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AN INTEGRAL REPRESENTATION, SOME INEQUALITIES, AND COMPLETE MONOTONICITY OF THE BERNOULLI NUMBERS OF THE SECOND KIND

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Abstract. In the paper, the authors discover an integral representation, some inequalities, and complete monotonicity of the Bernoulli numbers of the second kind.

1. Introduction

In number theory, the Bernoulli numbers of the second kind b_n for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ may be generated by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n, \tag{1}$$

where $\mathbb{N}$ denotes the set of positive integers. They are also known as the Cauchy numbers of the first kind (see [5, p. 294]), the Gregory coefficients, or logarithmic numbers. The first few Bernoulli numbers of the second kind b_n are

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \quad b_5 = \frac{3}{160}.$$

The first main result of this paper is the following integral representation of b_n for $n \in \mathbb{N}$.

Theorem 1. *The Bernoulli numbers of the second kind b_n may be represented as*

$$b_n = (-1)^{n+1} \int_1^{\infty} \frac{1}{\{[\ln(t-1)]^2 + \pi^2\} t^n} \,\mathrm{d}\, t, \quad n \in \mathbb{N}. \tag{2}$$

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Recall from [21, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0\le n\le\infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is

$$(-1)^k\Delta^k\mu_n\ge0,\quad k,n\in\mathbb{N}_0,$$

where

$$\Delta^k\mu_n=\sum_{m=0}^{k}(-1)^m\binom{k}{m}\mu_{n+k-m}.$$

Recall from [21, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n\ge0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$ and $\mu=(\mu_1,\mu_2,\dots,\mu_n)\in\mathbb{R}^n$. A sequence λ is said to be majorized by μ (in symbols $\lambda\preceq\mu$) if

$$\sum_{\ell=1}^{k}\lambda_{[\ell]}\le\sum_{\ell=1}^{k}\mu_{[\ell]},\quad k=1,2,\dots,n-1\quad\text{and}\quad\sum_{\ell=1}^{n}\lambda_\ell=\sum_{\ell=1}^{n}\mu_\ell,$$

where $\lambda_{[1]}\ge\lambda_{[2]}\ge\dots\ge\lambda_{[n]}$ and $\mu_{[1]}\ge\mu_{[2]}\ge\dots\ge\mu_{[n]}$ are respectively the components of λ and μ in decreasing order. A sequence λ is said to be strictly majorized by μ (in symbols $\lambda\prec\mu$) if λ is not a permutation of μ. For example,

$$\left(\frac1n,\dots,\frac1n\right)\prec\left(\frac1{n-1},\dots,\frac1{n-1},0\right)\prec\left(\frac12,\frac12,0,\dots,0\right)\prec(1,0,\dots,0).$$

For more information on the theory of majorization and its applications, please refer to monographs [8, 9] and closely related references therein.

Based on Theorem 1, the following inequalities and properties of the Bernoulli numbers of the second kind b_n are discovered.

Theorem 2. *The infinite sequence $\{(-1)^nb_{n+1}\}_{n\ge0}$ is completely monotonic and minimal.*

Theorem 3. *Let $m\in\mathbb{N}$ and a_k for $1\le k\le m$ be nonnegative integers. Then*

$$\bigl|(a_k+a_j)!b_{a_k+a_j+1}\bigr|_m\ge0 \tag{3}$$

and

$$\bigl|(-1)^{a_k+a_j}(a_k+a_j)!b_{a_k+a_j+1}\bigr|_m\ge0, \tag{4}$$

where $|a_{kj}|_m$ denotes a determinant of order m with elements a_{kj}.

Theorem 4. *Let $m\in\mathbb{N}$ and let λ and μ be two m-tuples of nonnegative numbers such that $\lambda\preceq\mu$. Then*

$$\left|\prod_{\ell=1}^{m}\lambda_\ell!b_{\lambda_\ell+1}\right|\le\left|\prod_{\ell=1}^{m}\mu_\ell!b_{\mu_\ell+1}\right|. \tag{5}$$

Corollary 1. *The infinite sequence $\{(-1)^nn!b_{n+1}\}_{n\ge0}$ is logarithmically convex.*

2. Lemmas

To prove our main results, we need the following two integral representations.

Lemma 1 ([3, p. 2130])**.** *Let $\mathbb{C}$ be the set of complex numbers and let*

$$\ln z = \ln|z| + i\arg z$$

be the principal branch of the holomorphic extension of $\ln x$ *from the open half-line* $(0,\infty)$ *to the cut plane*

$$\mathcal{A} = \mathbb{C} \setminus (-\infty, 0],$$

where $-\pi < \arg z < \pi$ *and* $i = \sqrt{-1}$ *is the imaginary unit. The function* $\frac{1}{\ln(1+z)}$ *for* $z \in \mathbb{C} \setminus (-\infty, 0]$ *has the integral representation*

$$\frac{1}{\ln(1+z)} = \frac{1}{z} + \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{\mathrm{d}\,t}{z+t}. \tag{6}$$

Lemma 2. *The function*

$$F(z) = \begin{cases} \dfrac{z}{(1+z)\ln(1+z)}, & z \in \mathbb{C} \setminus (-\infty, -1] \setminus \{0\} \\ 1, & z = 0 \end{cases}$$

has the integral representation

$$F(z) = \int_0^\infty \frac{t+1}{t[(\ln t)^2 + \pi^2]} \frac{\mathrm{d}\,t}{t+1+z}, \quad z \in \mathbb{C} \setminus (-\infty, -1]. \tag{7}$$

First proof of Lemma 2. For $z = \varepsilon e^{\theta i}$ with $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\varepsilon \in (0,1)$, by standard argument, we have

$$|zF(z-1)|^2 = \left|\frac{\varepsilon e^{\theta i} - 1}{\ln(\varepsilon e^{\theta i})}\right|^2 = \frac{1 - 2\varepsilon\cos\theta + \varepsilon^2}{(\ln\varepsilon)^2 + \theta^2} \to 0$$

uniformly as $\varepsilon \to 0^+$. Consequently,

$$\lim_{\varepsilon \to 0^+} [zF(z-1)] = 0 \tag{8}$$

uniformly.

For $\theta \in (-\pi, \pi)$ and $z = re^{\theta i}$, by standard argument, we have

$$|F(z-1)| = \left|\frac{re^{\theta i} - 1}{re^{\theta i}\ln(re^{\theta i})}\right| = \sqrt{\frac{1 + 2r\cos\theta + r^2}{r^2[(\ln r)^2 + \theta^2]}} \to 0 \tag{9}$$

uniformly as $r \to \infty$.

For $t \in (0,\infty)$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned} F(-t-1+\varepsilon i) &= \frac{-t-1+\varepsilon i}{(-t+\varepsilon i)\ln(-t+\varepsilon i)} \\ &= \frac{-t-1+\varepsilon i}{(-t+\varepsilon i)[\ln|-t+\varepsilon i| + i\arg(-t+\varepsilon i)]} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-t-1+\varepsilon i}{(-t+\varepsilon i)\bigl[\ln|-t+\varepsilon i| + i\bigl(\pi - \arctan\frac{\varepsilon}{t}\bigr)\bigr]}\\
&\to \frac{t+1}{t(\ln t+\pi i)}\\
&= \frac{(t+1)(\ln t-\pi i)}{t[(\ln t)^2+\pi^2]}
\end{aligned}
$$

as $\varepsilon \to 0^+$. In other words, for $t \in (0,\infty)$,

$$
\lim_{\varepsilon\to0^+} \Im F(-t-1+\varepsilon i) = -\frac{\pi(t+1)}{t[(\ln t)^2+\pi^2]}. \tag{10}
$$

Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D and extendable smoothly to the boundary of D, then

$$
f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(w)}{w-z}\,\mathrm{d}\,w, \quad z\in D, \tag{11}
$$

which is known as the Cauchy integral formula. See [7, p. 113]. For any fixed point $z_0 = x_0 + iy_0 \in \mathbb{C}\setminus(-\infty,0]$, choose ε and r such that

$$
\begin{cases}
0<\varepsilon<|y_0|\le|z_0|<r, & y_0\ne0,\\
0<\varepsilon<x_0=|z_0|<r, & y_0=0,
\end{cases}
$$

and consider the positively oriented contour $C(\varepsilon,r)$ in $\mathbb{C}\setminus(-\infty,-1]$ consisting of the half circle $z=-1+\varepsilon e^{\theta i}$ for $\theta\in\bigl[-\frac{\pi}{2},\frac{\pi}{2}\bigr]$ and the half lines $z=-1+x\pm\varepsilon i$ for $x\le0$ until they cut the circle $|z+1|=r$, which close the contour at the points $-1-r(\varepsilon)\pm\varepsilon i$, where $0<r(\varepsilon)\to r$ as $\varepsilon\to0$. By the formula (11), we have

(12)

$$
\begin{aligned}
F(z_0) = \frac{1}{2\pi i}\biggl[&\int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{\theta i}F\bigl(\varepsilon e^{\theta i}-1\bigr)}{\varepsilon e^{\theta i}-1-z_0}\,\mathrm{d}\,\theta + \int_{-r(\varepsilon)}^{0}\frac{F(x-1+\varepsilon i)}{x-1+\varepsilon i-z_0}\,\mathrm{d}\,x\\
&+\int_0^{-r(\varepsilon)}\frac{F(x-1-\varepsilon i)}{x-1-\varepsilon i-z_0}\,\mathrm{d}\,x + \int_{\arg[-r(\varepsilon)-\varepsilon i]}^{\arg[-r(\varepsilon)+\varepsilon i]}\frac{ire^{\theta i}F\bigl(re^{\theta i}-1\bigr)}{re^{\theta i}-1-z_0}\,\mathrm{d}\,\theta\biggr].
\end{aligned}
$$

By the formula (8), it follows that

$$
\lim_{\varepsilon\to0^+}\int_{\pi/2}^{-\pi/2}\frac{i\varepsilon e^{\theta i}F\bigl(\varepsilon e^{\theta i}-1\bigr)}{\varepsilon e^{\theta i}-1-z_0}\,\mathrm{d}\,\theta = 0. \tag{13}
$$

In virtue of the limit (9), it can be derived that

$$
\begin{aligned}
&\lim_{\substack{\varepsilon\to0^+\\ r\to\infty}}\int_{\arg[-r(\varepsilon)-\varepsilon i]}^{\arg[-r(\varepsilon)+\varepsilon i]}\frac{ire^{\theta i}F\bigl(re^{\theta i}-1\bigr)}{re^{\theta i}-1-z_0}\,\mathrm{d}\,\theta\\
&=\lim_{r\to\infty}\int_{-\pi}^{\pi}\frac{ire^{\theta i}F\bigl(re^{\theta i}-1\bigr)}{re^{\theta i}-1-z_0}\,\mathrm{d}\,\theta\\
&=0.
\end{aligned} \tag{14}
$$

Making use of the obvious fact that $F(\overline{z})=\overline{F(z)}$ and the limit (10) yields that

$$
\begin{aligned}
&\int_{-r(\varepsilon)}^{0}\frac{F(x-1+\varepsilon i)}{x-1+\varepsilon i-z_0}\,\mathrm{d}\,x+\int_{0}^{-r(\varepsilon)}\frac{F(x-1-\varepsilon i)}{x-1-\varepsilon i-z_0}\,\mathrm{d}\,x\\
&=\int_{-r(\varepsilon)}^{0}\biggl[\frac{F(x-1+\varepsilon i)}{x-1+\varepsilon i-z_0}-\frac{F(x-1-\varepsilon i)}{x-1-\varepsilon i-z_0}\biggr]\,\mathrm{d}\,x\\
&=2i\int_{-r(\varepsilon)}^{0}\frac{(x-1-z_0)\Im F(x-1+\varepsilon i)-\varepsilon\Re F(x-1+\varepsilon i)}{(x-1+\varepsilon i-z_0)(x-1-\varepsilon i-z_0)}\,\mathrm{d}\,x\\
&\to 2i\int_{-r}^{0}\frac{\lim_{\varepsilon\to0^+}\Im F(x-1+\varepsilon i)}{x-1-z_0}\,\mathrm{d}\,x\\
&=-2i\int_{0}^{r}\frac{\lim_{\varepsilon\to0^+}\Im F(-t-1+\varepsilon i)}{t+1+z_0}\,\mathrm{d}\,t\\
&\to -2i\int_{0}^{\infty}\frac{\lim_{\varepsilon\to0^+}\Im F(-t-1+\varepsilon i)}{t+1+z_0}\,\mathrm{d}\,t\\
&=2\pi i\int_{0}^{\infty}\frac{t+1}{t[(\ln t)^2+\pi^2]}\frac{\mathrm{d}\,t}{t+1+z_0}
\end{aligned}
\tag{15}
$$

as $\varepsilon\to0^+$ and $r\to\infty$. Substituting equations (13), (14), and (15) into (12) and simplifying produce the integral representation (7). The proof of Lemma 2 is complete. □

Second proof of Lemma 2. In all treatments of Pick functions, a main example is the principal logarithm ln defined in the cut plane $\mathcal{A}$ as well as

$$-\frac{1}{\ln z}=-\frac{1}{z-1}+\int_{-\infty}^{0}\frac{1}{(t-z)[(\ln t)^2+\pi^2]}\,\mathrm{d}\,t.$$

This formula is equivalent to [2, (1.4)]. Multiplying the identity

$$\int_0^\infty\frac{1}{t[(\ln t)^2+\pi^2]}=1$$

by $\frac{1}{z}$ and inserting it in the previous formula yield

$$\frac{z-1}{z\ln z}=\int_0^\infty\biggl[\frac{1}{tz}+\frac{z-1}{z(t+z)}\biggr]\frac{\mathrm{d}\,t}{(\ln t)^2+\pi^2}=\int_0^\infty\frac{1+t}{(t+z)[(\ln t)^2+\pi^2]}\,\mathrm{d}\,t,$$

which is the formula (7). The proof of Lemma 2 is complete. □

3. Proofs of theorems

Now we prove Theorems 1 to 4 and Corollary 1.

First proof of Theorem 1. By (6), we have

$$\frac{x}{\ln(1+x)}=1+\int_1^\infty\frac{1}{[\ln(t-1)]^2+\pi^2}\frac{x}{x+t}\,\mathrm{d}\,t\tag{16}$$

and

$$\begin{aligned}\left[\frac{x}{\ln(1+x)}\right]^{(k)} &= \int_1^\infty \frac{1}{[\ln(t-1)]^2+\pi^2}\left(\frac{x}{x+t}\right)^{(k)} \mathrm{d}\,t \\ &= \int_1^\infty \frac{1}{[\ln(t-1)]^2+\pi^2}\left(1-\frac{t}{x+t}\right)^{(k)} \mathrm{d}\,t \\ &= (-1)^{k+1}k!\int_1^\infty \frac{t}{[\ln(t-1)]^2+\pi^2}\frac{1}{(x+t)^{k+1}}\,\mathrm{d}\,t\end{aligned} \tag{17}$$

for $k\in\mathbb{N}$. On the other hand, by (1), we also have

$$\left[\frac{x}{\ln(1+x)}\right]^{(k)} = \sum_{n=k}^\infty b_n\frac{n!}{(n-k)!}x^{n-k}. \tag{18}$$

Combining (17) with (18) leads to

$$\sum_{n=k}^\infty b_n\frac{n!}{(n-k)!}x^{n-k} = (-1)^{k+1}k!\int_1^\infty \frac{t}{[\ln(t-1)]^2+\pi^2}\frac{1}{(x+t)^{k+1}}\,\mathrm{d}\,t.$$

Letting $x\to0^+$ on both sides of the above equation produces

$$k!b_k = (-1)^{k+1}k!\int_1^\infty \frac{1}{[\ln(t-1)]^2+\pi^2}\frac{1}{t^k}\,\mathrm{d}\,t.$$

Thus, the formula (2) is proved. □

Second proof of Theorem 1. By the integral representation (7), we have

$$\frac{x}{\ln(1+x)} = \int_1^\infty \frac{t}{(t-1)\{[\ln(t-1)]^2+\pi^2\}}\frac{1+x}{x+t}\,\mathrm{d}\,t$$

and

$$\begin{aligned}\left[\frac{x}{\ln(1+x)}\right]^{(k)} &= \int_1^\infty \frac{t}{(t-1)\{[\ln(t-1)]^2+\pi^2\}}\left(\frac{1+x}{x+t}\right)^{(k)} \mathrm{d}\,t \\ &= \int_1^\infty \frac{t}{(t-1)\{[\ln(t-1)]^2+\pi^2\}}\left(1+\frac{1-t}{x+t}\right)^{(k)} \mathrm{d}\,t \\ &= (-1)^{k+1}k!\int_1^\infty \frac{t}{[\ln(t-1)]^2+\pi^2}\frac{1}{(x+t)^{k+1}}\,\mathrm{d}\,t\end{aligned} \tag{19}$$

for $k\in\mathbb{N}$. Combining (19) with (18) leads to

$$\sum_{n=k}^\infty b_n\frac{n!}{(n-k)!}x^{n-k} = (-1)^{k+1}k!\int_1^\infty \frac{t}{[\ln(t-1)]^2+\pi^2}\frac{1}{(x+t)^{k+1}}\,\mathrm{d}\,t. \tag{20}$$

Letting $x\to0^+$ on both sides of (20) yields the formula (2). The proof of Theorem 1 is complete. □

First proof of Theorem 2. Theorem 4a in [21, p. 108] reads that a necessary and sufficient condition that the sequence $\{\mu_n\}_0^\infty$ should have the expression

$$\mu_n = \int_0^1 t^n \,\mathrm{d}\,\alpha(t) \tag{21}$$

for $n \ge 0$, where $\alpha(t)$ is non-decreasing and bounded for $0 \le t \le 1$, is that it should be completely monotonic. Theorem 14a in [21, p. 164] states that a completely monotonic sequence $\{\mu_n\}_{n\ge0}$ is minimal if and only if the equality (21) is valid for $n \ge 0$ and $\alpha(t)$ is a non-decreasing bounded function continuous at $t = 0$.

Setting in the equality (21)

$$\alpha(t) = \int_0^t \frac{1}{s\{[\ln(1/s-1)]^2+\pi^2\}}\,\mathrm{d}\,s$$

for $t \in [0,1]$ and $\alpha(1) = b_1 = \frac12$ yields the required complete monotonicity and minimality. $\square$

Second proof of Theorem 2. From (2), it follows that for $n \in \mathbb{N}$

$$\begin{aligned}
(-1)^{n+1}b_n &= \int_1^\infty \frac{1}{\{[\ln(t-1)]^2+\pi^2\}t^n}\,\mathrm{d}\,t \\
&= \int_1^0 \frac{1}{\{[\ln(1/s-1)]^2+\pi^2\}} s^n \,\mathrm{d}\biggl(\frac1s\biggr) \\
&= \int_0^1 \frac{1}{\{[\ln(1/s-1)]^2+\pi^2\}} s^{n-2}\,\mathrm{d}\,s \\
&= \int_0^1 \frac{1}{s\{[\ln(1/s-1)]^2+\pi^2\}} s^{n-1}\,\mathrm{d}\,s \\
&\triangleq c_{n-1}.
\end{aligned}$$

Since $c_0 = b_1 = \frac12$ and the function $\frac{1}{s\{[\ln(1/s-1)]^2+\pi^2\}}$ is positive on $(0,1)$, then the complete monotonicity and minimality of the sequence $\{c_n\}_0^\infty$ is readily obtained. The proof of Theorem 2 is complete. $\square$

Proof of Theorem 3. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$. See [11, Chapter XIII] and [21, Chapter IV].

From the proofs of Theorem 1, we observe that

$$b_n = (-1)^{n+1} \lim_{x\to0^+} h_n(x) \tag{22}$$

and

$$h_n(x) = \int_1^\infty \frac{1}{\{[\ln(t-1)]^2+\pi^2\}(t+x)^n}\,\mathrm{d}\,t \tag{23}$$

is completely monotonic on $[0,\infty)$.

In [10], or see [11, p. 367], it was obtained that if f is a completely monotonic function on $[0,\infty)$, then

$$\left|f^{(a_i+a_j)}(x)\right|_m \ge 0 \tag{24}$$

and

$$\left|(-1)^{a_i+a_j} f^{(a_i+a_j)}(x)\right|_m \ge 0, \tag{25}$$

where $|a_{ij}|_m$ denotes a determinant of order m with elements a_{ij} and a_i for $1 \le i \le m$ are nonnegative integers. Applying f in (24) and (25) to the function $h_n(x)$ yields

$$\left|h_n^{(a_i+a_j)}(x)\right|_m \ge 0$$

and

$$\left|(-1)^{a_i+a_j} h_n^{(a_i+a_j)}(x)\right|_m \ge 0,$$

that is,

$$\left|(-1)^{a_i+a_j}\frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)\right|_m \ge 0 \tag{26}$$

and

$$\left|\frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)\right|_m \ge 0. \tag{27}$$

Letting $x \to 0^+$ in (26) and (27) and making use of (22) produce

$$\left|(-1)^{a_i+a_j}\frac{(n+a_i+a_j-1)!}{(n-1)!}(-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}\right|_m \ge 0 \tag{28}$$

and

$$\left|\frac{(n+a_i+a_j-1)!}{(n-1)!}(-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}\right|_m \ge 0. \tag{29}$$

Further simplifying (28) and (29) leads to

$$\left|(-1)^{n+1}(n+a_i+a_j-1)! b_{n+a_i+a_j}\right|_m \ge 0$$

and

$$\left|(-1)^{n+a_i+a_j+1}(n+a_i+a_j-1)! b_{n+a_i+a_j}\right|_m \ge 0,$$

which are equivalent to (3) and (4). Theorem 3 is thus proved. □

Proof of Theorem 4. In [20, p. 106, Theorem A] and [11, p. 367, Theorem 2], a minor correction of [6, Theorem 1], it was obtained that if f is a completely monotonic function on $(0,\infty)$ and $\lambda \preceq \mu$, then

$$\left|\prod_{i=1}^{n} f^{(\lambda_i)}(x)\right| \le \left|\prod_{i=1}^{n} f^{(\mu_i)}(x)\right|. \tag{30}$$

Applying the inequality (30) to $h_n(x)$, defined by (23), creates

$$\left|\prod_{i=1}^{m}(-1)^{\lambda_i}\frac{(n+\lambda_i-1)!}{(n-1)!} h_{n+\lambda_i}(x)\right| \le \left|\prod_{i=1}^{m}(-1)^{\mu_i}\frac{(n+\mu_i-1)!}{(n-1)!} h_{n+\mu_i}(x)\right|$$

which can be simplified as

$$\left|\prod_{i=1}^{m}(n+\lambda_i-1)!h_{n+\lambda_i}(x)\right| \le \left|\prod_{i=1}^{m}(n+\mu_i-1)!h_{n+\mu_i}(x)\right|.$$

Further taking $x\to0^+$ and utilizing (22) turn out

$$\left|\prod_{i=1}^{m}(n+\lambda_i-1)!(-1)^{n+\lambda_i+1}b_{n+\lambda_i}\right| \le \left|\prod_{i=1}^{m}(n+\mu_i-1)!(-1)^{n+\mu_i+1}b_{n+\mu_i}\right|$$

which is equivalent to (5). The proof of Theorem 4 is complete. □

Proof of Corollary 1. It is clear that $(i,i+2)\succ(i+1,i+1)$ for $i\ge0$. Therefore, by virtue of (5), we have

$$(i!b_{i+1})[(i+2)!b_{i+3}]\ge[(i+1)!b_{i+2}]^2.$$

This implies the required logarithmic convexity.

This conclusion can also be deduced from Theorem 3. The proof of Theorem 1 is thus complete. □

4. Remarks

Finally, we would like to give some remarks on something related to the integral representations (6) and (7).

Remark 1. In [1, p. 230, 5.1.32], it is listed that

$$\ln\frac{b}{a}=\int_0^\infty\frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}\,u.$$

As a result, we have

$$\ln[\ln(1+x)]=\int_0^\infty\frac{e^{-u}-e^{-u\ln(1+x)}}{u}\,\mathrm{d}\,u=\int_0^\infty\frac{e^{-u}-(1+x)^{-u}}{u}\,\mathrm{d}\,u$$

and, by a differentiation,

$$\begin{aligned}\frac{1}{(1+x)\ln(1+x)}&=\int_0^\infty\frac{1}{(1+x)^{u+1}}\,\mathrm{d}\,u\\&=\int_0^\infty\left[\frac{1}{\Gamma(1+u)}\int_0^\infty t^ue^{-(1+x)t}\,\mathrm{d}\,t\right]\mathrm{d}\,u\\&=\int_0^\infty\left[\int_0^\infty\frac{t^u}{\Gamma(1+u)}\,\mathrm{d}\,u\right]e^{-(1+x)t}\,\mathrm{d}\,t,\end{aligned}\tag{31}$$

where $\Gamma(z)$ is the classical gamma function which may be defined by the Euler integral

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0.\tag{32}$$

The integral representation (31) means that $\frac{1}{(1+x)\ln(1+x)}$ is a completely monotonic function on $(0,\infty)$. In other words, the function $\frac{1}{\ln(1+x)}$ is logarithmically completely monotonic on $(0,\infty)$. More strongly, it was claimed in [3, p. 2130, (34)] and [4, p. 12, (33)] that the function $\frac{1}{\ln(1+x)}$ is a Stieltjes transform. For information on the notions "logarithmically completely monotonic function" and "Stieltjes transform", please refer to [14, Remark 8], [15, Section 1], [16, Remark 4.7], the monograph [18], and many other closely-related references therein.

From (31) and by integration by part, it is not difficult to obtain that

$$\frac{1}{\ln(1+x)}=\int_0^\infty\biggl[\int_0^\infty\frac{t^{u-1}}{\Gamma(u)}\,\mathrm{d}\,u\biggr]e^{-(1+x)t}\,\mathrm{d}\,t,\quad x>0.$$

By induction and integration by part, we can obtain

$$\begin{aligned}\frac{(1+x)^k}{\ln(1+x)}&=\int_0^\infty\biggl[\int_0^\infty\frac{t^{u-k-1}}{\Gamma(u-k)}\,\mathrm{d}\,u\biggr]e^{-(1+x)t}\,\mathrm{d}\,t\\&=\int_0^\infty\biggl[\int_{-k}^\infty\frac{t^{u-1}}{\Gamma(u)}\,\mathrm{d}\,u\biggr]e^{-(1+x)t}\,\mathrm{d}\,t\end{aligned}$$

for $x>0$ and $k\in\mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers and the classical gamma function $\Gamma(z)$ given in (32) may be extended analytically to $\mathbb{C}\setminus\{0,-1,-2,\dots\}$ by the Gauss formula

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

See [19, Section 1.1].

Remark 2. By the way, the term $\frac{1}{z}$ in (6) was lost in [3, p. 2130, (34)] and [4, p. 12, (33)] and was corrected in [17, 22].

Remark 3. The integral representation (7) in Lemma 2 has been utilized in the paper [13].

Remark 4. A function $f:I\subseteq(0,\infty)\to[0,\infty)$ is called a Bernstein function on I if $f(x)$ has derivatives of all orders and $f'(x)$ is completely monotonic on I. See the monograph [18]. We claim that the generating function $\frac{x}{\ln(1+x)}$ of the Bernoulli numbers of the second kind b_k is a Bernstein function on $(0,\infty)$. This can be proved by two approaches below.

(1) The integral representation (16) shows us that the function $\frac{x}{\ln(1+x)}$ is positive and increasing on $(0,\infty)$. The integral representation (17) reveals that the first derivative of $\frac{x}{\ln(1+x)}$ is completely monotonic on $(0,\infty)$. So the function $\frac{x}{\ln(1+x)}$ is a Bernstein function on $(0,\infty)$.
(2) It is not difficult to see that

$$\frac{x}{\ln(1+x)}=\int_0^1(1+x)^t\,\mathrm{d}\,t$$

and the function $(1+x)^t$ for $t\in(0,1)$ is a Bernstein function.

Remark 5. This paper is a combined and revised version of the preprints [12, 17] and Chapter 5 of the thesis [22].

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