31), (34), (46), and (47), as follows:

$$\begin{aligned} &\text{minimize } f(n_0, n_1, n_2) \\ &\text{subject to } \frac{n_0^2}{\Omega_r^2} + n_1^2 \geq \rho m_r^2 \\ &|n_2| \leq \frac{e_{\max}^{-1}(\omega)}{|c\omega_n^2|} \sqrt{\rho} \sqrt{\Omega_n^2 + (2\zeta\omega_n)^2} \\ &f_{OS}(\rho, n_0, n_1, n_2) \leq \Gamma, \quad \forall \tau \\ &f_{WT}(\rho, n_0, n_1, n_2) \geq \max(\omega_{ST}, \omega_{RT}), \quad \text{fixed } p, r \\ &n_0, n_1, n_2 > 0 \quad \text{for positive definite} \end{aligned} \quad (51)$$

where  $\omega_n$  and  $\zeta$  are given by the process, and  $\Omega_r$ ,  $m_r$ ,  $\Omega_n$ ,  $e_{\max}^{-1}(\omega)$ ,  $\Gamma$ ,  $\omega_{ST}$ ,  $\omega_{RT}$ , and  $\rho$  are given as the design specifications.

The design procedure is summarized as

#### Summary of Design Procedure

**Step 1:** Set parameter values of a given plant  $(\zeta, \omega_n)$ .

**Step 2:** Specify the frequency domain constraints with respect to the required design specifications ( $m_r, \Omega_r, \Omega_n, e_{\max}^{-1}(\omega)$ ) via the loop shaping method.

**Step 3:** Specify the time domain constraints with respect to the required design specifications ( $\Gamma, \omega_{ST}, \omega_{RT}$ ) via target function matching.

**Step 4:** Perform the optimization problem of Eq. (51) with the specific value of  $\rho$  to determine the weighting factor,  $Q = N^T N$ , subject to the constraints obtained by Step 2 and 3.

**Step 5:** Determine the optimal PID control gain,  $G$ , by solving the ARE with the system parameters and the weighting matrix  $Q$  and  $R = \rho I$  determined in Step 4.

**Step 6:** Finish the process to determine the adequate controller if all of required design specifications are satisfied, if not reset the constraints, beginning with Step 2.

## 4. Simulation Examples

**Example 1:** Consider the second order system with a small delay, referred in [27] as

$$g(s) = \frac{1}{(1+s)(1+0.5s)} e^{-0.5s}. \quad (52)$$

For the optimization, the constraints and weighting vector  $\gamma$  are given as follows

$$\alpha(\omega) = \begin{cases} 20dB, & \omega \leq 0.1 \\ 0, & \omega > 0.1 \end{cases},$$

$$\beta(\omega) = \begin{cases} 0, & \omega \leq 10^2 \\ -40dB, & \omega > 10^2 \end{cases},$$

**Table 1.** System performance comparisons of time and frequency responses for example 1.

Methods	Rising Time (sec)	Settling Time (sec)	Over-shoot (%)	$ g_{LQ}(0.1j) $ (db)	$ g_{LQ}(100j) $ (db)	Spec.	
						Time	Freq. (L/H)
Pro	1.0	3.0	3.9	20.0	-40.0	O	O/O
Grassi	0.5	4.0	26.6	25.9	-34.4	X	O/X
Olof	2.8	8.3	4.9	14.9	-45.9	O	X/O
Ho	0.5	4.5	28.7	23.9	-36.1	X	O/X
Yang	3.8	9.2	25.9	18.0	-30.2	X	X/X

$$f_{OS}(\rho, n_0, n_1, n_2) \leq 9\%$$

$$f_{WT}(\rho, n_0, n_1, n_2) \geq \omega_{ST}(8 \text{ sec})$$

$$\gamma = [1 \ 0.5], \quad \rho = 0.0001$$

The frequency and time domain constraints are chosen to avoid the loop shape from the frequency barriers  $Q$  and  $\beta(\omega)$ , and to satisfy that the overshoot and settling time of the target system are less than 9% and 8 sec, respectively.

According to the design procedures, design parameter  $N$  is determined as  $N = [0.0099 \ 0.0151 \ 0.005]$ , and PID controller parameters are obtained as  $K_p = 1.5027$ ,  $T_i = 0.9967$  and  $T_d = 0.5005$  via the ARE.

In Table 1, it is shown that the proposed method only yields the adequate loop shape required to avoid invading the frequency barrier, a good step response to meet the time response requirements of less than 9% overshoot and 8 sec

settling time, as well as simultaneously minimizing the cost function. Figs. 5 and 6 present the comparisons of the simulation results for frequency and time responses, respectively.

The results of the simulation are compared with other PID control methods by Yang [27], Ho [28], Grassi [29], and Olof [30]. As shown in the Table 1 and Fig. 5, the responses of the proposed and Olof methods satisfy the frequency-domain specifications. Among them, only the proposed method meets the required time-domain specifications described as settling time (8 sec) and overshoot (9%) in Fig. 6.

**Example 2:** A fired heater problem is taken from [31] as an example of practical process; this is one of the most common examples with long time delay for the practical process. The open loop dynamic of the system is described by the transfer function,  $G_I(s)$ , at low fuel gas feed rates:

$$G_I(s) = \frac{3}{(4s+1)^2} e^{-10s} \tag{53}$$

For the optimization, the constraints and weighting vector  $\gamma$  are given as follows.

$$\alpha(\omega) = \begin{cases} 15dB, & \omega \leq 0.01 \\ 0, & \omega > 0.01 \end{cases}, \quad \beta(\omega) = \begin{cases} 0, & \omega \leq 10 \\ -40dB, & \omega > 10 \end{cases}$$

$$f_{OS}(\rho, n_0, n_1, n_2) \leq 10\%$$

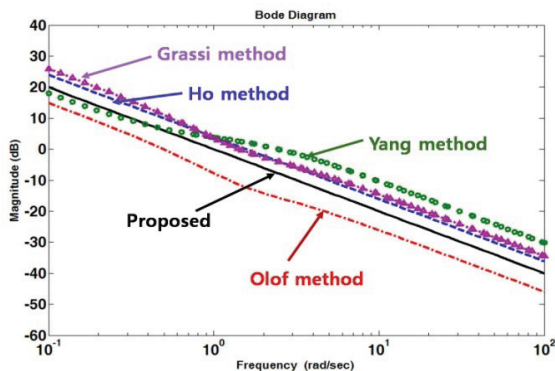
$$f_{WT}(\rho, n_0, n_1, n_2) \geq \omega_{ST}(100 \text{ sec})$$

$$\gamma = [1 \ 0.5], \quad \rho = 0.0001$$

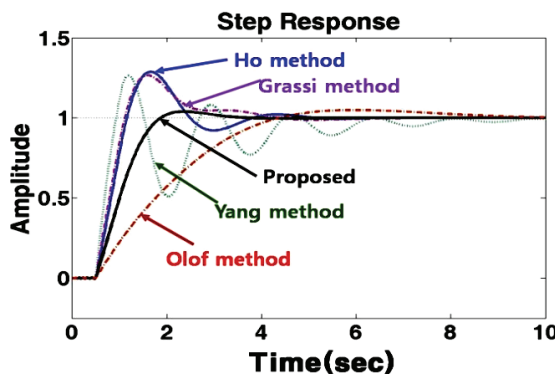
The design parameter  $N$  and PID controller are obtained as

$$N = [0.196 \ 3.08 \ 5.33] \times 10^{-3}, \quad K_p = 0.221, \quad T_i = 0.02, \quad \text{and} \\ T_d = 0.4567, \text{ respectively.}$$

Table 2 shows the results of the proposed controller and those by Grassi [29], Olof [30], Ho [28], and Huang [32]. With the exception of the Ho method, they satisfy the time domain specifications with respect to the percent overshoot (10%) and settling time (100 sec). On the other hand, the loop shapes at the low and high frequency are obtained only by the proposed method, and Ho's. Only by the proposed method presents the allowable good step



**Fig. 5.** Comparison of the frequency shape of the loop transfer function with various other methods

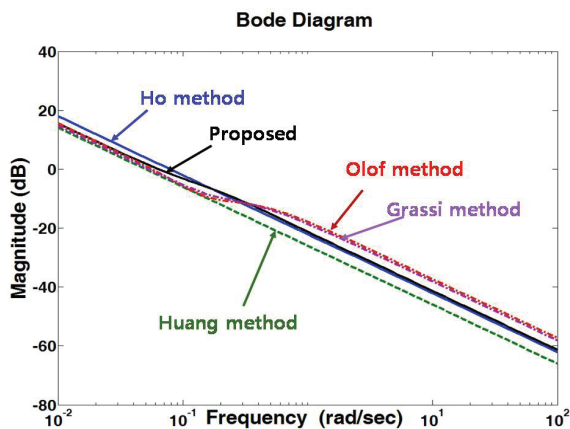


**Fig. 6.** Comparison of step response with various other methods for example 1

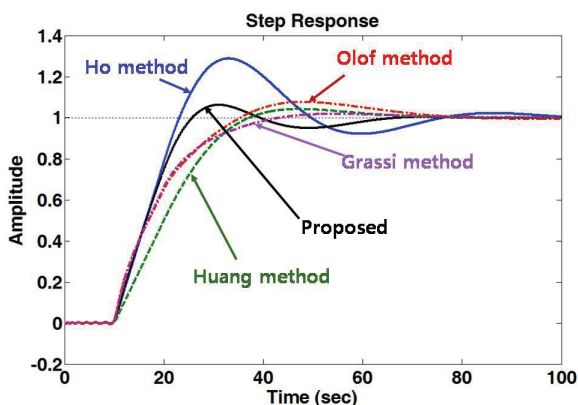


**Table 2.** System performance comparisons of time and frequency responses for example 2.

Methods	Rising Time (sec)	Settling Time (sec)	Overshoot (%)	$ g_{LQ}(0.01j) $ (db)	$ g_{LQ}(10j) $ (db)	Spec.	
						Time	Freq. (L/H)
Pro	11.6	60.0	6.2	15.4	-41.3	O	O/O
Grassi	19.1	39.3	1.8	14.8	-38.1	O	X/X
Olof	18.3	69.9	7.6	15.5	-37.3	O	O/X
Ho	10.3	89.6	28.8	17.9	-42.1	X	O/O
Huang	19.0	60.6	4.1	14.0	-46.0	O	X/O



**Fig. 7.** Comparison of the frequency shape of the loop transfer function with other methods for example 2



**Fig. 8.** Comparison of the step response with other methods for example 2

response for the time domain performance simultaneously satisfied in addition to the frequency response to guarantee a magnitude higher than 15 db at the low frequency and lower than -40 db at high frequency. Figs. 7 and 8 present a comparison of loop shapes in the frequency domain and step responses in the time domain, respectively.

### 5. Conclusion

This paper proposes a new tuning method to combine time- and frequency-domain requirements for an LQ-PID controller. The weighting matrices, Q and R, are formulated to improve limiting behaviour of the loop

transfer function by Kalman’s equality and target function matching is used for frequency and time domain performances, respectively. Both of them are simultaneously considered as the constraints of the optimal problem with the cost function in which the 2-norm of optimal control law and IAE index are applied. The PID control coefficients are obtained directly through the ARE, including the weighting factors Q and R. Simulation results have been presented to show the effectiveness of the proposed method in comparison to others. In conclusion, the proposed method guarantees not only the robust stability and design simplicity inherent in LQ-PID optimal control, but also considers the time- and frequency-domain performances for the second order system.

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