AN ANALOGUE OF THE HILTON-MILNER THEOREM FOR WEAK COMPOSITIONS

Cheng Yeaw Ku and Kok Bin Wong

ABSTRACT. Let $\mathbb{N}_0$ be the set of non-negative integers, and let $P(n, l)$ denote the set of all weak compositions of n with l parts, i.e., $P(n, l) = \{(x_1, x_2, \ldots, x_l) \in \mathbb{N}_0^l \ : \ x_1 + x_2 + \cdots + x_l = n\}$. For any element $\mathbf{u} = (u_1, u_2, \ldots, u_l) \in P(n, l)$, denote its ith-coordinate by $\mathbf{u}(i)$, i.e., $\mathbf{u}(i) = u_i$. A family $\mathcal{A} \subseteq P(n, l)$ is said to be t-intersecting if $|\{i : \mathbf{u}(i) = \mathbf{v}(i)\}| \geq t$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{A}$. A family $\mathcal{A} \subseteq P(n, l)$ is said to be trivially t-intersecting if there is a t-set T of $[l] = \{1, 2, \ldots, l\}$ and elements $y_s \in \mathbb{N}_0$ ($s \in T$) such that $\mathcal{A} = \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = y_j \text{ for all } j \in T\}$. We prove that given any positive integers l, t with $l \geq 2t + 3$, there exists a constant $n_0(l, t)$ depending only on l and t, such that for all $n \geq n_0(l, t)$, if $\mathcal{A} \subseteq P(n, l)$ is non-trivially t-intersecting, then

$$|\mathcal{A}| \leq \binom{n + l - t - 1}{l - t - 1} - \binom{n - 1}{l - t - 1} + t.$$

Moreover, equality holds if and only if there is a t-set T of $[l]$ such that

$$\mathcal{A} = \bigcup_{s \in [l] \setminus T} \mathcal{A}_s \cup \{\mathbf{q}_i : i \in T\},$$

where

$$\mathcal{A}_s = \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = 0 \text{ for all } j \in T \text{ and } \mathbf{u}(s) = 0\}$$

and $\mathbf{q}_i \in P(n, l)$ with $\mathbf{q}_i(j) = 0$ for all $j \in [l] \setminus \{i\}$ and $\mathbf{q}_i(i) = n$.

1. Introduction

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [14], Frankl [15], Wilson [43])**.** *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geq (k - t + 1)(t + 1)$, we*

have

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, if $n > (k-t+1)(t+1)$, then equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some t-set T.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [3, 17, 25, 29, 37, 39, 40, 41] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [38]. A complete solution for the t-intersection problem in the Hamming space is given in [2]. Intersecting families of permutations were initiated by Deza and Frankl in [10]. Some recent work done on this problem and its variants can be found in [5, 7, 8, 11, 12, 19, 26, 33, 34, 35, 36, 42]. The investigation of the Erdős-Ko-Rado property for graphs started in [23], and gave rise to [4, 6, 21, 22, 24, 44]. The Erdős-Ko-Rado type results also appear in vector spaces [9, 18], set partitions [27, 28, 31] and weak compositions [30, 32].

For a family $\mathcal{A}$ of k-subsets, $\mathcal{A}$ is said to be *trivially t-intersecting* if there exists a t-set $T = \{x_1, \ldots, x_t\}$ such that all members of $\mathcal{A}$ contain T. The Erdős-Ko-Rado theorem implies that a t-intersecting family of maximum size must be trivially t-intersecting when n is sufficiently large in terms of k and t.

Hilton and Milner [20] proved a strengthening of the Erdős-Ko-Rado theorem for $t = 1$ by determining the maximum size of a non-trivial 1-intersecting family. A short and elegant proof was later given by Frankl and Füredi [16] using the shifting technique.

Theorem 1.2 (Hilton-Milner). *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be a non-trivial 1-intersecting family with $k \geq 4$ and $n > 2k$. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality holds if and only if

$$\mathcal{A} = \left\{ X \in \binom{[n]}{k} : x \in X, X \cap Y \neq \emptyset \right\} \cup \{Y\}$$

for some k-subset $Y \in \binom{[n]}{k}$ and $x \in X \setminus Y$.

In this paper, we prove an analogue of Theorem 1.2 for weak compositions with fixed number of parts. Let $\mathbb{N}_0$ be the set of non-negative integers, and let $P(n, l)$ denote the set of all weak compositions of n with l parts, i.e., $P(n, l) = \{(x_1, x_2, \ldots, x_l) \in \mathbb{N}_0^l : x_1 + x_2 + \cdots + x_l = n\}$. Recall that

$$|P(n, l)| = \binom{n+l-1}{l-1}.$$

For any element $\mathbf{u} = (u_1, u_2, \ldots, u_l) \in P(n, l)$, denote its ith-coordinate by $\mathbf{u}(i)$, i.e., $\mathbf{u}(i) = u_i$. A family $\mathcal{A} \subseteq P(n, l)$ is said to be t-intersecting if $|\{i : \mathbf{u}(i) = \mathbf{v}(i)\}| \geq t$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{A}$.

A family $\mathcal{A} \subseteq P(n, l)$ is said to be *trivially* t-intersecting if there is a t-set T of $[l]$ and elements $y_s \in \mathbb{N}_0$ $(s \in T)$ such that $\mathcal{A} = \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = y_j$ for all $j \in T\}$. If $y_j = 0$ for all $j \in T$, then $\mathcal{A}$ is said to be *strong-trivially* t-intersecting. It has been shown that a t-intersecting family of maximum size must be strong-trivially t-intersecting when n is sufficiently large in terms of l and t [30, Theorem 1.2]. Similar problems have been considered by Engel and Frankl in [13] for t-intersecting families in $E_l(k, n)$, where the roles of l and n are interchanged from the notation in this paper. Although the case $k = l$ corresponds to $P(n, l)$, they did not address this case in their paper.

Our main result is the following.

Theorem 1.3. *Given any positive integers l, t with $l \geq 2t + 3$, there exists a constant $n_0(l, t)$ depending only on l and t, such that for all $n \geq n_0(l, t)$, if $\mathcal{A} \subseteq P(n, l)$ is non-trivially t-intersecting, then*

$$|\mathcal{A}| \leq \binom{n + l - t - 1}{l - t - 1} - \binom{n - 1}{l - t - 1} + t.$$

Moreover, equality holds if and only if there is a t-set T of $[l]$ such that

$$\mathcal{A} = \bigcup_{s \in [l] \setminus T} \mathcal{A}_s \cup \{\mathbf{q}_i : i \in T\},$$

where

$$\mathcal{A}_s = \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = 0 \text{ for all } j \in T \text{ and } \mathbf{u}(s) = 0\}$$

and $\mathbf{q}_i \in P(n, l)$ with $\mathbf{q}_i(j) = 0$ for all $j \in [l] \setminus \{i\}$ and $\mathbf{q}_i(i) = n$.

Note that

$$\bigcup_{s \in [l] \setminus T} \mathcal{A}_s = \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = 0 \text{ for all } j \in T\} \setminus \{\mathbf{u} \in P(n, l) : \mathbf{u}(j) = 0$$

for all $j \in T$ and $\mathbf{u}(s) \neq 0$ for all $s \in [l] \setminus T\}$.

Therefore,

$$\left| \bigcup_{s \in [l] \setminus T} \mathcal{A}_s \right| = \binom{n + l - t - 1}{l - t - 1} - \binom{n - 1}{l - t - 1}.$$

2. Certain intersecting conditions

The aim of this section is to prove Corollary 2.6 which will be used later in the proof of the main result. We first need some technical lemmas.

Lemma 2.1. *Let m be a positive integer, $r \geq 3$, and $k \in [r - 2]$. Let $y_1, y_2, \ldots, y_k \in \mathbb{N}_0$ and*

$$\mathcal{C} = \{\mathbf{u} \in P(m, r) : \mathbf{u}(i) = y_i \text{ for all } i \in [k - 1] \text{ and } \mathbf{u}(k) = 0\},$$

$$\mathcal{D} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = y_k\}.$$

Then

(a) $\mathcal{D} = \mathcal{C}$ *if* $m - \sum_{i=1}^{k-1} y_i < 0$ *or* $y_k = 0$;

(b) $|\mathcal{D}| < |\mathcal{C}|$ *if* $m - \sum_{i=1}^{k-1} y_i \geq 0$ *and* $y_k > 0$.

Proof. If $y_k = 0$, then $\mathcal{D} = \mathcal{C}$. If $m - \sum_{i=1}^{k-1} y_i < 0$, then $\mathcal{D} = \mathcal{C} = \varnothing$. Suppose $m - \sum_{i=1}^{k-1} y_i \geq 0$ and $y_k > 0$. Note that

$$|\mathcal{C}| = \binom{m - \sum_{i=1}^{k-1} y_i + r - k - 1}{r - k - 1}.$$

If $m - \sum_{i=1}^{k} y_i < 0$, then $\mathcal{D} = \varnothing$. So, $|\mathcal{D}| < |\mathcal{C}|$. If $m - \sum_{i=1}^{k} y_i \geq 0$, then

$$
\begin{aligned}
|\mathcal{D}| &= \binom{m - \sum_{i=1}^{k} y_i + r - k - 1}{r - k - 1} \\
&< \binom{m - \sum_{i=1}^{k-1} y_i + r - k - 1}{r - k - 1} = |\mathcal{C}|. \qquad \square
\end{aligned}
$$

Lemma 2.2. *Let* m *be a positive integer,* $r \geq 3$, $k \in [r]$, *and* $m \geq k$. *Let* $y_1, y_2, \ldots, y_k \in \mathbb{N}_0$ *and*

$$\mathcal{C} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = 0\},$$
$$\mathcal{D} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = y_k\}.$$

Then

(a) $\mathcal{D} = \mathcal{C}$ *if* $y_k = 0$;

(b) $|\mathcal{D}| < |\mathcal{C}|$ *if* $k \in [r-1]$ *and* $y_k > 0$;

(c) $|\mathcal{D}| \leq |\mathcal{C}|$ *if* $k = r$ *and* $y_r > 0$, $y_i = 0$ *for some* $1 \leq i \leq r-1$.

Proof. Clearly, part (a) is true. Suppose $y_k > 0$. If $k = 1$, then the result follows from Lemma 2.1. Let $k \geq 2$. Let $S_i = [m] \setminus \{y_i\}$ and set $S = S_1 \times S_2 \times \cdots \times S_{k-1}$. For each $(d_1, \ldots, d_{k-1}) \in S$, let

$$\mathcal{C}(d_1, \ldots, d_{k-1}) = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = d_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = 0\},$$
$$\mathcal{D}(d_1, \ldots, d_{k-1}) = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = d_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = y_k\}.$$

Note that

$$\mathcal{C} = \bigcup_{(d_1,\ldots,d_{k-1}) \in S} \mathcal{C}(d_1, \ldots, d_{k-1}),$$
$$\mathcal{D} = \bigcup_{(d_1,\ldots,d_{k-1}) \in S} \mathcal{D}(d_1, \ldots, d_{k-1}),$$

Furthermore,

$$\mathcal{C}(d_1, \ldots, d_{k-1}) \cap \mathcal{C}(d'_1, \ldots, d'_{k-1}) = \varnothing = \mathcal{D}(d_1, \ldots, d_{k-1}) \cap \mathcal{D}(d'_1, \ldots, d'_{k-1})$$

for all $(d_1, \ldots, d_{k-1}) \neq (d'_1, \ldots, d'_{k-1})$.

Suppose $k \in [r-2]$. By Lemma 2.1, $|\mathcal{D}(d_1, \ldots, d_{k-1})| \leq |\mathcal{C}(d_1, \ldots, d_{k-1})|$ for all $(d_1, \ldots, d_{k-1}) \in S$. It remains to show that

$$|\mathcal{D}(d_1, \ldots, d_{k-1})| < |\mathcal{C}(d_1, \ldots, d_{k-1})|$$

for at least one $(d_1, \ldots, d_{k-1}) \in S$. For each $i \in [k-1]$, set $z_i = 0$ if $y_i \neq 0$ and $z_i = 1$ if $y_i = 0$. Then $(z_1, \ldots, z_{k-1}) \in S$. Since $m - \sum_{i=1}^{k-1} z_i \geq m - (k-1) > 0$, by part (b) of Lemma 2.1, $|\mathcal{D}(z_1, \ldots, z_{k-1})| < |\mathcal{C}(z_1, \ldots, z_{k-1})|$.

Suppose $k = r - 1$. If $m - \sum_{i=1}^{r-2} d_i < 0$, then

$$\mathcal{D}(d_1, \ldots, d_{r-2}) = \mathcal{C}(d_1, \ldots, d_{r-2}) = \varnothing.$$

If $m - \sum_{i=1}^{r-2} d_i = 0$, then $|\mathcal{C}(d_1, \ldots, d_{r-2})| = 1$ and $\mathcal{D}(d_1, \ldots, d_{r-2}) = \varnothing$. If $m - \sum_{i=1}^{r-2} d_i > 0$, then $|\mathcal{C}(d_1, \ldots, d_{r-2})| = 1$ and $|\mathcal{D}(d_1, \ldots, d_{r-2})| \leq 1$. So, it remains to show that $m - \sum_{i=1}^{r-2} d_i = 0$ for at least one $(d_1, \ldots, d_{k-1}) \in S$.

Note that $m \geq k \geq 2$. Suppose $y_i \neq 0$ for all $i \in [r-2]$. If $y_{i_0} \neq m$ for some $i_0 \in [r-2]$, then set $d_i = 0$ for all $i \in [r-2] \setminus \{i_0\}$ and $d_{i_0} = m$. If $y_i = m$ for all $i \in [r-2]$, then set $d_1 = 1$, $d_2 = m - 1$, and $d_i = 0$ for all $i \in [r-2] \setminus \{1, 2\}$. Let $U \subseteq [r-2]$ and $U \neq \varnothing$. Suppose that $y_i = 0$ for all $i \in U$ and $y_i \neq 0$ for all $i \in [r-2] \setminus U$. Set $d_i = 0$ for all $i \in [r-2] \setminus U$. Let $a \in U$. Set $d_i = 1$ for all $i \in U \setminus \{a\}$ and $d_a = m - \sum_{i \in U \setminus \{a\}} d_i \geq m - (r-3) > 0$. In either case, $m - \sum_{i=1}^{r-2} d_i = 0$ and $(d_1, \ldots, d_{k-1}) \in S$. This concludes the proof of part (b).

The preceding argument does not apply to the case $k = r$. Instead, we will prove part (c) directly. Let $(d_1, \ldots, d_{r-1}, y_r) \in \mathcal{D}$. It is easy to see that the map $(d_1, \ldots, d_{r-1}, y_r) \mapsto (d_1, \ldots, d_{i-1}, d_i + y_r, d_{i+1}, \ldots, d_{r-1}, 0)$ is an injection from $\mathcal{D}$ to $\mathcal{C}$. Thus, $|\mathcal{D}| \leq |\mathcal{C}|$. $\qquad\square$

Observe that the inequality in part (c) of the preceding lemma may not be strict. For example, taking $m = 5$, $k = r = 3$ and $(y_1, y_2, y_3) = (4, 0, 1)$, we have $\mathcal{C} = \{(2, 3, 0), (3, 2, 0), (0, 5, 0), (1, 4, 0)\}$ and $\mathcal{D} = \{(2, 2, 1), (3, 1, 1), (1, 3, 1), (0, 4, 1)\}$. Furthermore, the condition that $y_i = 0$ for some $1 \leq i \leq r - 1$ is necessary. For example, taking $m = k = r = 3$ and $(y_1, y_2, y_3) = (3, 3, 1)$, we have $\mathcal{C} = \{(2, 1, 0), (1, 2, 0)\}$ and $\mathcal{D} = \{(1, 1, 1), (2, 0, 1), (0, 2, 1)\}$.

Lemma 2.3. *Let m be a positive integer, $r \geq 3$, $k \in [r]$, $k_0 \in [k]$, and $m \geq k$. Let $y_1, y_2, \ldots, y_k \in \mathbb{N}_0$ and*

$$\mathcal{C} = \{\mathbf{u} \in P(m, r) : \mathbf{u}(i) = y_i \text{ for some } i \in [k] \setminus \{k_0\} \text{ or } \mathbf{u}(k_0) = 0\},$$
$$\mathcal{D} = \{\mathbf{u} \in P(m, r) : \mathbf{u}(i) = y_i \text{ for some } i \in [k]\}.$$

Then

    (a) $\mathcal{D} = \mathcal{C}$ *if $y_{k_0} = 0$;*
    (b) $|\mathcal{D}| < |\mathcal{C}|$ *if $k \in [r-1]$ and $y_{k_0} > 0$;*
    (c) $|\mathcal{D}| \leq |\mathcal{C}|$ *if $k = r$ and $y_{k_0} > 0$, $y_i = 0$ for some $i \in [k] \setminus k_0$.*

Proof. By relabelling if necessary, we may assume that $k_0 = k$. Let

$$\mathcal{E}_0 = \{\mathbf{u} \in P(m, r) : \mathbf{u}(i) = y_i \text{ for some } i \in [k-1]\},$$

$$\mathcal{C}_1 = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = 0\},$$
$$\mathcal{D}_1 = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [k-1] \text{ and } \mathbf{u}(k) = y_k\}.$$

Then $\mathcal{C} = \mathcal{E}_0 \cup \mathcal{C}_1$ and $\mathcal{D} = \mathcal{E}_0 \cup \mathcal{D}_1$. Note that $\mathcal{E}_0 \cap \mathcal{C}_1 = \varnothing = \mathcal{E}_0 \cap \mathcal{D}_1$. The result now follows from Lemma 2.2. $\qquad\square$

Lemma 2.4. *Let $y_1, y_2, \ldots, y_r \in \mathbb{N}_0$ and m be a positive integer. Let $r \geq 3$ and*

$$\mathcal{C} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in [r-2] \text{ or } \mathbf{u}(r-1)=0 \text{ or } \mathbf{u}(r)=0\},$$
$$\mathcal{D} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in [r]\}.$$

Then $|\mathcal{D}| \leq |\mathcal{C}|$.

Proof. If one of y_{r-1} and y_r is 0, then the result follows immediately from part (a) or (c) of Lemma 2.3. So we may assume that $y_{r-1} > 0$ and $y_r > 0$. Let

$$\mathcal{C}' = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in [r-2] \text{ or } \mathbf{u}(r-1) = 0\},$$
$$\mathcal{C}'' = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [r-2] \text{ and } \mathbf{u}(r-1) \neq 0$$
$$\text{and } \mathbf{u}(r) = 0\},$$
$$\mathcal{D}' = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in [r-1]\},$$
$$\mathcal{D}'' = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) \neq y_i \text{ for all } i \in [r-2] \text{ and } \mathbf{u}(r-1) \neq y_{r-1}$$
$$\text{and } \mathbf{u}(r) = y_r\}.$$

Clearly, $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$, $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$ and $\mathcal{C}' \cap \mathcal{C}'' = \varnothing = \mathcal{D}' \cap \mathcal{D}''$. It follows from part (b) of Lemma 2.3 that $|\mathcal{D}'| < |\mathcal{C}'|$. On the other hand, the map $(d_1, \ldots, d_{r-1}, y_r) \mapsto (d_1, \ldots, d_{r-2}, d_{r-1} + y_r, 0)$ is an injection from $\mathcal{D}''$ to $\mathcal{C}''$. Hence, $|\mathcal{C}| \leq |\mathcal{D}|$. $\qquad\square$

Theorem 2.5. *Let $y_1, \ldots, y_r \in \mathbb{N}_0$ and $r \geq 3$. Let $S \subseteq [r]$, $S \neq \varnothing$, $m \geq |S|$, and*

$$\mathcal{C} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = 0 \text{ for some } i \in S\},$$
$$\mathcal{D} = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in S\}.$$

If $\sum_{s \in S} y_s > 0$, then $|\mathcal{D}| < |\mathcal{C}|$.

Proof. Without loss of generality, we may assume that $S = [k]$ for some $k \in [r]$. By relabelling if necessary, we may assume that $y_1 \geq y_2 \geq \cdots \geq y_k$. Since $\sum_{j=1}^{k} y_j > 0$, we have $y_1 > 0$. If $k = 1$, then by Lemma 2.1, the theorem holds. Suppose $k \geq 2$.

We shall distinguish two cases.

Case 1. Suppose $k \in [r-1]$. For each $j \in [k-1]$, let

$$\mathcal{F}_j = \{\mathbf{u} \in P(m,r) : \mathbf{u}(i) = y_i \text{ for some } i \in [j] \text{ or}$$
$$\mathbf{u}(i) = 0 \text{ for some } j+1 \leq i \leq k\}.$$

By Lemma 2.3, $|\mathcal{F}_1| < |\mathcal{C}|$. Again, by Lemma 2.3,

$$|\mathcal{D}| \leq |\mathcal{F}_{k-1}| \leq \cdots \leq |\mathcal{F}_2| \leq |\mathcal{F}_1|.$$

Hence, $|\mathcal{D}| < |\mathcal{C}|$.

Case 2. Suppose $k = r$. For each $j \in [r - 2]$, let

$$\mathcal{F}_j = \{\mathbf{u} \in P(m, r) : \mathbf{u}(i) = y_i \text{ for some } i \in [j] \text{ or}$$
$$\mathbf{u}(i) = 0 \text{ for some } j + 1 \leq i \leq r\}.$$

By Lemma 2.3, $|\mathcal{F}_1| < |\mathcal{C}|$. Again, by Lemma 2.3,

$$|\mathcal{F}_{r-2}| \leq \cdots \leq |\mathcal{F}_2| \leq |\mathcal{F}_1|.$$

By Lemma 2.4, $|\mathcal{D}| \leq |\mathcal{F}_{r-2}|$. Hence, $|\mathcal{D}| < |\mathcal{C}|$. $\qquad\qquad\square$

Let $\mathbf{u} = (u_1, u_2, \ldots, u_l) \in P(n, l)$. We define $R(i, \mathbf{u})$ to be the element obtained from $\mathbf{u}$ by removing the i-th coordinate, i.e.,

$$R(i; \mathbf{u}) = (u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_l).$$

Inductively, if $x_1, x_2, \ldots, x_t$ are distinct elements in $[l]$ with $x_1 < x_2 < \cdots < x_t$, we define

$$R(x_1, x_2, \ldots, x_t; \mathbf{u}) = R(x_1, x_2, \ldots, x_{t-1}; R(x_t; \mathbf{u})).$$

In other words, $R(x_1, x_2, \ldots, x_t; \mathbf{u})$ is the element obtained from $\mathbf{u}$ by removing the coordinates x_i.

Corollary 2.6. *Let* $w_1, \ldots, w_t, y_{t+1}, \ldots, y_{r+t} \in \mathbb{N}_0$, $t \geq 1$ *and* $r \geq 3$. *Let* $S \subseteq [r + t] \setminus [t]$, $S \neq \varnothing$, $m \geq |S| + \sum_{1 \leq i \leq t} w_i$, *and*

$$\mathcal{D} = \{\mathbf{u} \in P(m, r + t) : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \text{ and } \mathbf{u}(i) = y_i \text{ for some } i \in S\}.$$

If $\sum_{s \in S} y_s > 0$, *then*

$$|\mathcal{D}| < \sum_{0 \leq d \leq |S|-1} \binom{m - \sum_{1 \leq i \leq t} w_i - d + r - 2}{r - 2}.$$

Proof. Let

$$\mathcal{C} = \{\mathbf{u} \in P(m, r + t) : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \text{ and } \mathbf{u}(i) = 0 \text{ for some } i \in S\}.$$

Let $S^* = \{a - t : a \in S\}$, $y_i^* = y_{i+t}$ for all $i \in S^*$,

$$\mathcal{C}^* = \left\{\mathbf{u} \in P\left(m - \sum_{1 \leq i \leq t} w_i, r\right) : \mathbf{u}(i) = 0 \text{ for some } i \in S^*\right\},$$

$$\mathcal{D}^* = \left\{\mathbf{u} \in P\left(m - \sum_{1 \leq i \leq t} w_i, r\right) : \mathbf{u}(i) = y_i^* \text{ for some } i \in S^*\right\}.$$

By Theorem 2.5, $|\mathcal{D}^*| < |\mathcal{C}^*|$. For each $\mathbf{u} \in P(m, r + t)$, the mapping

$$\mathbf{u} \to R(1, \ldots, t; \mathbf{u}),$$

is a bijection from $\mathcal{C}$ onto $\mathcal{C}^*$ and from $\mathcal{D}$ onto $\mathcal{D}^*$. Thus, $|\mathcal{D}| < |\mathcal{C}|$.

Let
$$\mathcal{F} = \{\mathbf{u} \in P(m, r+t) : \mathbf{u}(i) = w_i \text{ for all } i \in [t]\}.$$
Then
$$\mathcal{C} = \mathcal{F} \setminus \{\mathbf{u} \in P(m, r+t) : \mathbf{u}(i) \neq 0 \text{ for all } i \in S\},$$
and
$$|\mathcal{C}| = \binom{m - \sum_{1 \leq i \leq t} w_i + r - 1}{r-1} - \binom{m - \sum_{1 \leq i \leq t} w_i - |S| + r - 1}{r-1}$$
$$= \sum_{0 \leq d \leq |S|-1} \binom{m - \sum_{1 \leq i \leq t} w_i - d + r - 2}{r-2}.$$

Hence, the corollary holds. $\qquad\square$

3. Main result

For any $\mathbf{u}, \mathbf{v} \in P(n, l)$, let $I(\mathbf{u}, \mathbf{v}) = \{i : \mathbf{u}(i) = \mathbf{v}(i)\}$. A family $\mathcal{B} \subseteq P(n, l)$ is said to be *independent* if $I(\mathbf{u}, \mathbf{v}) = \varnothing$, i.e., $|I(\mathbf{u}, \mathbf{v})| = 0$, for all $\mathbf{u}, \mathbf{v} \in \mathcal{B}$ with $\mathbf{u} \neq \mathbf{v}$.

We shall need the following theorem [30, Theorem 2.3].

Theorem 3.1. *Let m, n be positive integers satisfying $m \leq n$, and let q, r, s be positive integers with $r, s \geq 2$ and $n \geq (2s)^{2^{r-2}q} + 1$. If $\mathcal{A} \subseteq P(m, r)$ such that $|\mathcal{A}| \geq n^{\frac{1}{q}} \binom{n+r-2}{r-2}$, then there is an independent set $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| \geq s+1$.*

Let $\mathcal{A} \subseteq P(n, l)$. Let $x_1, x_2, \ldots, x_t$ be distinct elements in $[l]$ with $x_1 < x_2 < \cdots < x_t$, and $y_1, y_2, \ldots, y_t \in \mathbb{N}_0$ with $\sum_{j=1}^t y_j \leq n$. We set
$$\mathcal{A}(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t) = \{\mathbf{u} \in \mathcal{A} : \mathbf{u}(x_i) = y_i \text{ for all } i\},$$
$$\mathcal{A}^*(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t)$$
$$= \{R(x_1, x_2, \ldots, x_t; \mathbf{u}) : \mathbf{u} \in \mathcal{A}(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t)\}.$$
Note that
$$\mathcal{A}^*(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t) \subseteq P\left(n - \sum_{j=1}^t y_j, l - t\right),$$
and
$$|\mathcal{A}^*(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t)| = |\mathcal{A}(x_1, x_2, \ldots, x_t; y_1, y_2, \ldots, y_t)|.$$

Lemma 3.2. *Let $\mathcal{A} \subseteq P(n, l)$ be t-intersecting and $l \geq t + 3$. Let $x_1, x_2, \ldots, x_{t+1}$ be distinct elements in $[l]$ with $x_1 < x_2 < \cdots < x_{t+1}$, and $y_1, y_2, \ldots, y_{t+1} \in \mathbb{N}_0$ with $\sum_{j=1}^{t+1} y_j \leq n$. If $\mathcal{A}^*(x_1, \ldots, x_{t+1}; y_1, \ldots, y_{t+1})$ has an independent set of size at least $l - t$, then*
$$\mathcal{A} \subseteq \bigcup_{\substack{T \subseteq [t+1], \\ |T|=t}} \{\mathbf{u} \in P(n, l) : \mathbf{u}(x_s) = y_s \text{ for all } s \in T\}.$$

Proof. Let

$$\mathcal{B} = \{R(x_1, x_2, \ldots, x_{t+1}; \mathbf{u}_i) : i = 1, 2, \ldots, l - t\}$$

be an independent set of size $l - t$ in $\mathcal{A}^*(x_1, \ldots, x_{t+1}; y_1, \ldots, y_{t+1})$. Here each $\mathbf{u}_i$ is an element of $\mathcal{A}$ such that $\mathbf{u}_i(x_j) = y_j$ for all $1 \leq j \leq t$.

Let $\mathbf{v} \in \mathcal{A}$. Suppose

$$\mathbf{v} \notin \bigcup_{\substack{T \subseteq [t+1], \\ |T| = t}} \{\mathbf{u} \in P(n, l) : \mathbf{u}(x_s) = y_s \text{ for all } s \in T\}.$$

Then there exist j_1 and j_2 with $1 \leq j_1 < j_2 \leq t+1$ such that $\mathbf{v}(x_{j_1}) \neq y_{j_1}$ and $\mathbf{v}(x_{j_2}) \neq y_{j_2}$. Since $\mathcal{A}$ is t-intersecting,

$$|I(R(x_1, x_2, \ldots, x_{t+1}; \mathbf{v}), R(x_1, x_2, \ldots, x_{t+1}; \mathbf{u}_i))| \geq 1$$

for $i = 1, 2, \ldots, l - t$.

Set $\mathbf{z} = R(x_1, x_2, \ldots, x_{t+1}; \mathbf{v})$ and $\mathbf{w}_i = R(x_1, x_2, \ldots, x_{t+1}; \mathbf{u}_i)$. Since $\mathcal{B}$ is independent,

$$I(\mathbf{z}, \mathbf{w}_i) \cap I(\mathbf{z}, \mathbf{w}_{i'}) = \varnothing$$

for $i \neq i'$. Therefore $\left| \bigcup_{i=1}^{l-t} I(\mathbf{z}, \mathbf{w}_i) \right| = \sum_{i=1}^{l-t} |I(\mathbf{z}, \mathbf{w}_i)| \geq \sum_{i=1}^{l-t} 1 = l - t$, but on the other hand, $\bigcup_{i=1}^{l-t} I(\mathbf{z}, \mathbf{w}_i) \subseteq [l] \setminus \{x_j : j \in [t+1]\}$ which is of size at most $l - t - 1$, a contradiction. Hence,

$$\mathbf{v} \in \bigcup_{\substack{T \subseteq [t+1], \\ |T| = t}} \{\mathbf{u} \in P(n, l) : \mathbf{u}(x_s) = y_s \text{ for all } s \in T\},$$

and the lemma follows. $\qquad\square$

Lemma 3.3. *If $x_1, x_2, \ldots, x_r$ are positive real numbers, then*

$$\prod_{i=1}^{r}(1 + x_i) - \prod_{i=1}^{r}(1 - x_r) \geq 2\sum_{i=1}^{r} x_i.$$

Proof. Let $S \subseteq [r]$. If $|S|$ is even, then $\prod_{s \in S} x_s$ appears in the expansions of $\prod_{i=1}^{r}(1 + x_i)$ and $\prod_{i=1}^{r}(1 - x_r)$. So, $\prod_{s \in S} x_s$ does not appear in $\prod_{i=1}^{r}(1 + x_i) - \prod_{i=1}^{r}(1 - x_r)$. If $|S|$ is odd, then $\prod_{s \in S} x_s$ and $-\prod_{s \in S} x_s$ appear in the expansions of $\prod_{i=1}^{r}(1 + x_i)$ and $\prod_{i=1}^{r}(1 - x_r)$, respectively. So, $2\prod_{s \in S} x_s$ appears in $\prod_{i=1}^{r}(1 + x_i) - \prod_{i=1}^{r}(1 - x_r)$. The lemma follows by just considering all subsets S of $[r]$ with $|S| = 1$. $\qquad\square$

Lemma 3.4. *Let m, n be positive integers with $n \geq m + 1$. Then*

$$\binom{n + m}{m} - \binom{n - 1}{m} \geq \frac{(m+1)n^{m-1}}{(m-1)!}.$$

Proof. Note that

$$\binom{n + m}{m} = \frac{n^m}{m!}\prod_{i=1}^{m}\left(1 + \frac{i}{n}\right),$$

and

$$\binom{n-1}{m} = \frac{n^m}{m!} \prod_{i=1}^{m} \left(1 - \frac{i}{n}\right).$$

It then follows from Lemma 3.3 that

$$\binom{n+m}{m} - \binom{n-1}{m} \geq \frac{n^m}{m!} \left(2 \sum_{i=1}^{m} \frac{i}{n}\right) = \frac{(m+1)n^{m-1}}{(m-1)!}. \qquad \square$$

Lemma 3.5. *Let m, n be positive integers with $n \geq m$. Then*

$$\frac{n^m}{m!} < \binom{n+m}{m} < \frac{n^m}{m!} \left(1 + \frac{2^m m}{n}\right).$$

Proof. Note that

$$\binom{n+m}{m} = \frac{n^m}{m!} \prod_{i=1}^{m} \left(1 + \frac{i}{n}\right) > \frac{n^m}{m!}.$$

For the second inequality, it is sufficient to show that

$$\prod_{i=1}^{m} \left(1 + \frac{i}{n}\right) < \left(1 + \frac{2^m m}{n}\right).$$

Now,

$$\prod_{i=1}^{m} \left(1 + \frac{i}{n}\right) \leq \left(1 + \frac{m}{n}\right)^m$$

$$= \left(1 + \sum_{i=1}^{m} \binom{m}{i} \left(\frac{m}{n}\right)^i\right)$$

$$\leq \left(1 + \frac{m}{n} \left(\sum_{i=1}^{m} \binom{m}{i}\right)\right)$$

$$< \left(1 + \frac{2^m m}{n}\right). \qquad \square$$

Lemma 3.6. *Let m be a positive integer and f, g be positive real numbers. There exists a constant $n_0 = n_0(f, g, m)$ depending on f, g and m such that if $n \geq n_0$, then*

$$f \binom{n+m}{m} + g\sqrt{n} \binom{n+m-1}{m-1} < \frac{n^m(f+1)}{m!}.$$

Proof. Suppose $n > \max\left(m, 2^{m-1}(m-1), \left(\frac{2^m f}{g}\right)^2, (3gm)^2\right)$. By Lemma 3.5,

$$f \binom{n+m}{m} + g\sqrt{n} \binom{n+m-1}{m-1}$$

$$< f\frac{n^m}{m!}\left(1+\frac{2^m m}{n}\right) + g\sqrt{n}\frac{n^{m-1}}{(m-1)!}\left(1+\frac{2^{m-1}(m-1)}{n}\right)$$

$$\leq f\frac{n^m}{m!}\left(1+\frac{2^m m}{n}\right) + \frac{2gn^{m-\frac{1}{2}}}{(m-1)!}$$

$$= f\frac{n^m}{m!}\left(1+\frac{2^m m}{n}+\frac{2gm}{f\sqrt{n}}\right)$$

$$\leq f\frac{n^m}{m!}\left(1+\frac{3gm}{f\sqrt{n}}\right)$$

$$= \frac{n^m}{m!}\left(f+\frac{3gm}{\sqrt{n}}\right)$$

$$< \frac{n^m(f+1)}{m!}. \qquad\qquad \square$$

Proof of Theorem 1.3. We may assume that $\mathcal{A}$ is maximally t-intersecting in the sense that $\mathcal{A}\cup\{\mathbf{u}\}$ is not t-intersecting for any $\mathbf{u}\in P(n,l)\setminus\mathcal{A}$.

Suppose

$$|\mathcal{A}| < \frac{n^{l-t-2}(t+3)}{(l-t-2)!}.$$

By Lemma 3.4,

$$\binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1} \geq \frac{(l-t)n^{l-t-2}}{(l-t-2)!}.$$

Since $l \geq 2t+3$, $|\mathcal{A}| < \binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1}$. So, we may assume that

$$(1) \qquad\qquad |\mathcal{A}| \geq \frac{n^{l-t-2}(t+3)}{(l-t-2)!}.$$

Let $\mathbf{w} = (w_1, w_2, \ldots, w_l) \in \mathcal{A}$ be fixed. Then

$$\mathcal{A} = \bigcup_{\substack{\{x_1,x_2,\ldots,x_t\}\subseteq[l], \\ x_1<x_2<\cdots<x_t}} \mathcal{A}(x_1, x_2, \ldots, x_t; w_{x_1}, w_{x_2}, \ldots, w_{x_t}).$$

Let $\{x'_1, x'_2, \ldots, x'_t\} \subseteq [l]$ with $x'_1 < x'_2 < \cdots < x'_t$ be fixed and

$$\mathcal{C} = \mathcal{A}(x'_1, x'_2, \ldots, x'_t; w_{x'_1}, w_{x'_2}, \ldots, w_{x'_t}).$$

We may assume that $|\mathcal{C}|$ is maximum in the sense that

$$|\mathcal{A}(x_1, x_2, \ldots, x_t; w_{x_1}, w_{x_2}, \ldots, w_{x_t})| \leq |\mathcal{C}|$$

for all $\{x_1, x_2, \ldots, x_t\} \subseteq [l]$ with $x_1 < x_2 < \cdots < x_t$.

By relabelling if necessary, we may assume that $x'_i = i$ for all i, i.e.,

$$\mathcal{C} = \mathcal{A}(1, 2, \ldots, t; w_1, w_2, \ldots, w_t).$$

Let $P = P(n,l)$. Since $\mathcal{A}$ is non-trivially t-intersecting,

$$\mathcal{A} \nsubseteq P(1, 2, \ldots, t; w_1, w_2, \ldots, w_t).$$

Therefore there is a $\mathbf{y} = (y_1, y_2, \ldots, y_l) \in \mathcal{A}$ with $y_i \neq w_i$ for some $1 \leq i \leq t$. Since $\mathcal{C} \cup \{\mathbf{y}\}$ is t-intersecting,

$$\mathcal{C} = \bigcup_{t+1 \leq s \leq l} \mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s).$$

We shall distinguish 5 cases.

Case 1. Suppose that

$$|\mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)| \leq n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3}$$

for all $t + 1 \leq s \leq l$. Then

$$|\mathcal{C}| \leq (l - t) n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3}.$$

By the maximality of $|\mathcal{C}|$,

$$|\mathcal{A}| \leq \binom{l}{t} (l - t) n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3}.$$

By Lemma 3.6,

$$|\mathcal{A}| < \frac{n^{l - t - 2}}{(l - t - 2)!},$$

contradicting equation (1). Hence, Case 1 cannot happen.

Now, by relabelling if necessary we may assume that there is a $k \in \{t + 1, t + 2, \ldots, l\}$ such that

$$|\mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)| \geq n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3}$$

for all $t + 1 \leq s \leq k$ and

$$|\mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)| \leq n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3}$$

for all $k + 1 \leq s \leq l$. Note that if $k = l$, then there will be no family satisfying the preceding inequality.

Case 2. Suppose $k = t + 1$. Since

$$|\mathcal{A}^*(1, 2, \ldots, t, t + 1; w_1, w_2, \ldots, w_t, y_{t+1})|$$
$$= |\mathcal{A}(1, 2, \ldots, t, t + 1; w_1, w_2, \ldots, w_t, y_{t+1})|,$$

and

$$\mathcal{A}^*(1, 2, \ldots, t, t + 1; w_1, w_2, \ldots, w_t, y_{t+1}) \subseteq P\left(n - y_{t+1} - \sum_{j=1}^{t} w_j, l - t - 1\right),$$

by Theorem 3.1, $\mathcal{A}^*(1, 2, \ldots, t, t+1; w_1, w_2, \ldots, w_t, y_{t+1})$ has an independent set of size at least $l - t$, if $n \geq (2(l - t - 1))^{2^{l-t-2}} + 1$. Then it follows from Lemma 3.2 that

$$\mathcal{A} = \mathcal{C} \cup \bigcup_{1 \leq j \leq t} \mathcal{A}(1, \ldots, j-1, j+1, \ldots, t, t+1; w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_t, y_{t+1}).$$

By the choice of $|\mathcal{C}|$, $|\mathcal{A}| \leq (t + 1)|\mathcal{C}|$.

Now,

$$|\mathcal{A}(1, 2, \ldots, t, t+1; w_1, w_2, \ldots, w_t, y_{t+1})|$$
$$\leq \binom{n - y_{t+1} - (\sum_{j=1}^{t} w_j) + l - t - 2}{l - t - 2}$$
$$\leq \binom{n + l - t - 2}{l - t - 2}.$$

Therefore

$$|\mathcal{C}| \leq \binom{n + l - t - 2}{l - t - 2} + (l - t - 1)n^{\frac{1}{2}}\binom{n + l - t - 3}{l - t - 3},$$

and

$$|\mathcal{A}| \leq (t + 1)\binom{n + l - t - 2}{l - t - 2} + (t + 1)(l - t - 1)n^{\frac{1}{2}}\binom{n + l - t - 3}{l - t - 3}.$$

By Lemma 3.6,

$$|\mathcal{A}| < \frac{n^{l-t-2}(t + 2)}{(l - t - 2)!},$$

contradicting equation (1). Hence, Case 2 cannot happen.

Case 3. Suppose $k = t+2$. Again, by Theorem 3.1, $\mathcal{A}^*(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)$ has an independent of size at least $l - t$ for $s \in \{t + 1, t + 2\}$, if $n \geq (2(l - t - 1))^{2^{l-t-2}} + 1$. For each $j \in [t]$, let

$$\mathcal{Q}_j = \{\mathbf{u} \in \mathcal{A} : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \setminus \{j\},\ \mathbf{u}(j) \neq w_j,$$
$$\mathbf{u}(t + 1) = y_{t+1} \text{ and } \mathbf{u}(t + 2) = y_{t+2}\}.$$

It follows from Lemma 3.2 that

$$\mathcal{A} = \mathcal{C} \cup \bigcup_{1 \leq j \leq t} \mathcal{Q}_j.$$

Let $n_j' = n - y_{t+1} - y_{t+2} - \left(\sum_{1 \leq i \leq t, i \neq j} w_i\right)$. If $n_j' < 0$, then $|\mathcal{Q}_j| = 0$. If $n_j' \geq 0$, then

$$|\mathcal{Q}_j| \leq \sum_{\substack{0 \leq d \leq n_j', \\ d \neq w_j}} \binom{n_j' - d + l - t - 3}{l - t - 3}$$

$$\leq \sum_{0 \leq d \leq n'_j} \binom{n'_j - d + l - t - 3}{l - t - 3}$$

$$= \binom{n'_j + l - t - 2}{l - t - 2}$$

$$\leq \binom{n + l - t - 2}{l - t - 2}.$$

Therefore $|\mathcal{A}| \leq |\mathcal{C}| + t\binom{n+l-t-2}{l-t-2}$.

Now, for each $s \in \{t+1, t+2\}$,

$$|\mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)| \leq \binom{n - y_s - (\sum_{j=1}^{t} w_j) + l - t - 2}{l - t - 2}$$

$$\leq \binom{n + l - t - 2}{l - t - 2}.$$

Therefore

$$|\mathcal{C}| \leq 2\binom{n + l - t - 2}{l - t - 2} + (l - t - 2)n^{\frac{1}{2}}\binom{n + l - t - 3}{l - t - 3},$$

and

$$|\mathcal{A}| \leq (t + 2)\binom{n + l - t - 2}{l - t - 2} + (l - t - 2)n^{\frac{1}{2}}\binom{n + l - t - 3}{l - t - 3}.$$

By Lemma 3.6,

$$|\mathcal{A}| < \frac{n^{l-t-2}(t + 3)}{(l - t - 2)!},$$

contradicting equation (1). Hence, Case 3 cannot happen.

We may assume that $t + 3 \leq k \leq l$. In particular,

$$|\mathcal{A}(1, 2, \ldots, t, t+1; w_1, w_2, \ldots, w_t, y_{t+1})| \geq n^{\frac{1}{2}}\binom{n + l - t - 3}{l - t - 3}.$$

On the other hand,

$$|\mathcal{A}(1, 2, \ldots, t, t+1; w_1, w_2, \ldots, w_t, y_{t+1})|$$

$$\leq \binom{n - y_{t+1} - (\sum_{j=1}^{t} w_j) + l - t - 2}{l - t - 2}$$

$$\leq \binom{n - (\sum_{j=1}^{t} w_j) + l - t - 2}{l - t - 2}.$$

If $n - \sum_{j=1}^{t} w_j < l - t$, then $\binom{n - (\sum_{j=1}^{t} w_j) + l - t - 2}{l - t - 2} < \binom{2l - 2t - 2}{l - t - 2}$, which is impossible for large n. So, we may assume that

$$(2) \qquad\qquad n - \sum_{j=1}^{t} w_j \geq l - t.$$

Case 4. Suppose $t + 3 \leq k \leq l - 1$. Again, by Theorem 3.1, $\mathcal{A}^*(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)$ has an independent set of size at least $l - t$ for $s \in \{t + 1, t + 2, t + 3\}$, if $n \geq (2(l - t - 1))^{2^{l-t-2}} + 1$. For each $j \in [t]$, let

$$\mathcal{Q}_j = \{\mathbf{u} \in \mathcal{A} : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \setminus \{j\}, \ \mathbf{u}(j) \neq w_j,$$
$$\mathbf{u}(t + 1) = y_{t+1}, \ \mathbf{u}(t + 2) = y_{t+2}, \text{ and } \mathbf{u}(t + 3) = y_{t+3}\}.$$

It follows from Lemma 3.2 that

$$\mathcal{A} = \mathcal{C} \cup \bigcup_{1 \leq j \leq t} \mathcal{Q}_j.$$

Let $n'_j = n - y_{t+1} - y_{t+2} - y_{t+3} - \left(\sum_{1 \leq i \leq t, i \neq j} w_i\right)$. If $n'_j < 0$, then $|\mathcal{Q}_j| = 0$. If $n'_j \geq 0$, then

$$|\mathcal{Q}_j| \leq \sum_{\substack{0 \leq d \leq n'_j, \\ d \neq w_j}} \binom{n'_j - d + l - t - 4}{l - t - 4}$$

$$\leq \sum_{0 \leq d \leq n'_j} \binom{n'_j - d + l - t - 4}{l - t - 4}$$

$$= \binom{n'_j + l - t - 3}{l - t - 3}$$

$$\leq \binom{n + l - t - 3}{l - t - 3}.$$

Therefore $|\mathcal{A}| \leq |\mathcal{C}| + t\binom{n+l-t-3}{l-t-3}$.

Let

$$\mathcal{D} = \{\mathbf{u} \in P(n, l) : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \text{ and}$$
$$\mathbf{u}(i) = y_i \text{ for some } t + 1 \leq i \leq l - 1\}.$$

By equation (2) and Corollary 2.6,

$$|\mathcal{D}| \leq \sum_{0 \leq d \leq l-t-2} \binom{n - \sum_{1 \leq i \leq t} w_i - d + l - t - 2}{l - t - 2}$$

$$\leq \sum_{0 \leq d \leq l-t-2} \binom{n - d + l - t - 2}{l - t - 2}.$$

Since

$$\bigcup_{t+1 \leq s \leq l-1} \mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s) \subseteq \mathcal{D},$$

we have

$$|\mathcal{C}| \leq \sum_{0 \leq d \leq l-t-2} \binom{n - d + l - t - 2}{l - t - 2} + n^{\frac{1}{2}} \binom{n + l - t - 3}{l - t - 3},$$

and

$$|\mathcal{A}| \leq \sum_{0 \leq d \leq l-t-2} \binom{n-d+l-t-2}{l-t-2} + (n^{\frac{1}{2}}+t)\binom{n+l-t-3}{l-t-3}$$

$$\leq \sum_{0 \leq d \leq l-t-2} \binom{n-d+l-t-2}{l-t-2} + 2n^{\frac{1}{2}}\binom{n+l-t-3}{l-t-3}.$$

Since

$$\binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1} = \sum_{0 \leq d \leq l-t-1} \binom{n-d+l-t-2}{l-t-2},$$

we have $|\mathcal{A}| < \binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1}$ if and only if

$$2n^{\frac{1}{2}}\binom{n+l-t-3}{l-t-3} < \binom{n-1}{l-t-2}.$$

By Lemma 3.5,

$$2n^{\frac{1}{2}}\binom{n+l-t-3}{l-t-3} < \frac{4n^{l-t-\frac{5}{2}}}{(l-t-3)!}$$

$$< \frac{n^{l-t-2}}{(l-t-2)!}\left(1 - \frac{l-t-1}{n}\right)^{l-t-2}$$

$$< \binom{n-1}{l-t-2}.$$

Hence, $|\mathcal{A}| < \binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1}$ and Case 4 is done.

Case 5. Suppose $k = l$. Again, by Theorem 3.1, $\mathcal{A}^*(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s)$ has an independent set of size at least $l - t$ for $s \in [l] \setminus [t]$, if $n \geq (2(l-t-1))^{2^{l-t-2}} + 1$. For each $j \in [t]$, let

$$\mathcal{Q}_j = \{\mathbf{u} \in \mathcal{A} : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \setminus \{j\}, \ \mathbf{u}(j) \neq w_j,$$

$$\text{and } \mathbf{u}(i) = y_i \text{ for all } i \in [l] \setminus [t]\}.$$

It follows from Lemma 3.2 that

$$\mathcal{A} = \mathcal{C} \cup \bigcup_{1 \leq j \leq t} \mathcal{Q}_j.$$

Let $n'_j = n - \sum_{t+1 \leq i \leq l} y_i - \left(\sum_{1 \leq i \leq t, i \neq j} w_i\right)$. If $n'_j < 0$, then $|\mathcal{Q}_j| = 0$. If $n'_j \geq 0$, then $|\mathcal{Q}_j| = 1$. Therefore $|\mathcal{A}| \leq |\mathcal{C}| + t$.

Let

$$\mathcal{D} = \{\mathbf{u} \in P(n, l) : \mathbf{u}(i) = w_i \text{ for all } i \in [t] \text{ and}$$

$$\mathbf{u}(i) = y_i \text{ for some } t+1 \leq i \leq l\}.$$

By equation (2) and Corollary 2.6,

$$|\mathcal{D}| \leq \sum_{0 \leq d \leq l-t-1} \binom{n - \sum_{1 \leq i \leq t} w_i - d + l - t - 2}{l - t - 2}$$

$$\leq \sum_{0 \leq d \leq l-t-1} \binom{n - d + l - t - 2}{l - t - 2}.$$

Since

$$\mathcal{C} = \bigcup_{t+1 \leq s \leq l} \mathcal{A}(1, 2, \ldots, t, s; w_1, w_2, \ldots, w_t, y_s) \subseteq \mathcal{D},$$

we have

$$|\mathcal{C}| \leq \sum_{0 \leq d \leq l-t-1} \binom{n - d + l - t - 2}{l - t - 2},$$

and

$$|\mathcal{A}| \leq \sum_{0 \leq d \leq l-t-1} \binom{n - d + l - t - 2}{l - t - 2} + t.$$

Since

$$\binom{n + l - t - 1}{l - t - 1} - \binom{n - 1}{l - t - 1} = \sum_{0 \leq d \leq l-t-1} \binom{n - d + l - t - 2}{l - t - 2},$$

we have $|\mathcal{A}| \leq \binom{n+l-t-1}{l-t-1} - \binom{n-1}{l-t-1} + t$. Furthermore, equality holds if and only if $w_i = 0$ for all $i \in [t]$ and $y_i = 0$ for all $i \in [l] \setminus [t]$. By the maximality of $\mathcal{A}$,

$$\mathcal{C} = \bigcup_{s \in [l] \setminus [t]} \mathcal{A}_s,$$

and $\mathcal{Q}_j = \{\mathbf{q}_j\}$ for all $j \in [t]$. This completes the proof of the theorem. $\square$

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Cheng Yeaw Ku
Department of Mathematics
National University of Singapore
117543, Singapore
E-mail address: matkcy@nus.edu.sg

Kok Bin Wong
Institute of Mathematical Sciences
University of Malaya
50603 Kuala Lumpur, Malaysia
E-mail address: kbwong@um.edu.my