



A complete S-shape feed rate scheduling approach for NURBS interpolator

Xu Du, Jie Huang, Li-Min Zhu*

State Key Laboratory of Mechanical System and Vibration, School of Mechanical Engineering, Shanghai Jiao Tong University, Shanghai 200240, PR China

Received 28 March 2015; received in revised form 7 May 2015; accepted 8 June 2015

Available online 20 June 2015

Abstract

This paper presents a complete S-shape feed rate scheduling approach (CSFA) with confined jerk, acceleration and command feed rate for parametric tool path. For a Non-Uniform Rational B-Spline (NURBS) tool path, the critical points of the tool path where the radius of curvature reaches extreme values are found firstly. Then, the NURBS curve is split into several NURBS sub-curves or blocks by the critical points. A bidirectional scanning strategy with the limitations of chord error, normal/tangential acceleration/jerk and command feed rate is employed to make the feed rate at the junctions between different NURBS blocks continuous. To improve the efficiency of the feed rate scheduling, the NURBS block is classified into three types: short block, medium block and long block. The feed rate profile corresponding to each NURBS block is generated according to the start/end feed rates and the arc length of the block and the limitations of tangential acceleration/jerk. In addition, two compensation strategies are proposed to make the feed rate more continuous and the arc increment more precise. Once the feed rate profile is determined, a second-order Taylor's expansion interpolation method is applied to generate the position commands. Finally, experiments with two free-form NURBS curves are conducted to verify the applicability and accuracy of the proposed method.

© 2015 Society of CAD/CAM Engineers. Production and hosting by Elsevier. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Keywords: Feed rate scheduling; Acceleration/deceleration; Bidirectional scanning; NURBS

1. Introduction

Smooth feed rate scheduling of parametric interpolator for CNC machine tools plays a key role in high-speed and high-accuracy machining of sculptured surfaces, such as dies, molds and aerospace parts. The discontinuities in feed motion can evoke the natural modes of the mechanical structure and the servo control system, which will degrade the contouring accuracy [1]. In order to achieve high-speed and high-accuracy machining, many scholars devote to the fields such as parametric curve interpolation, feed rate profile scheduling and servo-loop control techniques. The acceleration/deceleration feed rate profiles, e.g. the trapezoidal and the jerk-limited profile, are most widely used in feed rate scheduling.

The trapezoidal feed rate scheduling method [2] is simple to implement, but it will result in various oscillations on the feed and acceleration when interpolating along complex tool paths,

compared with jerk-limited feed rate scheduling methods. Hence, many jerk-limited methods, such as Sine-shape method [3], Time-optimal method [4,5] and S-shape method [6–11], have been developed to schedule the feed rate with the advantages of improved smoothness. Lee et al. [3] proposed an off-line feed rate scheduling method that considers the constraints imposed by chord error, acceleration and jerk limitations, however, the acceleration function obtained with this method is discontinuous comparing to those of the time-optimal and the S-shape methods. Dong et al. [4] and Beudaert et al. [5] proposed a similar time-optimal method, respectively; however, the iterative algorithm applied in this method is hard to be implemented due to its heavy computational burden and non-feasibility in some situations. Erkorkmaz et al. [8] proposed the S-shape feed rate scheduling method aimed at avoiding the excitation of structural dynamics due to the high bandwidth of the drive system.

The S-shape method is widely used in smooth feed rate scheduling [9–11] because of its simplicity and smoothness, but the complete procedure of S-shape method for the complex reference trajectory has not been fully studied. Here, an

*Corresponding author.

E-mail address: zhulm@sjtu.edu.cn (L.-M. Zhu).

approach for S-shape feed rate scheduling is proposed, of which the procedure is complete and manifested as follows:

1. Split NURBS curve into several NURBS sub-curves according to the critical points.
2. Calculate the arc length of each NURBS sub-curve based on an adaptive quadrature method.
3. Obtain the final feed rate at the junctions between successive NURBS sub-curves by bidirectional scanning method.
4. Calculate the arc length criterion and decide the feed rate scheduling algorithm.
5. Compensate with the variable-jerk and the arc length error compensation strategies.
6. Finally, obtain the reference positions by a second-order Taylor interpolation method.

The proposed approach has three main advantages:

1. The bidirectional scanning method guarantees the continuity of the feed rate even if there exists the short block.
2. The arc length criterion makes the feed rate scheduling approach easy for implementation.
3. The variable-jerk compensation strategy eliminates the feed rate discontinuities caused by the round-off error of the duration quantization.

This paper is organized as follows. In the next section, the definition of the NURBS curve and the method to calculate its arc length are briefly introduced. In Section 3, a complete feed rate scheduling approach for NURBS curve is presented. Experiments whose layout of the three-axis experimental system is shown in Fig. 1 are conducted in Section 4, and the conclusions are given in Section 5.

2. Definition and arc length of a NURBS curve

2.1. Definition of a NURBS curve

A NURBS curve $C(u)$ can be expressed as [12]

$$C(u) = \frac{\sum_{i=0}^n N_{i,p}(u)\omega_i P_i}{\sum_{i=0}^n N_{i,p}(u)\omega_i} \quad (1)$$

where $\{P_i\}$ are the control points which form a control polygon, $\{\omega_i\}$ are the weights, p is the degree of a NURBS curve and $\{N_{i,p}(u)\}$ are the p th-degree B-spline basis functions defined on the non-uniform knot vector, i.e.

$$U = \{u_0, u_1, \dots, u_{n+p+1}\} \quad (2)$$

The curvature of a NURBS curve is defined as [13]

$$\kappa = \frac{\|C'(u) \times C''(u)\|}{\|C'(u)\|^3} \quad (3)$$

where $C'(u)$ and $C''(u)$ are the first- and second-order derivatives of a NURBS curve, respectively.

2.2. Arc length of a NURBS curve

The arc length function $s(u)$ of a NURBS curve $C(u)$ over parameter interval $[u_i, u_{i+1}]$ is given as [13]

$$s(u) = \int_{u_i}^{u_{i+1}} \|C'(u)\| du \quad (4)$$

The first-order derivative of the arc length $s(u)$ can be expressed as

$$s'(u) = \|C'(u)\| > 0 \quad (5)$$

An adaptive quadrature method [14] based on Simpson's rule is used to calculate the desired arc length over a very small parameter interval $[u_i, u_{i+1}]$ as follows:

$$s(u_i, u_{i+1}) = \frac{(u_{i+1} - u_i)}{6} \left[s'(u_i) + 4s'\left(\frac{u_{i+1} + u_i}{2}\right) + s'(u_{i+1}) \right] \quad (6)$$

A composite Simpson rule using four subintervals of $[u_i, u_{i+1}]$ can be performed by bisecting parameter interval above into two equal subintervals (i.e. $[u_i, (u_i + u_{i+1})/2]$ and $[(u_i + u_{i+1})/2, u_{i+1}]$). Eq. (6) is recursively applied over every subinterval as follows:

$$s\left[u_i, \frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] = s\left[u_i, \frac{(u_i + u_{i+1})}{2}\right] + s\left[\frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] \quad (7)$$

where

$$s\left[u_i, \frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] = \frac{u_{i+1} - u_i}{12} \left[s'(u_i) + 4s'\left(\frac{u_i + u_{i+1}}{2}\right) + s'\left(\frac{u_i + u_{i+1}}{2}\right) \right]$$

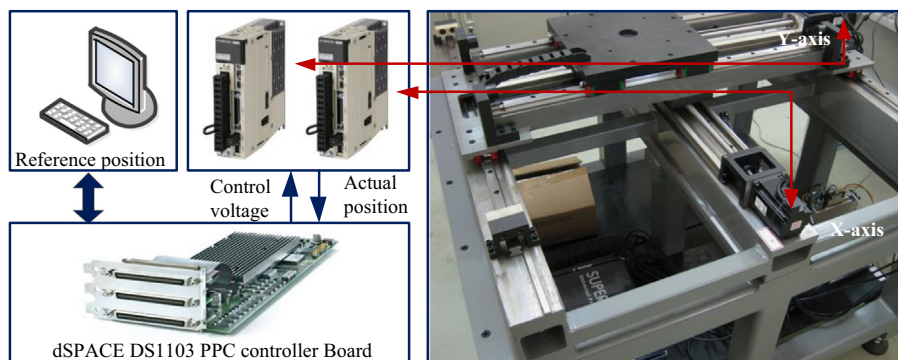


Fig. 1. Layout of the three-axis experimental system.

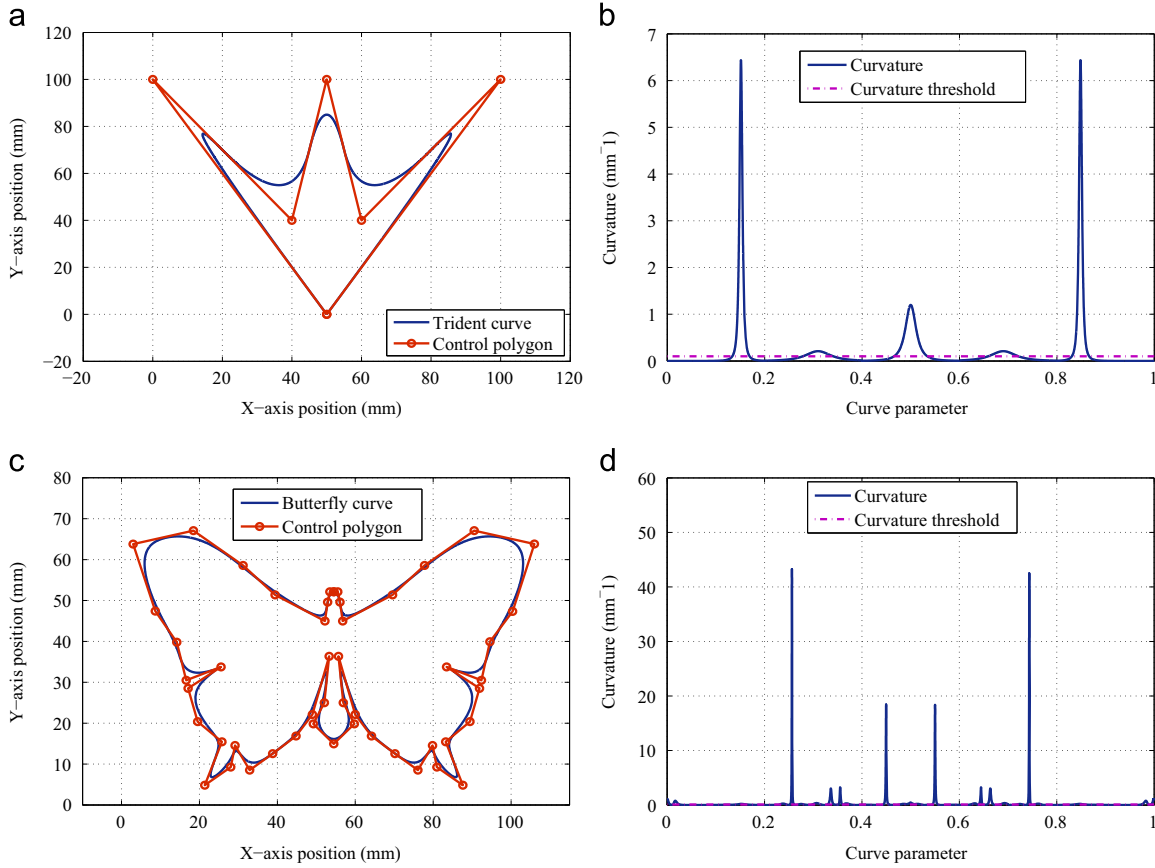


Fig. 2. Critical point distribution for trident curve. (a) Trident, (b) trident curvature, (c) butterfly, (d) butterfly curvature.

$$s\left[\frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] = \frac{u_i - u_{i+1}}{12} \left[s'\left(\frac{u_i + u_{i+1}}{2}\right) + 4s'\left(\frac{u_i + 3u_{i+1}}{2}\right) + s'(u_{i+1}) \right]$$

Assuming that the tolerance of ζ is specified for the parameter interval, if

$$\frac{1}{10} \left[s\left[u_i, \frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] - s(u_i, u_{i+1}) \right] < \zeta \quad (8)$$

Then it can be proved that

$$\left| \int_{u_i}^{u_{i+1}} s'(u) du - s\left[u_i, \frac{(u_i + u_{i+1})}{2}, u_{i+1}\right] \right| < \zeta \quad (9)$$

Thus the arc length of NURBS is equal to $s[u_i, (u_i + u_{i+1})/2, u_{i+1}]$. If Eq. (8) holds, the arc length calculated by Eq. (7) is defined as the desired arc length in $[u_i, u_{i+1}]$ for the feed rate scheduling. If Eq. (8) fails, the two subintervals are labeled as $[u_i, u_i + u_{i+1}/2]$ and $[u_i + u_{i+1}/2, u_{i+1}]$ for further subdivision and testing until Eq. (8) holds for each subinterval in $[u_i, u_{i+1}]$. Finally, the sum of the arc length calculated by Eq. (7) in each subinterval is defined as the desired arc length in $[u_i, u_{i+1}]$ for the feed rate scheduling.

3. NURBS interpolator with feed rate scheduling

In this section, a complete S-shape feed rate scheduling approach for NURBS curve is proposed. It consists of three

stages: the preprocessing stage, the scheduling feed rate stage and the interpolating stage.

In the stage of preprocessing, the critical points of the tool path are found out by the threshold curvature. Then, the NURBS curve is split into several NURBS sub-curves by the critical points. The arc length of each NURBS sub-curve, which is defined as the desired arc length, is approximately calculated by the adaptive quadrature method. A bidirectional scanning with the limitations of chord error, normal/tangential acceleration/jerk and command feed rate is employed to make the feed rate at the junctions between successive NURBS blocks continuous. Finally, the feed rate at critical points determined by the directional scanning is accepted for the practical CNC machining.

In the stage of scheduling feed rate, the NURBS block is classified into three types (i.e. short block, medium block and long block) by the arc length criterion. Three kinds of algorithms are developed to schedule feed rate for each type of the NURBS block, respectively. The feed rate profile corresponding to each NURBS block is generated according to the start/end feed rates, the desired arc length of the NURBS block and the limitations of tangential acceleration/jerk. A variable-jerk compensation strategy is proposed to make the feed rate more continuous. An arc length error compensation strategy is proposed to reduce the difference between the scheduling arc length and the desired arc length. Finally, the modified arc length increment is output for the interpolation algorithm.

In the stage of interpolating, the second-order Taylor's expansion of the spline parameter with respect to the arc-length using the compensated arc length is adopted to generate more accurate position commands.

3.1. Critical points and curve splitting

The point, where its curvature is larger than the curvature threshold, is called a candidate point. The candidate point, which has the local maximum curvature, is called a critical point. To find out the critical points, the curvature threshold κ_{cr} is determined firstly according to the chord error, normal acceleration and jerk [3].

$$\kappa_{cr} = \min \left(\frac{8\delta}{(v_{max}T_s)^2 + 4\delta^2}, \frac{A_n}{v_{max}^2}, \sqrt{\frac{J_n}{v_{max}^3}} \right) \quad (10)$$

where δ is the chord error, v_{max} is the command feed rate and A_n/J_n are the normal acceleration/jerk. The curve parameter u corresponding to the local maximum value of the curvature among the candidate points is computed by

$$\kappa'(u) = 0 \quad (11)$$

The interpolation points corresponding to the solutions of the equation are defined as the critical points as shown in Fig. 2.

When the critical points are determined, the arc length between two successive points is calculated by the composite Simpson method.

3.2. Bidirectional scanning

Bidirectional scanning, which consists of the backward scanning and the forward scanning, is an important approach to generate smooth trajectory. As illustrated in Fig. 3, the objective of the backward scanning is to determine the maximum allowable end feed rate for the NURBS block. After the backward scanning, the forward scanning is used to determine the final feed rates at the critical points.

3.2.1. Backward scanning

Backward scanning is performed from the final to the first NURBS blocks. At the beginning of the backward scanning, the end feed rate for the final NURBS block is set to zero to make the feed motion stop at the end of the NURBS curve. The S-shape acceleration profiles as shown in Fig. 4(a) and (b), are used to restrict the feed rate at the junctions between different blocks. In the backward scanning, the chord error, the normal acceleration/jerk and the command feed rate are considered simultaneously. The details of the algorithm are described as follows, where N is the number of the NURBS blocks, ${}^b v_{i+1}$ is the end feed rate of the i th NURBS block determined by the backward scanning for the $(i+1)$ th NURBS block and A_t/J_t are the tangential acceleration/jerk.

- (1) Let $i=N$ and ${}^b v_{i+1}$.
- (2) If $i=1$, go to (3). Otherwise, go to (2).
 - (2a) Apply jerk-limited S-shape acceleration from the

end to the start of the NURBS block. When considering the tangential jerk limitation, the cubic equation below is constructed as

$$({}^b v_i^T)^3 + {}^b v_{i+1}({}^b v_i^T)^2 - ({}^b v_{i+1})^2 {}^b v_i^T - ({}^b v_{i+1})^3 - J_t s_i^2 = 0 \quad (12)$$

To solve for ${}^b v_i$ from Eq. (12), the Newton–Raphson method is applied and the initial value is set to ${}^b v_{i+1}$. If ${}^b v_i^T \leq A_t^2/J_t + {}^b v_{i+1}$, it implies that the tangential acceleration limitation is satisfied (i.e. $a_m < A_t$ showed in Fig. 4(a)), and go to (2b). Otherwise, the tangential acceleration limitation is violated, so Eq. (12) does not hold, a new relation between ${}^b v_i^T$ and ${}^b v_{i+1}$ is constructed according as

$$J_t ({}^b v_i^T)^2 + A_t^2 ({}^b v_i^T) - J_t ({}^b v_{i+1})^2 + A_t^2 ({}^b v_{i+1}) - 2A_t J_t s_i = 0 \quad (13)$$

To solve for ${}^b v_i^T$ from Eq. (13), the Newton–Raphson method is applied and the initial value is set to ${}^b v_i$, and go to (2b). ${}^b v_i^T$ resolved by Eqs. (12) or (13) is plotted in pentagonal polygon in Fig. 3.

- (2b) Consider the chord error (i.e. \square in Fig. 3), normal acceleration/jerk (i.e. \diamond in Fig. 3) [3]. The maximum normal acceleration/jerk is supposed to be equal to the tangential ones. The feed rate ${}^b v_i$ for the backward scanning should be determined by

$${}^b v_i = \min \left\{ {}^b v_i^T, \frac{2}{T_s} \sqrt{\frac{2\delta}{\kappa_i} - \delta^2}, \sqrt{\frac{A_n}{\kappa_i}}, \sqrt[3]{\frac{J_n}{\kappa_i^2}}, v_{max} \right\} \quad (14)$$

where T_s is the sampling period, κ_i is the curvature computed using Eq. (10), δ is the chord error, and v_{max} is the command feed rate (i.e. \circ in Fig. 3), and go to (2c).

- (2c) Let $i=i-1$ and go to (2).
- (3) Store ${}^b v_i (i=1, 2, \dots, N)$ into the buffer.

3.2.2. Forward scanning

Forward scanning is performed from the first to the final NURBS blocks. Since the chord error, the normal acceleration/jerk and the command feed rate have been already considered in the backward scanning, tangential acceleration/jerk is taken into account in the forward scanning. The details of algorithm are described as follows, where v_i obtained by the forward scanning for the $(i-1)$ th NURBS block is used as the start feed rate for the i th NURBS block, and where ${}^f v_{i+1}^T$ for the i th NURBS block is derived from the tangential acceleration/jerk limitations.

- (1) Let $i=1$ and $v_i=0$.
- (2) If $i=N$, go to (3). Otherwise, go to (2).
 - (2a) Solve for ${}^f v_{i+1}^T$ from Eq. (12). If ${}^f v_{i+1}^T \leq A_t^2/J_t + v_i$, the acceleration profile is showed in Fig. 4(a) and go

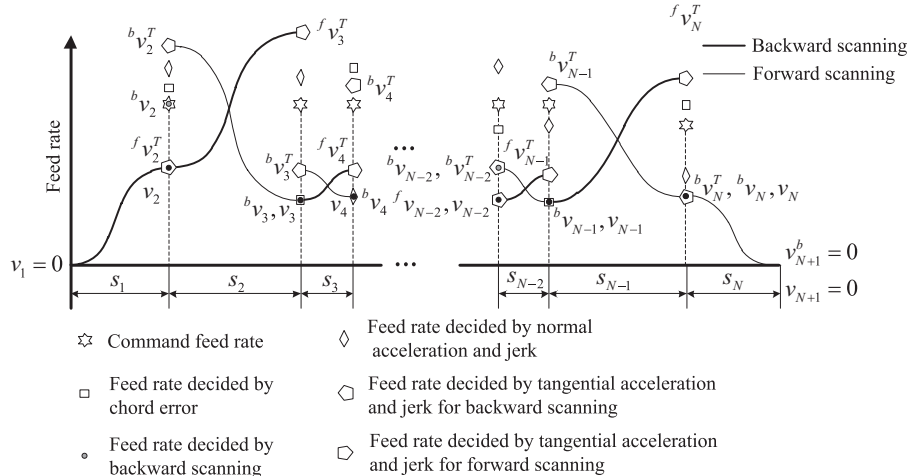


Fig. 3. Process of smooth feed rate scheduling.

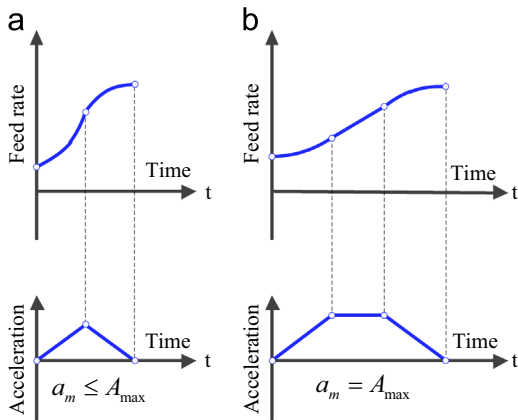


Fig. 4. Two cases for S-shape acceleration.

to (2b). Otherwise, the acceleration profile is showed in Fig. 4(b). Solve for $f v_{i+1}^T$ from Eq. (13), and go to (2b). $f v_{i+1}^T$ resolved by Eq. (12) or Eq. (13) is plotted in pentagonal polygon (i.e. Δ) in Fig. 3.

(2b) Determine the feed rate at the end of the NURBS block by

$$v_{i+1} = \min \{ {}^b v_{i+1}, {}^f v_{i+1}^T \} \quad (15)$$

(2c) Let $i = i + 1$ and go to (2).

(3) Store $v_{i+1} (i = 1, 2, \dots, N)$ into the buffer.

Finally, the feed rates $v_{i+1} (i = 1, 2, \dots, N)$ output by the forward scanning are used for the feed rate scheduling below.

3.3. S-shape feed rate profile generation

When the process of the bidirectional scanning is finished, the NURBS block feature data $\{s_i, v_s, v_e\}$ (where $v_s = v_i$ and $v_e = v_{i+1}$) to make the feed rate profile at the critical points continuous are generated. In the stage of scheduling feed rate,

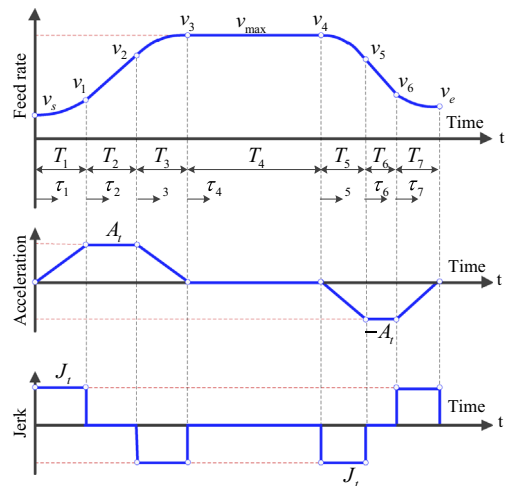


Fig. 5. S-shape ACC/DEC profile with seven phases.

Table 1
Expressions for the feed rate profile.

τ	$\dot{s}_k(t)$	$\ddot{s}_k(t)$	$\dot{s}_k(t)$	$s_k(t)$
τ_1	J_i	$J_i \tau_1$	$v_s + \frac{1}{2} J_i \tau_1^2$	$v_s \tau_1 + \frac{1}{6} J_i \tau_1^3$
τ_2	0	A_i	$v_1 + A_i \tau_2$	$s_{a1} + v_1 \tau_2 + \frac{1}{2} A_i \tau_2^2$
τ_3	$-J_i$	$A_i - J_i \tau_3$	$v_2 + A_i \tau_3 - \frac{1}{2} J_i \tau_3^2$	$s_{a2} + v_2 \tau_3 + \frac{1}{2} A_i \tau_3^2 - \frac{1}{6} J_i \tau_3^3$
τ_4	0	0	v_3	$s_a + v_3 \tau_4$
τ_5	$-J_i$	$-J_i \tau_5$	$v_4 - \frac{1}{2} J_i \tau_5^2$	$s_a + s_c + v_4 \tau_5 - \frac{1}{6} J_i \tau_5^3$
τ_6	0	$-A_i$	$v_5 - A_i \tau_6$	$s_a + s_c + s_{d1} + v_5 \tau_6 - \frac{1}{2} A_i \tau_6^2$
τ_7	J_i	$-A_i + J_i \tau_7$	$v_6 - A_i \tau_7 + \frac{1}{2} J_i \tau_7^2$	$s_a + s_c + s_{d2} + v_6 \tau_7 - \frac{1}{2} A_i \tau_7^2 + \frac{1}{6} J_i \tau_7^3$

the main concern is to construct an S-shape feed rate profile through the NURBS block feature data where $C(u_i)$ and $C(u_{i+1})$ are the start/end points, v_s and v_e are the corresponding start/end feed rates and s_i is the arc length of the i th NURBS block. The S-shape ACC/DEC profile, which consists of seven phases, is shown in Fig. 5. The expression of the feed

rate profile in the seven phases is summarized in Table 1. The symbols $\tau_i (i = 1, 2, \dots, 7)$ denote the relative time parameters as illustrated in Fig. 5.

The durations $T_k (k = 1, 2, \dots, 7)$ of the seven phases are calculated by the start/end feed rates v_s/v_e , the command feed rate v_{max} , the tangential acceleration/jerk A_t/J_t and the desired arc length s_i , as follows:

$$\begin{cases} T_1(v_s, v_e, A_t, J_t) = \frac{A_t}{J_t} \\ T_2(v_s, v_e, A_t, J_t) = \frac{v_{max} - v_s}{A_t} - \frac{A_t}{J_t} \\ T_3(v_s, v_e, A_t, J_t) = \frac{A_t}{J_t} \\ T_4(v_s, v_e, A_t, J_t) = \frac{(s_i - s_a - s_d)}{v_{max}} \\ T_5(v_s, v_e, A_t, J_t) = \frac{A_t}{J_t} \\ T_6(v_s, v_e, A_t, J_t) = \frac{v_{max} - v_e}{A_t} - \frac{A_t}{J_t} \\ T_7(v_s, v_e, A_t, J_t) = \frac{A_t}{J_t} \end{cases} \quad (16)$$

However, the durations might be zero for a certain phase or several phases. So, the feed rate profiles are classified into 17 types as shown in Fig. 6. For the S-shape feed rate scheduling, the first step is to determine the type of the feed rate profile. A criterion arc length s_{cri}^s of the NURBS block, is given as follows: if $v_s < v_e$, s_{cri}^s is equal to the acceleration arc length $s_a(v_s, v_e)$ which is computed by

$$s_a(v_s, v_e) = \begin{cases} (v_s + v_e) \sqrt{\frac{v_e - v_s}{J_t}} & v_e - v_s \leq \frac{A_t^2}{J_t} \\ \frac{1}{2}(v_s + v_e) \left[\frac{A_t}{J_t} + \frac{(v_e - v_s)}{A_t} \right] & v_e - v_s > \frac{A_t^2}{J_t} \end{cases} \quad (17)$$

If $v_s > v_e$, s_{cri}^s is equal to the deceleration arc length $s_d(v_s, v_e)$ which is computed by

$$s_d(v_s, v_e) = \begin{cases} (v_s + v_e) \sqrt{\frac{v_s - v_e}{J_t}} & v_s - v_e \leq \frac{A_t^2}{J_t} \\ \frac{1}{2}(v_s + v_e) \left[\frac{A_t}{J_t} + \frac{(v_s - v_e)}{A_t} \right] & v_s - v_e > \frac{A_t^2}{J_t} \end{cases} \quad (18)$$

If the desired arc length s_i is equal to the criterion arc length s_{cri}^s , the feed rate profile belongs to one of Fig. 6(10)–(13) and the corresponding NURBS block is called a short block. Otherwise, the NURBS block is called as a medium or long block. To further check the block type, a criterion arc length s_{cri}^l of the NURBS block is defined as

$$s_{cri}^l = s_a(v_s, v_{max}) + s_d(v_{max}, v_e) \quad (19)$$

where the acceleration and deceleration arc lengths $s_a(v_s, v_{max})$ and $s_d(v_{max}, v_e)$ are calculated using Eqs. (17) and (18). If the desired arc length is larger than the criterion arc length s_{cri}^l , the corresponding NURBS block is called a long block whose feed rate profile belongs to those shown in Fig. 6(1)–(9). Otherwise,

the NURBS block is called as a medium block whose feed rate profile falls into those shown in Fig. 6(14)–(17). Three feed rate scheduling algorithms are proposed as follows to generate the feed rate profiles for the three kinds of blocks.

Algorithm 3.1. Feed rate scheduling for a short block.

Input: The start/end feed rates v_s/v_e .

Output: The durations $T_k (k = 1, 2, \dots, 7)$ for the S-shape feed rate scheduling.

- (1) If $v_s < v_e$ and $v_e - v_s \leq A_t^2/J_t$, then the feed rate profile is shown in Fig. 6(10) and the durations $T_k (k = 1, 3)$ are computed using Eq. (16).
- (2) If $v_s < v_e$ and $v_e - v_s > A_t^2/J_t$, then the feed rate profile is shown in Fig. 6(11) and the durations $T_k (k = 1, 2, 3)$ are computed using Eq. (16).
- (3) If $v_s > v_e$ and $v_s - v_e \leq A_t^2/J_t$, then the feed rate profile is shown in Fig. 6(12) and the durations $T_k (k = 5, 7)$ are computed using Eq. (16).
- (4) If $v_s > v_e$ and $v_s - v_e > A_t^2/J_t$, then the feed rate profile is shown in Fig. 6(13) and the durations $T_k (k = 5, 6, 7)$ are computed using Eq. (16).

Algorithm 3.2. Feed rate scheduling for a long block.

Input: The start/end feed rate v_s/v_e , the command feed rate v_{max} and the desired arc length s_i .

Output: The durations $T_k (k = 1, 2, \dots, 7)$ for the S-shape feed rate scheduling.

- (1) Compute the durations $T_k (k = 1, 2, 3)$ and $T_k (k = 5, 6, 7)$ using Algorithm 3.1.
- (2) Compute the acceleration and deceleration arc length $s_a(v_s, v_{max})$ and $s_d(v_{max}, v_e)$ using Eqs. (17) and (18).
- (3) Compute the constant feed rate duration T_4 using Eq. (16).

Algorithm 3.3. Feed rate scheduling for a medium block.

The maximum feed rate v is searched with bisection method, where $Arr = [v_0, v_1]$ represents an array to store temporary feed rates and the initial values of v_0/v_1 are set to $\max(v_s, v_e)/v_{max}$.

Input: The start/end feed rate v_s/v_e , the command feed rate v_{max} , the desired arc length s_i and the searching precision ϵ .

Output: The durations $T_k (k = 1, 2, \dots, 7)$ for the S-shape feed rate scheduling.

- (1) Let $v = \frac{(v_0 + v_1)}{2}$.
- (2) Calculate $s_a(v_s, v)$ and $s_d(v, v_e)$ using Eqs. (17) and (18).
- (3) Compare the arc length.
 - (2a) If $s_a(v_s, v) + s_d(v, v_e) - s_i > \epsilon$, then the maximum feed rate v is decreased to $\begin{cases} v_1 = v \\ v = \frac{v_0 + v}{2} \end{cases}$ (20)

And go to (2).

- (2b) If $s_a(v_s, v) + s_d(v, v_e) - s_i < -\epsilon$, then the maximum feed rate v is increased to $\begin{cases} v_0 = v \\ v = \frac{v + v_1}{2} \end{cases}$ (21)

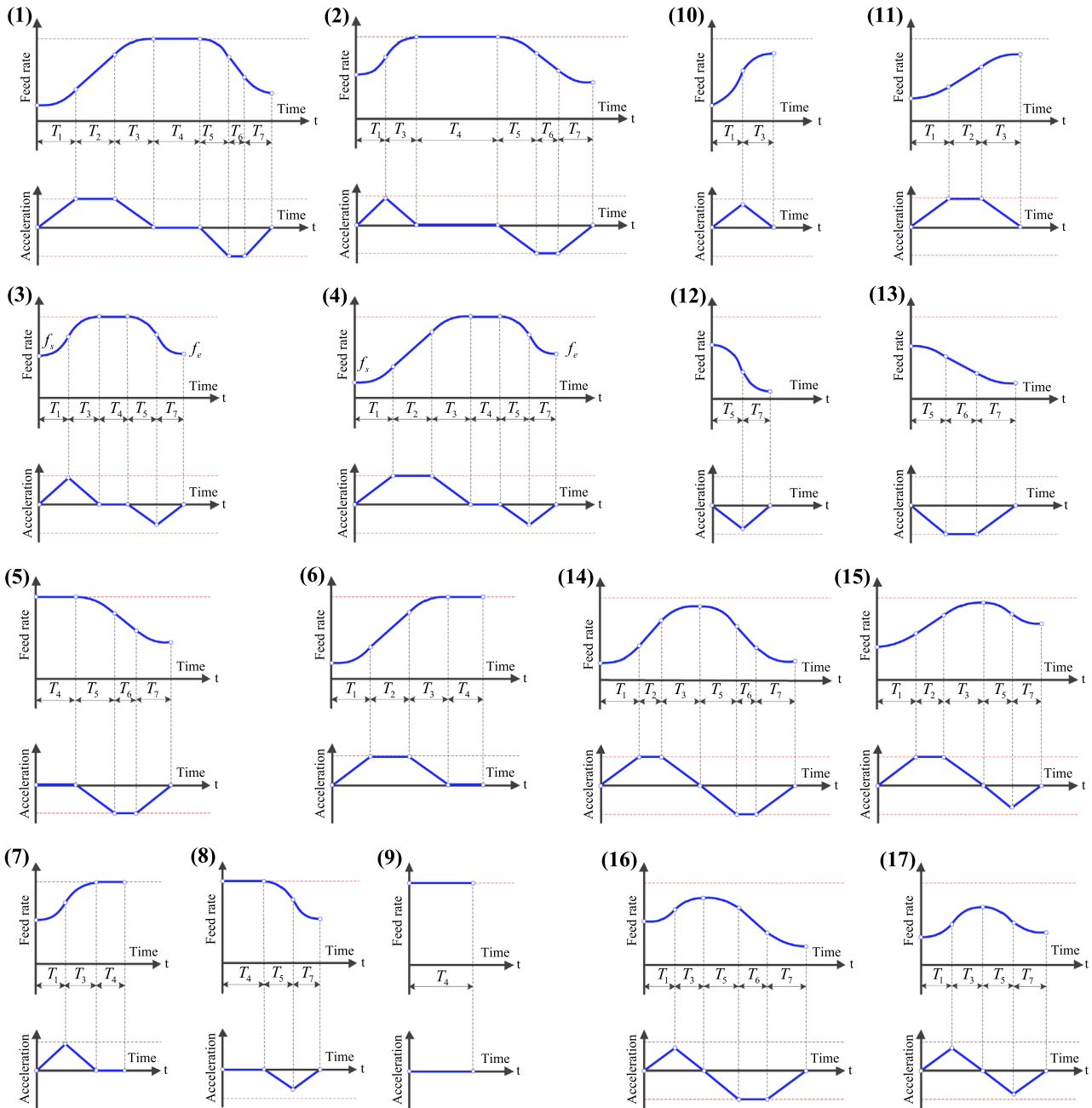


Fig. 6. Seventeen types of feed rate profiles.

- (2c) If $|s_d(v_s, v) + s_d(v, v_e) - s_i| \leq \varepsilon$, then $v_{max} = v$ and the bisection algorithm terminates. And go to (4).
- (4) Compute the durations $T_k (k = 1, 2, 3)$ and $T_k (k = 4, 5, 6)$ using Algorithm 3.1.

3.4. Variable-jerk compensation

After the feed rate scheduling in Section 3.3, the duration of each phase should be expressed by the integer multiples of the sampling period T_s . The acceleration feed rate profile as shown

in Fig. 6(10) is taken as an example.

$$\begin{cases} v_1 = v_s + \frac{1}{2}J_t T_1^2 \\ v_2 = v_1 \\ v_e = v_2 + \left[J_t T_1 - \frac{1}{2}J_t T_3 \right] T_3 \end{cases} \quad (22)$$

Since the absolute value of the jerk $|J_t|$ in the first phase is equal to that $| -J_t |$ in the third phase, the duration T_1 is equal to the duration T_3 . The number of interpolation steps the first

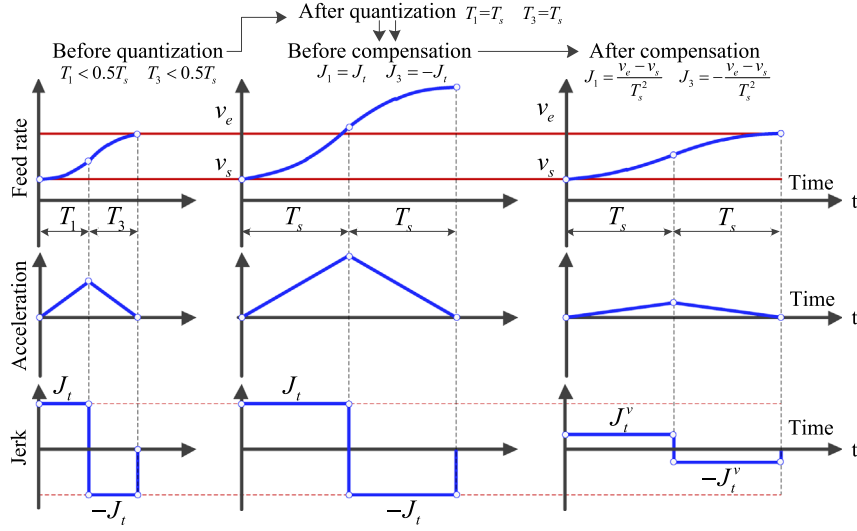


Fig. 7. Variable-jerk compensation.

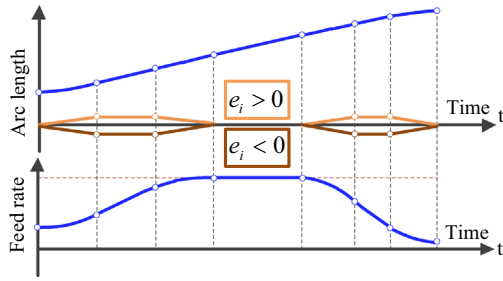


Fig. 8. Arc length error compensation.

Table 2
Parameters of S-shape feed rate scheduling.

Parameters	Symbol	Units
Command feed rate	v_{max}	100 mm/s
Normal acceleration	A_n	1000 mm/s ²
Normal jerk	J_n	30,000 mm/s ³
Tangential acceleration	A_t	1000 mm/s ²
Tangential jerk	J_t	30,000 mm/s ³
Chord error	δ	5 μ m
Sampling period	T_s	1 ms
Searching precision	ϵ	0.4 mm

phase is quantized as follows:

$$\begin{cases} N_{a1} = \text{round}\left(\frac{T_1}{T_s}\right) \\ N_{a3} = \text{round}\left(\frac{T_3}{T_s}\right) \end{cases} \quad (23)$$

If N_{a1} and N_{a3} are zeros for corresponding nonzero duration T_1 and T_3 , the feed rate scheduling failed for the current NURBS block. Therefore, a variable-jerk compensation strategy, as shown in Fig. 7, where T_1 and T_3 are set to T_s , is proposed to make the feed rate continuous among adjacent NURBS blocks. Substituting v_s , v_e , T_1 and T_3 into Eq. (22) to calculate the suitable jerk, yielding

$$J_t^v = \frac{v_e - v_s}{T_s^2} \quad (24)$$

where the modified jerk J_t^v is only adapted to the current NURBS block.

3.5. Arc length error compensation

After the feed rate scheduling, there is round-off error e_i which is defined as the difference between the desired arc length and the sum of the arc length increment calculated by the feed rate and the

sampling period. It is calculated via the following formula [15]:

$$e_i = s_i - (s_a + s_c + s_d) \quad (25)$$

where s_a , s_c and s_d are the acceleration/constant/deceleration arc lengths, respectively. The acceleration arc length is calculated by

$$\begin{cases} v_1 = v_s + \frac{1}{2}J_t(N_{a1}T_s)^2 \\ v_2 = v_1 + A_t(N_{a2}T_s) \\ v_3 = v_2 + A_t(N_{a3}T_s) - \frac{1}{2}J_t(N_{a3}T_s)^2 \end{cases} \quad (26)$$

$$\begin{cases} s_{a1} = v_s(N_{a1}T_s) + \frac{1}{6}J_t(N_{a1}T_s)^3 \\ s_{a2} = s_{a1} + v_1(N_{a2}T_s) + \frac{1}{2}A_t(N_{a2}T_s)^2 \\ s_a = s_{a2} + v_2(N_{a3}T_s) + \frac{1}{2}A_t(N_{a3}T_s)^2 - \frac{1}{6}J_t(N_{a3}T_s)^3 \end{cases} \quad (27)$$

where N_{a1} , N_{a2} and N_{a3} are the number of interpolation steps after the quantization. The deceleration arc length is obtained by replacing v_s , N_{a1} , N_{a2} and N_{a3} with v_e , N_{d3} , N_{d2} and N_{d1} in Eqs. (26) and (27), respectively. The constant feed rate arc length is calculated by

$$s_c = N_c v_{max} \quad (28)$$

where N_c is the number of interpolation steps for the constant feed rate phase.

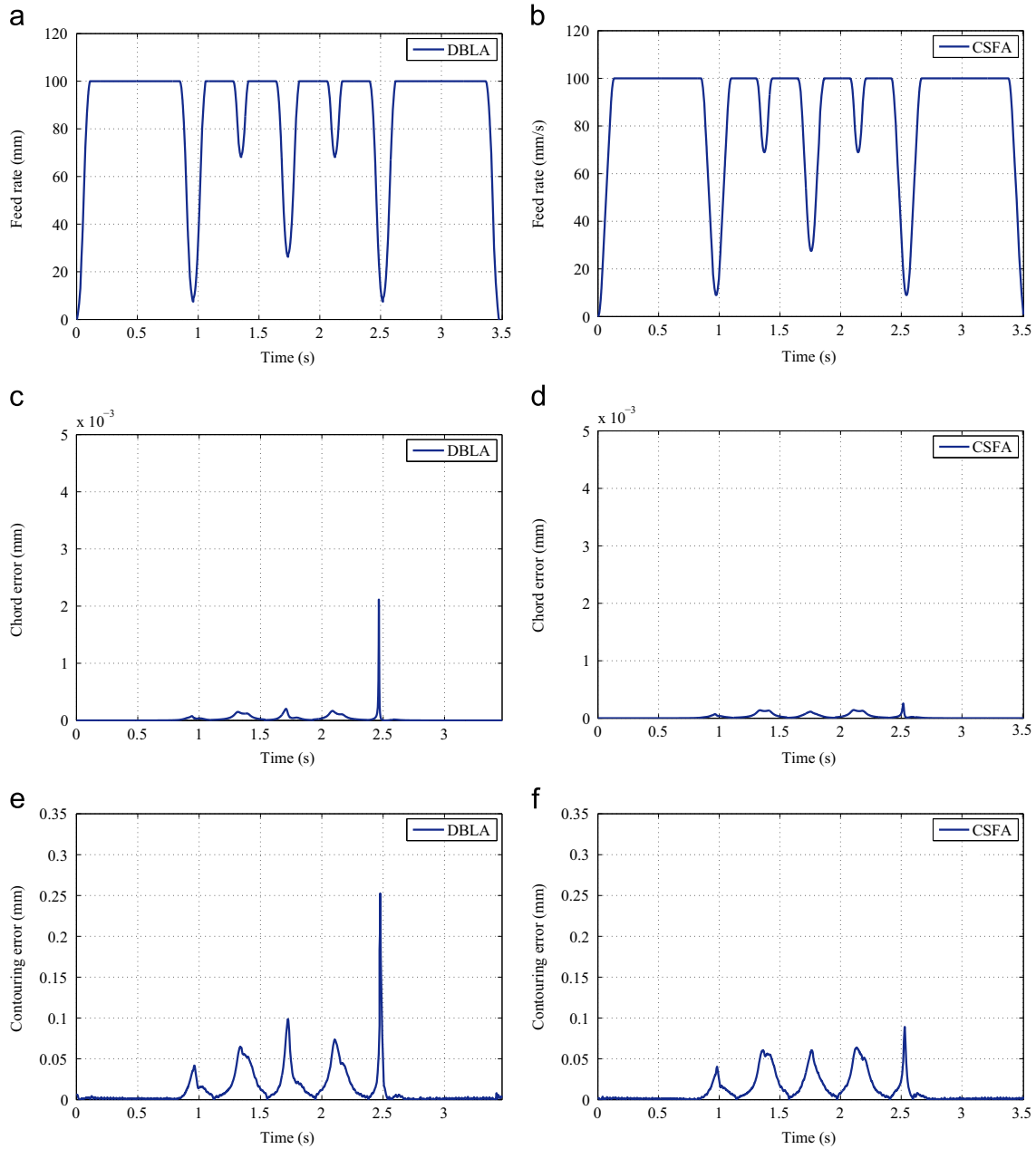


Fig. 9. Trident curve for experimental validation. (a) Feed rate with DBLA, (b) feed rate with CSFA, (c) chord error with DBLA, (d) chord error with CSFA, (e) contouring error with DBLA, (f) contouring error with CSFA.

In order to improve the accuracy of interpolation, an arc length error compensation strategy is proposed. The acceleration feed rate profiles shown in Fig. 6(10)-(11) are taken as examples. If the constant acceleration phase does not exist, round-off error is split into $N_{a1} + N_{a3}$ sections, i.e.

$$0 \underbrace{\Delta e_i \ 2\Delta e_i \cdots (N_{a1} - 1)\Delta e_i}_{N_{a1}} \underbrace{(N_{a3} - 1)\Delta e_i \cdots 2\Delta e_i \Delta e_i 0}_{N_{a3}} \quad (29)$$

Due to $T_1 = T_3$ presented in Section 3.4, so $N_{a1} = N_{a3}$ and ${}^a e_i$ is computed by

$$\Delta e_i = \frac{e_i}{N_{a1}(N_{a1} - 1)} \quad (30)$$

The arc length increment ${}^a s_j$ during every interpolation period can be modified by

$$\Delta s'_j = \begin{cases} \Delta s_j + (j - 1)\Delta e_i & \text{if } j \leq N_{a1} \\ \Delta s_j + (2N_{a1} - j)\Delta e_i & \text{if } N_{a1} + 1 \leq j < 2N_{a1} \end{cases} \quad (31)$$

When there is the constant acceleration in the feed rate profile, the compensation strategy above is expanded and the arc length error is split into $N_{a1} + N_{a2} + N_{a3}$ sections, i.e.

$$0 \underbrace{\Delta e_i 2\Delta e_i \cdots (N_{a1} - 1)\Delta e_i}_{N_{a1}} \underbrace{N_{a1}\Delta e_i N_{a1}\Delta e_i N_{a1}\Delta e_i}_{N_{a2}} \underbrace{(N_{a3} - 1)\Delta e_i \cdots 2\Delta e_i \Delta e_i 0}_{N_{a3}} \quad (32)$$

Hence, Δe_i is computed by

$$\Delta e_i = \frac{e_i}{N_{a1}(N_{a1} - 1) + N_{a1}N_{a2}} \quad (33)$$

Table 3
Experimental results for NURBS curves.

NURBS curve	Method	Machining time (s)	RMS of contouring error (mm)
Trident curve	DBLA	3.469	0.0167
	CSFA	3.507	0.0135
Butterfly curve	DBLA	5.397	0.0369
	CSFA	5.547	0.0334

The arc length increment Δs_j during every interpolation period can be modified by

$$\Delta s'_j = \begin{cases} \Delta s_j + (j-1)\Delta e_i & \text{if } j \leq N_{a1} \\ \Delta s_j + N_{a1}\Delta e_i & \text{if } N_{a1} + 1 \leq j < N_{a1} + N_{a2} \\ \Delta s_j + (2N_{a1} + N_{a2} - j)\Delta e_i & \text{if } N_{a1} + N_{a2} + 1 \leq j < 2N_{a1} + N_{a2} \end{cases} \quad (34)$$

Since the feed rate for the constant feed rate phase is the command feed rate, the arc length error compensation is applied to the acceleration/deceleration to avoid feed rate jump at high feed rate region, as shown in Fig. 8.

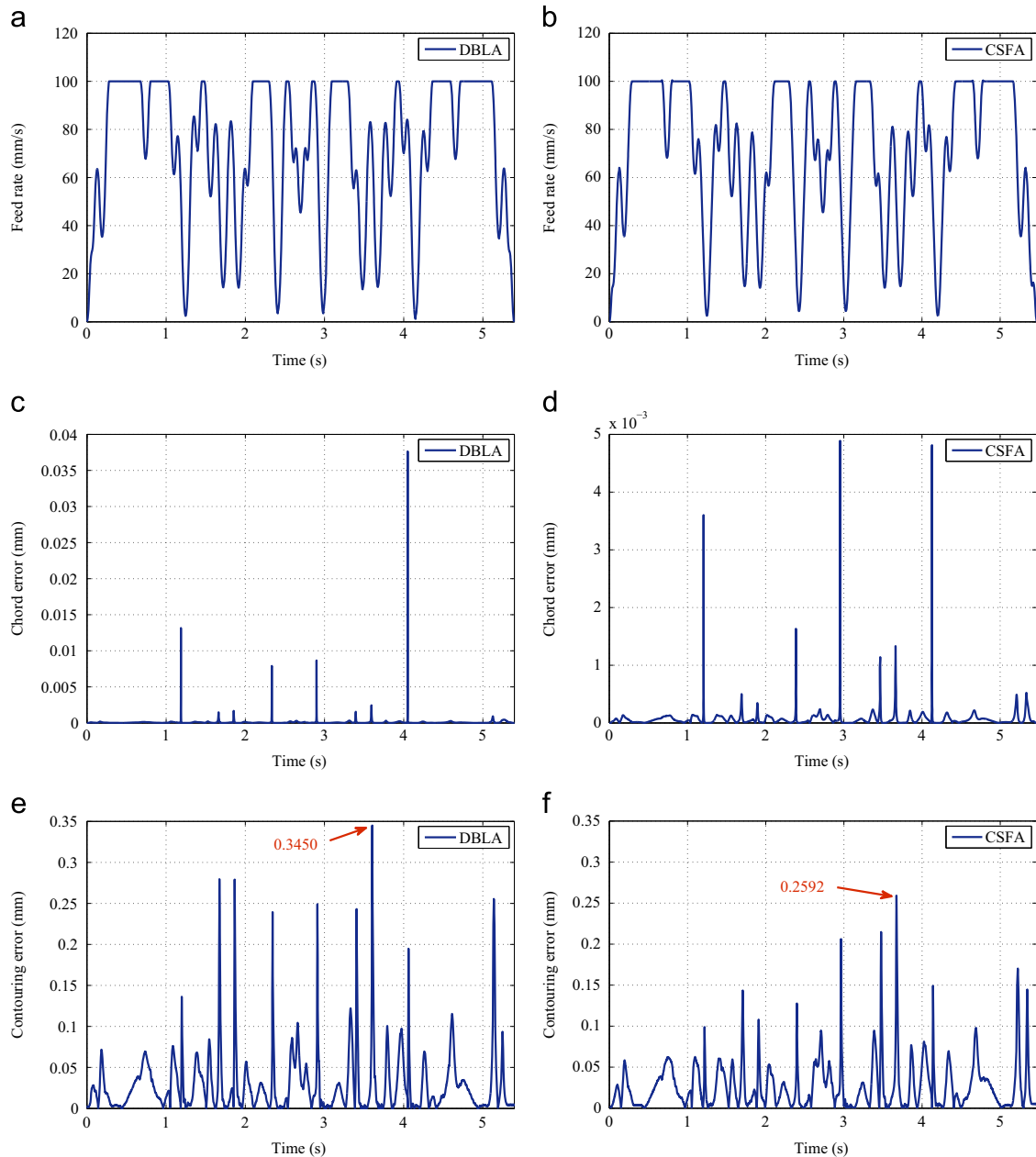


Fig. 10. Butterfly curve for experimental validation. (a) Feed rate with DBLA, (b) feed rate with CSFA, (c) chord error with DBLA, (d) chord error with CSFA, (e) contouring error with DBLA, (f) contouring error with CSFA.

3.6. Interpolation algorithm

After the arc length error compensation, the interpolation task for NURBS curve needs to be completed to output the position commands. The second-order Taylor's expansion of the spline parameter with respect to the arc-length is adopted, which is given as [16]

$$u(s_{j+1}) = u_{s_j} + \frac{du}{ds} \Big|_{s=s_j} \Delta s_j + \frac{1}{2} \frac{d^2u}{ds^2} \Big|_{s=s_j} \Delta s_j^2 \quad (35)$$

The arc length increment Δs_j is replaced by the modified arc length increment $\Delta s'_j$ in Section 3.5 to generate more accurate position commands.

4. Implementation and experimental results

In this section, experiments are conducted on a three-axis CNC machine with YASKAWA SGD series servo drives and SGMJV motors. The position of the feed drive system is measured by the servo motor photoelectric encoder attached to each feed drive axis, whose resolution is set to 2500 pulse/rev. This complete S-shape feed rate scheduling approach is implemented on MATLAB. The trident and butterfly curves are tested under the command feed rate of, and the other parameters are listed in Table 2.

Comparative analysis between the present CSFA and the dynamic-based interpolator with real-time look-ahead algorithm (DBLA) [11] are made to verify the effectiveness of the new approach. Experimental validation for the trident curve is shown in Fig. 9. The statistical data summarized in Table 3, the real machining times for performing the CSFA and the DBLA are 3.507 s and 3.469 s, respectively. The main reason for causing shorter machining time for performing DBLA is that the tangential acceleration is violated. The maximum contouring errors and the RMS of the contouring errors obtained by the CSFA is 0.0892 mm and 0.0135 mm which are much smaller than 0.2526 mm and 0.0167 mm, respectively. For the butterfly curve, the real machining times for performing the CSFA and the DBLA are 5.467 s and 5.397 s, respectively. The maximum contouring errors and the RMS of the contouring errors obtained by the CSFA are 0.2592 mm and 0.0334 mm which are much smaller than 0.3450 mm and 0.0369 mm, respectively. The different experiments for the trident curve and the butterfly curve have the consistent results, i.e. the CSFA has better stability compared with the DBLA (Fig. 10).

5. Conclusions

In this paper, a complete S-shape feed rate scheduling approach is proposed for NURBS curve. Firstly, a bidirectional scanning algorithm is designed to guarantee the continuity of the feed rate at the junctions between successive NURBS blocks. The result shows that the feed rate continuities are guaranteed with the limitations. Then, the NURBS blocks are classified into three types (i.e., the short blocks, the long blocks and the medium blocks) according to

two threshold arc length, which makes the scheduling method easy for implementation. Secondly, a variable-jerk compensation strategy and an arc length error compensation strategy are developed in the feed rate scheduling to eliminate the round-off error of the duration quantization. The effectiveness of the proposed approach is verified by a series of comparative experiments conducted on a three-axis stage with different NURBS curves.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grant no. 51325502, and the Science & Technology Commission of Shanghai Municipality under Grant nos. 13JC1408400 and 14111104801.

References

- [1] Fagali de Souza Adriano, Teixeira Coelho Reginaldo. Experimental investigation of feed rate limitations on high speed milling aimed at industrial applications. *International Journal of Manufacture Technology* 2007;**32**:1104–14.
- [2] Bollinger John G, Duffie Neil A. *Computer Control of Machines and Processes*. Inc. Boston, MA, USA: Addison-Wesley Longman Publishing Co; 1988.
- [3] Lee An-Chen, Lin Ming-Tzong, Pan Yi-Ren, Pan WenYi-Ren, Lin Wen-Yu. The feed rate scheduling of NURBS interpolator for CNC machine tools. *Computer-Aided Design* 2011;**43**:612–28.
- [4] Dong J, Stori JA. Optimal feed rate scheduling for high-speed contouring. *Journal of Manufacturing Science and Engineering* 2004;**129**(1):612–28.
- [5] Beudaert Xavier, Lavernhe Sylvain, Tournier Christophe. Feed rate interpolation with axis jerk constraints on 5-axis NURBS and G1 tool path. *International Journal of Machine Tools and Manufacture* 2012;**57**: 73–82.
- [6] Lai Jiing-Yih, Lin Kuan-Yuan, Tseng Sheng-Jung, Ueng Wen-Der. On the development of a parametric interpolator with confined chord error, feed rate, acceleration and jerk. *International Journal of Advanced Manufacture Technology* 2008;**37**:104–21.
- [7] Nam Sung-Ho, Yang Min-Yang. A study on a generalized parametric interpolator with real-time jerk-limited acceleration. *Computer-Aided Design* 2004;**36**(1):27–36.
- [8] Erkorkmaz K, Altintas Y. High speed CNC system design. Part I: *jerk limited trajectory generation and quantice spline interpolation*. *International Journal of Machine Tools and Manufacture* 2001;**41**:1323–45.
- [9] Li Yuyao, Feng Jingchun, Wang Yuhan, Yang Jianguo. Variable-period feed interpolation algorithm for high-speed five-axis machining. *International Journal of Manufacture Technology* 2009;**40**:769–75.
- [10] Park Jinho, Nam Sungho, Yang Minyang. Development of a real-time trajectory generator for NURBS interpolation method. *The International Journal of Advanced Manufacturing Technology* 2005;**26**(4):359–65.
- [11] Lin Ming-Tzong, Tsai Meng-Shiun, Yau Hong-Tzong. Development of a dynamics-based NURBS interpolator with real-time look-ahead algorithm. *International Journal of Machine Tools and Manufacture* 2007;**47**: 2246–62.
- [12] Les Piegl. *Tiller Wayne. The NURBS Books*, Second, Berlin, Heidelberg: Springer; 1997.
- [13] Ding Han, Zhu LiMin. *Geometric Theories and Methods For Digital Manufacturing of Complex Surfaces*, First, BeiJing: Science Press; 2011.

- [14] Lei WT, Sung MP, Lin LY, Huang JJ. Fast real-time NURBS path interpolation for CNC machine tools. *International Journal of Machine Tools and Manufacture* 2007;**47**(10):1530–41.
- [15] Fu-Yuan Luo, Yun-Fei Zhou, Juan-Yin. A universal velocity profile generation approach for high-speed machining of small line segments with look-ahead. *International Journal Advanced Manufacture Technology* 2007;**35** 505–518.
- [16] Zhao Huan, Zhu Li-Min, Ding Han. A real-time look-ahead interpolation methodology with curvature-continuous B-spline transition scheme for CNC machining of short line segment. *International Journal of Machine Tools and Manufacture* 2013;**65**:88–98.