Bull. Korean Math. Soc. **52** (2015), No. 3, pp. 999–1006
http://dx.doi.org/10.4134/BKMS.2015.52.3.999

POSETS ADMITTING THE LINEARITY OF ISOMETRIES

Jong Youn Hyun, Jeongjin Kim, and Sang-Mok Kim

Abstract. In this paper, we deal with a characterization of the posets with the property that every poset isometry of $\mathbb{F}_q^n$ fixing the origin is a linear map. We say such a poset to be *admitting the linearity of isometries*. We show that a poset P admits the linearity of isometries over $\mathbb{F}_q^n$ if and only if P is a disjoint sum of chains of cardinality 2 or 1 when $q = 2$, or P is an anti-chain otherwise.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, and $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$. In 1995, Brualdi *et al.* [1] introduced a non-Hamming metric on $\mathbb{F}_q^n$ which is associated to an arbitrary poset on $[n] = \{1, 2, \ldots, n\}$. It is called a poset metric. The poset metric spaces have been extensively studied in [1, 3, 4, 5, 8]. We briefly introduce basic notions for poset metric on $\mathbb{F}_q^n$.

Let $P = ([n], \leq)$ be a poset on $[n]$ of coordinate positions of vectors on $\mathbb{F}_q^n$. A subset I of P is called an order ideal (or a down-set) if $x \in I$ and $y \leq x$ imply $y \in I$. For an arbitrary subset A of P, we denote by $\langle A \rangle$ the smallest order ideal of P containing A. The P-weight of a vector $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ is defined as the cardinality

$$w_P(u) = |\langle \mathrm{supp}(u) \rangle| \tag{1}$$

of the smallest order ideal of P containing $\mathrm{supp}(u)$, where $\mathrm{supp}(u) = \{i \mid u_i \neq 0\}$. For $u, v \in \mathbb{F}_q^n$, the P-distance $d_P(u, v)$ between u and v is defined by

$$d_P(u, v) = |\langle \mathrm{supp}(u - v) \rangle|. \tag{2}$$

It is well known from [1] that d_P defines a metric on $\mathbb{F}_q^n$. If P is an anti-chain, then d_P coincides with the Hamming metric. Therefore, we may view that poset metric is a generalization of the Hamming metric. The theory of poset code generally plays with the properties of the poset metric space $(\mathbb{F}_q^n, d_P)$.

The isometry group of $(\mathbb{F}_q^n, d_P)$ is defined by

$$\mathrm{Iso}_P(\mathbb{F}_q^n) = \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid d_P(f(u), f(v)) = d_P(u, v) \text{ for all } u, v \in \mathbb{F}_q^n\}. \tag{3}$$

Received June 26, 2014; Revised October 31, 2014.
2010 *Mathematics Subject Classification.* 94B05.
Key words and phrases. P-isometries, P-isometry group, poset metric.

An element of the isometry group of $(\mathbb{F}_q^n, d_P)$ is called a P-isometry of $\mathbb{F}_q^n$. It is easy to see that if f is a P-isometry of $\mathbb{F}_q^n$, then it is a bijective map. Moreover, it is easily seen that a linear transformation f of $\mathbb{F}_q^n$ into itself preserves P-distance if and only if it preserves the P-weight. Thus we define the automorphism group of $(\mathbb{F}_q^n, d_P)$, as follows:

(4)
$$\mathrm{Aut}_P(\mathbb{F}_q^n) = \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f \text{ is linear and } w_P(f(u)) = w_P(u) \text{ for all } u \in \mathbb{F}_q^n\}.$$

An element of the automorphism group of $(\mathbb{F}_q^n, d_P)$ is called a P-automorphism of $\mathbb{F}_q^n$. It is obvious that $\mathrm{Aut}_P(\mathbb{F}_q^n)$ is a subgroup of $\mathrm{Iso}_P(\mathbb{F}_q^n)$.

In [8], Panek *et al.* determine the structure of such automorphism group $\mathrm{Aut}_P(\mathbb{F}_q^n)$, as follows:

$$\mathrm{Aut}_P(\mathbb{F}_q^n) \simeq G(P) \rtimes \mathrm{Aut}(P), \tag{5}$$

where, if $M_{n\times n}(\mathbb{F}_q)$ denotes the set of all $n \times n$ matrices over $\mathbb{F}_q$, then

$$G(P) = \left\{ (a_{ij}) \in M_{n\times n}(\mathbb{F}_q) \;\middle|\; a_{ij} \in \begin{array}{ccc} \mathbb{F}_q & \text{if} & i <_P j \\ \mathbb{F}_q^* & \text{if} & i = j \\ \{0\} & & \text{otherwise} \end{array} \right\},$$

and $\mathrm{Aut}(P)$ is the set of all order-preserving-bijections on P, i.e., $f \in \mathrm{Aut}(P)$ is a bijection on P, provided that if $x \le y$, then $f(x) \le f(y)$ for x and y in P. In contrast to the early settled result on $\mathrm{Aut}_P(\mathbb{F}_q^n)$, the generalized question on the structure of $\mathrm{Iso}_P(\mathbb{F}_q^n)$ is known to be very difficult, as described in [3, 7].

Now, we define $\mathrm{Iso}_P^0(\mathbb{F}_q^n)$ as the set of P-isometries of $\mathbb{F}_q^n$ fixing the zero vector $\mathbf{0}$, i.e.,

$$\mathrm{Iso}_P^0(\mathbb{F}_q^n) = \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f \text{ is a } P\text{-isometry and } f(\mathbf{0}) = \mathbf{0}\}. \tag{6}$$

Note that $\mathrm{Aut}_P(\mathbb{F}_q^n)$ is a subgroup of $\mathrm{Iso}_P^0(\mathbb{F}_q^n)$, and it follows from [3, 8] that $\mathrm{Aut}_P(\mathbb{F}_q^n) = \mathrm{Iso}_P^0(\mathbb{F}_q^n)$ if P is an anti-chain. Conversely, it is worth studying to determine the posets P which satisfy the property that $\mathrm{Aut}_P(\mathbb{F}_q^n) = \mathrm{Iso}_P^0(\mathbb{F}_q^n)$. We now say that *a poset P on $[n]$ admits the linearity of isometries over $\mathbb{F}_q^n$* if $\mathrm{Aut}_P(\mathbb{F}_q^n) = \mathrm{Iso}_P^0(\mathbb{F}_q^n)$. In this paper, we characterize the posets on $[n]$ which admit the linearity of isometries over $\mathbb{F}_q^n$.

Before we state our main result, we give some terminologies in poset theory, as follows. For two disjoint poset $P = (X, \le)$ and $Q = (Y, \le)$, the disjoint sum $P + Q$ of P and Q denotes the ordered set on $X \bigcup Y$ such that $x \le y$ if and only if either $x \le y$ in P or $x \le y$ in Q. The linear sum $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the new order relations $x \le y$ for all $x \in X$ and $y \in Y$. We now state our main result.

Theorem 1.1. *Let P be a poset on $[n]$. Then a poset P admits the linearity of isometries over $\mathbb{F}_q^n$ if and only if P is a disjoint sum of chains of cardinality 2 or 1 when $q = 2$, or P is an anti-chain otherwise.*

2. Proof of Theorem 1.1

In this section we give a complete proof of Theorem 1.1. For convenience to prove, we define the simple notations for chains and anti-chains as follows. **1** denotes the poset with one element. The disjoint sum of n **1**'s, written as $n\mathbf{1}$, denotes the anti-chain of cardinality n. The linear sum of n **1**'s, written as **n**, denotes the chain of cardinality n. We use the notation $\{e_1, e_2, \ldots, e_n\}$ to denote the canonical basis of $\mathbb{F}_2^n$.

We first begin with giving a proof of the theorem for the binary case by giving four consecutive propositions and their proofs. Among these propositions, we remark that Propositions 2.3, 2.5, and 2.6 can be also obtained from Theorems 3.3 and 3.9 in [3], through the long and complicate process. However, we now give more direct and simpler proofs, instead.

Proposition 2.1. *If P contains* **3** *as a subposet, then* $\mathrm{Aut}_P(\mathbb{F}_2^n)$ *is a proper subgroup of* $\mathrm{Iso}_P^0(\mathbb{F}_2^n)$.

Proof. We proceed by induction on n. Let P be a poset of cardinality n which contains **3** as a subposet. Recall that **3** is the poset on $[3] = \{1, 2, 3\}$ with order relation $1 < 2 < 3$. We take P_3 to be **3** and define a function f_3 of $\mathbb{F}_2^3$ into $\mathbb{F}_2^3$ by $f_3(\mathbf{0}) = \mathbf{0}$, $f_3(e_1) = e_1$, $f_3(e_2) = e_1 + e_2$, $f_3(e_1 + e_2) = e_2$, $f_3(e_3) = e_1 + e_2 + e_3$, $f_3(e_1 + e_3) = e_2 + e_3$, $f_3(e_2 + e_3) = e_1 + e_3$, $f_3(e_1 + e_2 + e_3) = e_3$. It is easy to check that

$$d_{P_3}(f_s(u), f_3(v)) = \langle \mathrm{supp}(f_3(u) + f_3(v)) \rangle = \langle \mathrm{supp}(u + v) \rangle = d_{P_3}(u, v) \tag{7}$$

for all $u, v \in \mathbb{F}_2^3$. Using (7), one has that f_3 is a P-isometry of $\mathbb{F}_2^3$ which is not a linear map (for example, $f_3(e_2 + e_3) = e_1 + e_3 \neq e_3 = f_3(e_2) + f_3(e_3)$). To avoid confusion, let $\langle A \rangle_P$ denote the order ideal of P generated by A. Assume that a poset P_n on $[n]$ and a function f_n of $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$ have been constructed for which

$$\langle \mathrm{supp}(f_n(u) + f_n(v)) \rangle_{P_n} = \langle \mathrm{supp}(u + v) \rangle_{P_n} \tag{8}$$

for all $u, v \in \mathbb{F}_2^n$. We define P_{n+1} as a poset on $[n+1]$ which contains P_n as a subposet, and a function $f_{n+1} : \mathbb{F}_2^{n+1} \to \mathbb{F}_2^{n+1}$ as follows:

$$f_{n+1}(u) = \begin{cases} f_n(u) & \text{if } \mathrm{supp}(u) \subseteq [n], \\ f_n(u') + e_{n+1} & \text{if } u = u' + e_{n+1},\ \mathrm{supp}(u') \subseteq [n]. \end{cases}$$

Note that for subsets A, B of P_n, we have that

$$\langle A \rangle_{P_{n+1}} = \langle B \rangle_{P_{n+1}}, \tag{9}$$

if $\langle A \rangle_{P_n} = \langle B \rangle_{P_n}$. Now we will prove by induction that

$$\langle \mathrm{supp}(f_{n+1}(u) + f_{n+1}(v)) \rangle_{P_{n+1}} = \langle \mathrm{supp}(u + v) \rangle_{P_{n+1}} \tag{10}$$

for all $u, v \in \mathbb{F}_2^{n+1}$. We have three cases for these vectors u, v in $\mathbb{F}_2^{n+1}$:
Case 1: $\mathrm{supp}(u)$, $\mathrm{supp}(v) \subseteq [n]$.
Case 2: $\mathrm{supp}(u) \subseteq [n]$, $v = v' + e_{n+1}$, $\mathrm{supp}(v') \subseteq [n]$.

Case 3: $u = u' + e_{n+1}$, $\text{supp}(u) \subseteq [n]$, $v = v' + e_{n+1}$, $\text{supp}(v') \subseteq [n]$.
We only give a proof for Case 2 since the other cases can be treated similarly. One has that

$$\begin{aligned} &\langle \text{supp}(f_{n+1}(u) + f_{n+1}(v))\rangle_{P_{n+1}} \\ =\ &\langle \text{supp}(f_n(u') + f_n(v') + e_{n+1})\rangle_{P_{n+1}} \\ =\ &\langle \text{supp}(f_n(u') + f_n(v'))\rangle_{P_{n+1}} \cup \langle n+1\rangle_{P_{n+1}} \\ =\ &\langle \text{supp}(u' + v')\rangle_{P_{n+1}} \cup \langle n+1\rangle_{P_{n+1}} \text{ (by (8), (9))} \\ =\ &\langle \text{supp}(u' + v' + e_{n+1})\rangle_{P_{n+1}} \\ =\ &\langle \text{supp}(u+v)\rangle_{P_{n+1}}. \end{aligned}$$

It follows from (10) that f_{n+1} is a P_{n+1}-isometry of $\mathbb{F}_2^{n+1}$ which is not a linear map (for example, $f_{n+1}(e_2 + e_3) = f_3(e_2 + e_3) \neq f_3(e_2) + f_3(e_3) = f_{n+1}(e_2) + f_{n+1}(e_3)$). This proves the proposition. □

Let P be a poset on $[n]$. For an order ideal I of P, we denote by $\max(I)$ (resp. $\min(I)$) the set of maximal (resp. minimal) elements of I. For a subset A of P, we define $e_A = \sum_{i\in A} e_i$. By convention, $e_\varnothing = \mathbf{0}$.

We begin with the following simple lemma which follows easily from the definition of the P-distance.

Lemma 2.2. *Let A, B be subsets of a poset P. If $\langle A\rangle \subseteq \langle B\rangle$ and $\max(\langle A\rangle) \cap \max(\langle B\rangle) = \varnothing$, then $d_P(e_A, e_B) = w_P(e_B)$.*

From now on, P is a poset which does not contain $\mathbf{3}$ as a subposet. Hence every subset A of P has a decomposition $A = B \cup C$, where $B = A \cap \max(P)$ and $C = A \cap \min(P)$ so that $e_A = e_B + e_C$.

Proposition 2.3. *If P contains $\mathbf{1} \oplus \mathbf{21}$ as a subposet, then $\text{Aut}_P(\mathbb{F}_2^n)$ is a proper subgroup of $\text{Iso}_P^0(\mathbb{F}_2^n)$.*

Proof. Let A be a subset of P. As mentioned above, we may write

$$A = B \cup C, \text{ where } B = A \cap \max(P) \text{ and } C = A \cap \min(P).$$

Define a function f of $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$ as follows:

$$f(e_A) = f(e_B + e_C) = f(e_B) + e_C,$$

where

$$f(e_B) = \begin{cases} e_B + \sum\limits_{k,l\in B, k\neq l} e_{\langle k\rangle\cap\langle l\rangle} & \text{if } |B| \geq 2, \\ e_B & \text{otherwise.} \end{cases}$$

First, we will show that f is a P-isometry of $\mathbb{F}_2^n$. For an arbitrary subset U of P, we put

$$t_U = \sum_{k,l\in U, k\neq l} e_{\langle k\rangle\cap\langle l\rangle}.$$

Then

$$\langle \mathrm{supp}(t_U)\rangle \subseteq \langle U\rangle \text{ and } t_U = 0 \text{ if } |U| \leq 1. \tag{11}$$

Let A' be a subset of P such that

$$A' = B' \cup C', \text{ where } B' = A' \cap \max(P) \text{ and } C' = A' \cap \min(P).$$

It follows from the definition of f that

$$\begin{aligned} d_P(f(e_A), f(e_{A'})) &= d_P(f(e_B + e_C), f(e_{B'} + e_{C'})) \\ &= d_P(e_B + t_B + e_C, e_{B'} + t_{B'} + e_{C'}) \\ &= d_P(e_{B \triangle B'} + e_{C \triangle C'}, t_B + t_{B'}), \end{aligned} \tag{12}$$

where $A \triangle B$ denote the symmetric difference of sets A and B. Note that $t_B + t_{B'}$ can be represented by the sum of four parts as follows:

$$t_{B\setminus B'} + t_{B'\setminus B} + \sum_{k\in B\setminus B', l\in B\cap B'} e_{\langle k\rangle\cap\langle l\rangle} + \sum_{k\in B'\setminus B, l\in B\cap B'} e_{\langle k\rangle\cap\langle l\rangle}. \tag{13}$$

Using (11) the supports of each part in the right hand side of (13) are contained in the order ideal generated by $B \triangle B'$. By Lemma 2.2, (12) becomes $w_P(e_{B \triangle B'} + e_{C \triangle C'}) = d_P(e_{B \triangle B'}, e_{C \triangle C'}) = d_P(e_A, e_{A'})$. Next, we will show that f is not a linear map if P contains $\mathbf{1} \oplus \mathbf{21}$. Let $\mathbf{1} \oplus \mathbf{21} = \{j\} \oplus \{i_1, i_2\}$. For a contradiction, assume that f is a linear. Then

$$e_{i_1} + e_{i_2} = f(e_{i_1}) + f(e_{i_2}) = f(e_{i_1} + e_{i_2}) = e_{i_1} + e_{i_2} + e_{\langle i_1\rangle\cap\langle i_2\rangle}.$$

So we have $e_{\langle i_1\rangle\cap\langle i_2\rangle} = \varnothing$, i.e., $\langle i_1\rangle \cap \langle i_2\rangle = \varnothing$, a contradiction to $j \in \langle i_1\rangle \cap \langle i_2\rangle$. This proves the proposition. □

The following lemma (See [2]) is essentially the same as that an anti-chain admits the linearity of isometries.

Lemma 2.4. *Let f be a P-isometry of $\mathbb{F}_2^n$ which fixes the origin. Let A be a subset of* $\min(P)$. *Then* $f(e_A) = \sum_{i\in A} f(e_i)$.

From now on, we assume that P does not contain both $\mathbf{3}$ and $\mathbf{1} \oplus 2\mathbf{1}$. Therefore P is the disjoint union of Q_s's and an anti-chain, where $Q_s = s\mathbf{1} \oplus \mathbf{1}$ $(s \geq 1)$.

Proposition 2.5. *If P contains $2\mathbf{1} \oplus \mathbf{1}$ as a subposet, then* $\mathrm{Aut}_P(\mathbb{F}_2^n)$ *is a proper subgroup of* $\mathrm{Iso}_P^0(\mathbb{F}_2^n)$.

Proof. Let $s \geq 2$ and σ a nontrivial permutation on $\min(Q_s)$. Let us write $\sigma(u)$ for $\sum_{i\in \mathrm{supp}(u)} e_{\sigma(i)}$. Define a map g of $\mathbb{F}_2^{s+1}$ into $\mathbb{F}_2^{s+1}$ by

$$g(u) = \begin{cases} \sigma(u) & \text{if } \mathrm{supp}(u) \subseteq \min(Q_s), \\ u & \text{otherwise.} \end{cases}$$

We claim that g is a Q_s-isometry which is not a linear map. If $\mathrm{supp}(u)$ and $\mathrm{supp}(v)$ are both subsets of the anti-chain $\min(Q_s)$, then by Lemma 2.4, σ can be extended to the Q_s-isometry of $\mathbb{F}_2^{s+1}$ by fixing the maximal element of Q_s. If neither is a subset of $\min(Q_s)$, then $g(u) = u$ and $g(v) = v$, so $d_{Q_s}(g(u), g(v)) = d_{Q_s}(u, v)$. Thus, to prove that g preserves the Q_s-distance, it is sufficient to consider vectors u, v in $\mathbb{F}_2^{s+1}$ which satisfy $\mathrm{supp}(u) \subseteq \min(Q_s)$ and $\mathrm{supp}(v) \nsubseteq \min(Q_s)$. For such u and v in $\mathbb{F}_2^{s+1}$, one obtains $d_P(g(u), g(v)) = d_P(\sigma(u), v) = s + 1 = d_P(u, v)$. It remains to show that g is not a linear map. Let $\max(Q_s) = \{i\}$. Since σ is a nontrivial permutation on $\min(Q_s)$, there are distinct elements j, k in $\min(Q_s)$ such that $\sigma(j) = k$. From this, one has $g(e_i + e_j) = e_i + e_j \neq e_i + e_k = g(e_i) + g(e_j)$. This proves the claim. It follows from the assumption of the proposition that $P = Q_s \mathring{\cup} P'$ with $s \geq 2$. We may write $u \in \mathbb{F}_2^n$ as $u = (u^1, u^2)$ with $\mathrm{supp}(u^1) \subseteq Q_s$ and $\mathrm{supp}(u^2) \subseteq P'$. Define $f(u) = f(u^1, u^2) = (g(u^1), u^2)$. It is easy to see that f is a P-isometry which is not a linear map. This proves the proposition. □

Now assume that P contains none of $\mathbf{3}, \mathit{2}\mathbf{1} \oplus \mathbf{1}, \mathbf{1} \oplus \mathit{2}\mathbf{1}$ as a subposet. Then $P = m\mathbf{1} \mathring{\cup} n\mathbf{2}$ for some $m, n \geq 0$.

Proposition 2.6. *If $P = m\mathbf{1} \mathring{\cup} n\mathbf{2}$ for some $m, n \geq 0$, then every P-isometry of $\mathbb{F}_2^n$ which fixes the origin is a linear map.*

Proof. We will show that

$$f(e_A) = \sum_{i \in A} f(e_i) \tag{14}$$

for any subset A of P. We proceed by induction on the cardinality of A. It is clear that (15) holds for every A with $|A| = 1$. Assume that (15) holds for every subset of cardinality less than s and that $|A| = s$. If $A \subseteq \min(P)$, then the result follows from Lemma 2.4. So we may assume that $A \cap \max(P) \neq \varnothing$. Take an element $i \in A \cap \max(P)$. Since f is a bijection of $\mathbb{F}_2^n$, we may write

$$f(e_A) = f(e_{A \setminus \{i\}} + e_i) = f(e_{A \setminus \{i\}}) + f(u) \tag{15}$$

for some $u \in \mathbb{F}_2^n$. By induction hypothesis,

$$f(e_A) = \sum_{i \in A \setminus \{i\}} f(e_i) + f(u).$$

We claim that $u = e_i$. Since f preserves the P-distance, we obtain the following three relations from (15):

$$|\langle A \rangle| = d_P(e_{A \setminus \{i\}}, u), \tag{16}$$

$$2 = |\langle i \rangle| = |\langle \mathrm{supp}(u) \rangle|, \tag{17}$$

$$w_P(e_A + u) = |\langle A \setminus \{i\} \rangle|. \tag{18}$$

For example, the relation (17) is derived as follows:

$$
\begin{aligned}
|\langle i\rangle| = w_P(e_i) &= d_P(e_A, e_{A\setminus\{i\}}) \\
&= d_P(f(e_A), f(e_{A\setminus\{i\}})) \text{ (since } f \text{ is a } P\text{-isometry)} \\
&= w_P(f(e_A) + f(e_{A\setminus\{i\}})) \\
&= w_P(f(u)) \text{ (by (15))} \\
&= w_P(u) \text{ (since } f \text{ is a } P\text{-isometry)} \\
&= |\langle \mathrm{supp}(u)\rangle|.
\end{aligned}
$$

Since $|\langle \mathrm{supp}(u)\rangle| = |\langle i\rangle| = 2$, we have two possibilities:

$$\text{(i)} \quad \langle \mathrm{supp}(u)\rangle = \langle k\rangle \ \text{ for some } k \in \max(P), \text{ and} \tag{19}$$

$$\text{(ii)} \quad \mathrm{supp}(u) = \{k, l\} \ \text{ for some } k, l \in \min(P). \tag{20}$$

Notice that

$$|\langle A \setminus \{i\}\rangle| = \begin{cases} |\langle A\rangle| - 2 & \text{if } \langle i\rangle \nsubseteq A, \\ |\langle A\rangle| - 1 & \text{if } \langle i\rangle \subseteq A. \end{cases} \tag{21}$$

If (20) happens, by (18), we have $|\langle A \setminus \{i\}\rangle| = w_P(e_A + e_k + e_l) \geq |\langle A\rangle|$, a contradiction to (21). Thus (19) must hold. So we may write $u = e_k + ae_l$, $a \in \mathbb{F}_2$ and $l < k$ in P. It follows from (18) that

$$|\langle A \setminus \{i\}\rangle| = w_P(e_A + e_k + ae_l). \tag{22}$$

From (21) and (22) we deduce that $k \in A$. It now follows from (16) that

$$|\langle A\rangle| = d_P(e_{A\setminus\{i\}}, e_k + ae_l) = d_P(e_{(A\setminus\{i\})} + e_k, ae_l). \tag{23}$$

From (23) and $k \in A$ we deduce that $i = k$. From $i = k$ and (22) we see that

$$|\langle A \setminus \{i\}\rangle| = w_P(e_{A\setminus\{i\}} + ae_l) \text{ and } l < i \text{ in } P.$$

Therefore, $a = 0$ and so $u = e_k$. This proves the proposition. □

Theorem 1.1 follows from the four previous propositions.

Next, we deal with the q-ary case for $q > 2$. Recall that the height of an element j in a poset P is the maximum of lengths of chains descending from j.

Proposition 2.7. *Let P be a poset on $[n]$ and let $q > 2$. Then P admits the linearity of isometries over $\mathbb{F}_q^n$ if and only if P is an anti-chain.*

Proof. As mentioned in Introduction, if P is an anti-chain, it admits the linearity of isometries over $\mathbb{F}_q^n$. Conversely, let P be a poset on $[n]$ which contains an element j of height one. Then we can choose a minimal element i in P so that $i <_P j$. We may assume that $i = 1$, $j = 2$. Define a function f of $\mathbb{F}_q^n$ into $\mathbb{F}_q^n$ by $f(e_1 + e_2) = e_2, f(e_2) = e_1 + e_2$ and $f(u) = u$ otherwise. It is easy to check that f is not a linear map. Indeed, $f(2e_2) = 2e_2 \neq 2(e_1 + e_2) = 2f(e_2)$. To show that f is a P-isometry of $\mathbb{F}_q^n$, we just consider the following cases since the other cases are clearly satisfied: For $u \neq e_2, e_2 + e_1$ we have $d_P(e_1, e_2 + u) = w_P(e_2 + u)$ by Lemma 2.2. Therefore, we get $d_P(f(e_2), f(u)) = d_P(e_2, u)$ and $d_P(f(e_1 + e_2), f(u)) = d_P(e_1 + e_2, u)$. This proves the proposition. □

Acknowledgment. The present research has been conducted by the Research Grant of Kwangwoon University in 2014.

References

[1] R. A. Brualdi, J. Graves, and K. M. Lawrence, *Codes with a poset metric*, Discrete Math. **147** (1995), no. 1-3, 57–72.
[2] H. Fripertinger, *Enumeration of the semilinear isometry classes of linear codes*, Bayreuth. Math. Schr. **74** (2005), 100–122.
[3] J. Y. Hyun, *A subgroup of the full poset-isometry group*, SIAM J. Discrete Math. **24** (2010), no. 2, 589–599.
[4] J. Y. Hyun and H. K. Kim, *Maximum distance separable poset codes*, Des. Codes Cryptogr. **48** (2008), no. 3, 247–261.
[5] H. K. Kim and D. S. Krotov, *The poset metrics that allow binary codes of codimension m to be m-, $(m-1)$-, or $(m-2)$-perfect*, IEEE Trans. Inform. Theory **54** (2008), no. 11, 5241–5246.
[6] F. J. MacWilliams and N. J. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1998.
[7] L. Panek, M. Firer, and M. M. S. Alves, *Symmetry groups of Rosenbloom-Tsfasman spaces*, Discrete Math. **309** (2009), no. 4, 763–771.
[8] L. Panek, M. Firer, H. K. Kim, and J. Y. Hyun, *Groups of linear isometries on poset structures*, Discrete Math. **308** (2008), no. 18, 4116–4123.

Jong Youn Hyun
Institute of Mathematical Sciences
Ewha Womans University
Seoul 120-750, Korea
E-mail address: hyun33@postech.ac.kr

Jeongjin Kim
Department of Mathematics
Myungji University
Yongin 449-728, Korea
E-mail address: jjkim@mju.ac.kr

Sang-Mok Kim
Department of Mathematics
Kwangwoon University
Seoul 139-701, Korea
E-mail address: smkim@kw.ac.kr