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RESOLUTION OF UNMIXED BIPARTITE GRAPHS

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Abstract. Let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ with the edge set $E(G)$, and let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. Two monomial ideals are associated to G, the *edge ideal* $I(G)$ generated by all monomials x_ix_j with $\{x_i, x_j\} \in E(G)$, and the *vertex cover ideal* I_G generated by monomials $\prod_{x_i \in C} x_i$ for all minimal vertex covers C of G. A *minimal vertex cover* of G is a subset $C \subset V(G)$ such that each edge has at least one vertex in C and no proper subset of C has the same property. Indeed, the vertex cover ideal of G is the Alexander dual of the edge ideal of G. In this paper, for an unmixed bipartite graph G we consider the lattice of vertex covers $\mathcal{L}_G$ and we explicitly describe the minimal free resolution of the ideal associated to $\mathcal{L}_G$ which is exactly the vertex cover ideal of G. Then we compute depth, projective dimension, regularity and extremal Betti numbers of $R/I(G)$ in terms of the associated lattice.

Introduction

In recent years the ideals associated to graphs have been intensively studied. One central question in this context is to describe the minimal free resolution of these ideals, and to give some explicit combinatorial formulas for their homological invariants. In general it is hard to give uniform formulas for the projective dimension or the regularity for all graphs. Nevertheless there are some classes of graphs where the homological data of the edge ideals can be described. The problem has been studied for two well-known families of graphs, chordal and bipartite graphs. For Cohen-Macaulay bipartite graphs a nice and complete answer is known, see [5]. In particular, in these cases one knows the projective dimension of the edge ideal. There are also several papers in which the regularity of edge ideals has been studied, see [3, 4, 11, 12, 15, 17].

In this paper we consider *unmixed* bipartite graphs in which all minimal vertex covers are of the same cardinality. Let G be an unmixed bipartite graph with vertex set $V(G) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$. Then there exists a perfect matching for G, see [16]. Therefore we may assume that $\{x_i, y_i\}$ is an edge of G for all i and it follows that $m = n$. So each minimal vertex cover of G is of

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the form $\{x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_n}\}$, where $\{i_1, \ldots, i_n\} = [n]$. Let B_n be the Boolean lattice on the set $X = \{x_1, \ldots, x_n\}$. In [6] it is shown that the set

$$\mathcal{L}_G = \{C \cap \{x_1, \ldots, x_n\} : C \text{ is a minimal vertex cover of } G\}$$

is a sublattice of B_n which contains $\emptyset$ and X, and for any such sublattice $\mathcal{L}$ of B_n, there exists an unmixed bipartite graph G such that $\mathcal{L} = \mathcal{L}_G$. Here $\wedge$ and $\vee$ in $\mathcal{L}_G$ are just taking the intersection and union. In our description of the resolution of the vertex cover ideal of an unmixed bipartite graph, we use in a substantial way this lattice associated to the graph. Attached to the lattice $\mathcal{L}_G$ is a monomial ideal $H_{\mathcal{L}_G} := (u_p : \ p \in \mathcal{L}_G)$ in the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where $u_p = X_p Y_{[n]\setminus p}$ for $X_p = \prod_{i \in p} x_i$ and $Y_{[n]\setminus p} = \prod_{j \in [n]\setminus p} y_j$.

In Section 1 we describe the multigraded minimal free resolution of the vertex cover ideal of G in terms of its associated lattice. In fact, the resolution we describe in Theorem 1.2 is a variation of the resolution of Hibi ideals of meet-distributive semilattices, given in [8, Theorem 2.1]. More important is the fact, that the multigraded basis elements of the resolution, can be identified with the Boolean sublattices of $\mathcal{L}_G$, see Proposition 2.1. Having this identification, it turns out that the multigraded extremal Betti numbers correspond to the maximal Boolean sublattices of $\mathcal{L}_G$.

In Section 2 we compute the homological invariants of $I(G)$ in terms of the lattice. Since the ideal $H_{\mathcal{L}_G} = I_G$ is the Alexander dual of $I(G)$, we may apply the Bayer-Charalambous-Popescu theorem [1, Theorem 2.8] which relates the multigraded extremal Betti numbers of $H_{\mathcal{L}_G}$ and $I(G)$. With this information at hand, we can express the depth and regularity of $R/I(G)$ in terms of the lattice $\mathcal{L}_G$ and invariants of the graph G, see Corollaries 2.3 and 2.4. One can also obtain a lower bound for the last nonzero total Betti number of $R/I(G)$. We do not know of any example where this lower bound is not achieved. It would always be obtained if one could prove the following: all nonzero multigraded Betti numbers in the last step of the resolution are extremal (in the multigraded sense). There is a simple argument, given in the proof of Proposition 2.10 that whenever I is a Cohen-Macaulay monomial ideal, then all multigraded extremal Betti numbers appear at the end of the resolution of I.

Kummini in [13] has also computed the depth and regularity of unmixed bipartite graphs, but his approach and the terms in which he expresses these invariants differ from ours.

1. Minimal free resolution of $H_{\mathcal{L}_G}$

The purpose of this section is to construct a resolution of the ideal $H_{\mathcal{L}_G}$ which is a modification of the resolution given [8, Theorem 2.1] adopted to our situation. The differences between our resolution and the one given in [8] arise from the fact that the lattices under consideration are embedded differently into Boolean lattices. This fact is important, because the multidegrees of the resolution depend upon the embedding. For $p \in \mathcal{L}_G$, the set of lower neighbors of p in $\mathcal{L}_G$ is denoted by $N(p)$. In order to guarantee the minimality of the resolution which we are going to describe, we need the following result.

Lemma 1.1. *Let $p \in \mathcal{L}_G$. For any two distinct subsets $S, S' \subseteq N(p)$, we have $\wedge\{q : q \in S\} \neq \wedge\{q : q \in S'\}$.*

Proof. Let S and S' be distinct subsets of $N(p)$. One can assume that $S \nsubseteq S'$. Let $q_1 \in S \setminus S'$. If $\wedge\{q : q \in S\} = \wedge\{q : q \in S'\}$, then

$$(\wedge\{q : q \in S\}) \vee q_1 = (\wedge\{q : q \in S'\}) \vee q_1.$$

Since $\mathcal{L}_G$ is distributive $\wedge\{q \vee q_1 : q \in S\} = \wedge\{q \vee q_1 : q \in S'\}$. For any $q \in S'$, $q \vee q_1 = p$, therefore $\wedge\{q \vee q_1 : q \in S'\} = p$. But then $\wedge\{q \vee q_1 : q \in S\} = q_1$, a contradiction. $\square$

From the above lemma it is easy to see that for every pair of subsets S and S' of $N(p)$ with $S \subseteq S'$, we have $|S'| - |S| \leq |\wedge\{q : q \in S\}| - |\wedge\{q : q \in S'\}|$. For $p, q \in \mathcal{L}_G$, the sublattice of $\mathcal{L}_G$ with maximal element q and minimal element p is called an interval of $\mathcal{L}_G$ and is denoted by $[p, q]$. In the following we denote by $\hat{0}$ and $\hat{1}$ the minimal and maximal element of $\mathcal{L}_G$, respectively. For any $p \in \mathcal{L}$, the rank of p which is denoted by $\operatorname{rank}(p)$, is the maximal length of chains descending from p. We extend the partial rank order on $\mathcal{L}_G$ to a total order $\prec$.

The proof of the following theorem is similar, up to some modification, to the proof of [8, Theorem 2.1], which was proved for a meet-semilattice. For more emphasis we state the proof in the lattice case.

Theorem 1.2. *There exists a minimal multigraded free resolution $\mathbb{F}$ of $H_{\mathcal{L}_G}$ such that for each $i \geq 0$, the free module $\mathbb{F}_i$ has a basis with basis elements $b(p; S)$, where $p \in \mathcal{L}_G$ and S is a subset of the set of lower neighbors $N(p)$ of p with $|S| = i$ and multidegree of $b(p; S)$ is the least common multiple of u_p and all u_q with $q \in S$.*

Proof. The construction of resolution is as in the proof of [8, Theorem 2.1] by mapping cone. For any $p \in \mathcal{L}_G$ we construct inductively a complex $\mathbb{F}(p)$ which is a multigraded free resolution of the ideal $H_{\mathcal{L}_G}(p) = (u_q : q \preceq p)$. The complex $\mathbb{F}(\hat{0})$ is defined as $\mathbb{F}_i(\hat{0}) = 0$ for $i > 0$ and $\mathbb{F}_0(\hat{0}) = S$. Now, let $p \in \mathcal{L}_G$ and $q \in \mathcal{L}_G$, $q \prec p$ be the element preceding p. Then $H_{\mathcal{L}_G}(p) = (H_{\mathcal{L}_G}(q), u_p)$, and hence we have the exact sequence of multigraded S-modules

$$0 \longrightarrow (S/L)(-\operatorname{multideg} u_p) \longrightarrow S/H_{\mathcal{L}_G}(q) \longrightarrow S/H_{\mathcal{L}_G}(p) \longrightarrow 0,$$

where L is the colon ideal $H_{\mathcal{L}_G}(q) : u_p$. Let $u_{q'}/\gcd(u_{q'}, u_p) \in L$, where $q' \prec q$ and let $t \in [q' \wedge q, p] \cap N(p)$. Then $u_t/\gcd(u_t, u_p)$ divides $u_{q' \wedge p}/\gcd(u_{q' \wedge p}, u_p)$ and $u_{q' \wedge p}/\gcd(u_{q' \wedge p}, u_p)$ divides $u'_q/\gcd(u'_q, u_p)$. Therefore we have

$$L = (\{u_t/\gcd(u_t, u_p)\}_{t \in N(p)}).$$

Let $\mathbb{T}$ be the Taylor complex associated with the sequence $u_t/\gcd(u_t,u_p)$, $t\in N(p)$, where the order of the sequence is given by the order $\prec$ on the elements of $\mathcal{L}_G$. Then T_i has a basis with elements $e_{t_1}\wedge e_{t_2}\wedge\cdots\wedge e_{t_i}$, where $t_1\prec t_2\prec\cdots\prec t_i$. The multidegree of $e_{t_1}\wedge e_{t_2}\wedge\cdots\wedge e_{t_i}$ is the least common multiple of the elements $u_{t_j}/\gcd(u_{t_j},u_p)$ for $j=1,\dots,i$. The shifted complex $\mathbb{T}(-\operatorname{multideg}(u_p))$ is a multigraded free resolution of $(S/L)(-\operatorname{multideg}(u_p))$. Let $b(p;t_1,\dots,t_i)$ be the basis element of $\mathbb{T}_i(-\operatorname{multideg}(u_p))$ corresponding to $e_{t_1}\wedge e_{t_2}\wedge\cdots\wedge e_{t_i}$. Then

$$\begin{aligned}\operatorname{multideg}(b(p;t_1,\dots,t_i)) &= \operatorname{multideg}(u_p)+\operatorname{multideg}(e_{t_1}\wedge e_{t_2}\wedge\cdots\wedge e_{t_i})\\ &= \operatorname{lcm}(u_p,u_{t_1},\dots,u_{t_i}).\end{aligned}$$

This resolution is minimal since for any $t_1\prec t_2\prec\cdots\prec t_i$ we have

$$\operatorname{lcm}(f_1,\dots,f_i)/\operatorname{lcm}(f_1,\dots,\hat{f}_j,\dots,f_i)=Y_{p\setminus\bigwedge_{l=1}^{i}t_l}/Y_{p\setminus\bigwedge_{\substack{l=1\\ l\neq j}}^{i}t_l}\neq 1.$$

The monomorphism $(S/L)(-\operatorname{multideg}(u_p))\longrightarrow S/H_{\mathcal{L}_G}(q)$ induces a comparison map $\alpha:\mathbb{T}(-\operatorname{multideg}(u_p))\longrightarrow\mathbb{F}(q)$ of multigraded complexes. Let $\mathbb{F}(p)$ be the mapping cone of α. Then $\mathbb{F}(p)$ is a multigraded free S-resolution of $H_{\mathcal{L}_G}(p)$ with the desired multigraded basis.

We claim that this resolution is minimal. For any two basis elements $b(p;S)$ and $b(q;T)$ with $|T|=|S|-1$, we show that the coefficient of $b(q;T)$ in $\partial b(p;S)$ is either zero or a monomial $\neq 1$. First assume that $p=q$. If $T\subseteq S$, then the coefficient is $\operatorname{multideg}(b(p;S))/\operatorname{multideg}(b(p,T))=Y_A$, where $A=\wedge\{r:r\in T\}\setminus\wedge\{r:r\in S\}$. Since A is a nonempty set by Lemma 1.1, then $Y_A\neq 1$. If $T\nsubseteq S$ and $\operatorname{multideg}(b(p;T))$ divides $\operatorname{multideg}(b(p;S))$, then $\wedge\{r:r\in S\}\preceq\wedge\{r:r\in T\}$. Therefore $\wedge\{r:r\in N(p)\}=\wedge\{r:r\in N(p)\setminus(T\setminus S)\}$, which is a contradiction by Lemma 1.1. Now, assume that $q\prec p$. If $\operatorname{multideg}(b(q;T))$ divides $\operatorname{multideg}(b(p;S))$, then the coefficient is $X_{p\setminus q}Y_B$ for some set $B\subseteq[n]$, and so it is not 1. In the remaining case $q\not\prec p$, $\operatorname{multideg}(b(q;T))$ does not divide $\operatorname{multideg}(b(p;S))$. □

2. Some homological invariants of $I(G)$

Here we apply the results of the first section to compute the homological invariants of the edge ideal of an unmixed bipartite graph. The next observation is of crucial importance for understanding the i-extremal and extremal Betti numbers of $H_{\mathcal{L}_G}$ and $I(G)$.

Proposition 2.1. *There is a correspondence between the basis elements $b(p;S)$ and intervals in $\mathcal{L}_G$, which are isomorphic to a Boolean lattice.*

Proof. For any basis element $b(p;S)$, we show that the interval $[\wedge\{q:q\in S\},p]$ is isomorphic to $B_{|S|}$. Let $S=\{q_1,\dots,q_n\}$ and $v_i=\wedge\{q:q\in S\setminus\{q_i\}\}$. The interval $[\wedge\{q:q\in S\},p]$ is a Boolean lattice on $v_1,\dots,v_n$. For any $I\in[\wedge\{q:q\in S\},p]$, let $i_1,\dots,i_k$ be the indices such that $I\leq q_{i_j}$, $1\leq j\leq k$. Then $I\preceq\wedge\{q_{i_j}:1\leq j\leq k\}$. If $I\neq\wedge\{q_{i_j}:1\leq j\leq k\}$, then there exists an

$x \in \wedge\{q_{i_j} : 1 \leq j \leq k\} \setminus I$. Since $\wedge\{q : q \in S\} \preceq I$, there exists $1 \leq l \leq n$, $l \neq i_1, \ldots, i_k$ such that $x \notin q_l$. But then $x \notin I \vee q_l = p$, a contradiction. Thus $I = \wedge\{q_{i_j} : 1 \leq j \leq k\} = \vee\{v_j : 1 \leq j \leq n,\ j \neq i_1, \ldots, i_k\}$.

Let $\mathfrak{N}$ be a function from the set of basis elements to intervals in $\mathcal{L}_G$, which are isomorphic to a Boolean lattice such that $\mathfrak{N}(b(p;S)) = [\wedge\{q : q \in S\}, p]$. From Lemma 1.1 we know that $\mathfrak{N}$ is a monomorphism. For any interval $[J, I]$ in $\mathcal{L}_G$ isomorphic to a Boolean lattice, set $S = N(I) \cap [J, I]$. Then $[J, I] = [\wedge\{q : q \in S\}, I]$ and $\mathfrak{N}$ is surjective. $\square$

In the following we denote by A_G the set of elements $p \in \mathcal{L}_G$ such that the interval $[\wedge\{r : r \in N(p)\}, p]$ is isomorphic to a maximal Boolean lattice in $\mathcal{L}_G$. By a maximal Boolean lattice we mean a Boolean lattice in $\mathcal{L}_G$ which is not a sublattice of another Boolean lattice in $\mathcal{L}_G$.

Lemma 2.2. *Let $p \in A_G$ and $q \in \mathcal{L}_G$. If $[\wedge\{r : r \in N(q)\}, q] \subseteq [\wedge\{r : r \in N(p)\}, p]$, then we have*

$$|q| - |N(q)| - | \wedge \{r : r \in N(q)\}| \leq |p| - |N(p)| - | \wedge \{r : r \in N(p)\}|.$$

Proof. Let $[\wedge\{r : r \in N(p)\}, p]$ be a Boolean lattice on the elements $v_1, \ldots, v_n$. Then $|N(p)| = n$, $|N(q)| \leq n$ and $v_i \wedge v_j = \wedge\{r : r \in N(p)\}$ for any $1 \leq i < j \leq n$. Also $v_1 \vee \cdots \vee v_n = p$. Without loss of generality assume that $q = v_1 \vee \cdots \vee v_m$ for some $m < n$. Then $|N(q)| = m$. We claim for every $1 \leq i \leq n$, there exists an element $x_i \in v_i$ such that x_i is not in any other v_j, $1 \leq j \leq n$. Otherwise let $1 \leq i \leq n$ be such that for any $x \in v_i$, there exists $j \neq i$ with $x \in v_j$. This means that $v_i \leq \wedge\{r : r \in N(p)\}$ which is a contradiction. Thus $|p| - |q| \geq n - m$. Since $| \wedge \{r : r \in N(p)\}| \leq | \wedge \{r : r \in N(q)\}|$, we get the inequality. $\square$

As a first corollary we obtain:

Corollary 2.3. *Let G be an unmixed bipartite graph on (X, Y) such that $|X| = |Y| = n$ and $A_G \subseteq \mathcal{L}_G$ be the set defined above. Then*

(i) $\operatorname{depth}(R/I(G)) = n - \max_{p \in A_G}\{|p| - |N(p)| - | \wedge \{r : r \in N(p)\}|\}$;
(ii) $\operatorname{reg}(R/I(G)) = \max_{p \in \mathcal{L}_G} |N(p)|$.

Proof. For any basis element $b(p;S)$ in $\mathbb{F}_{|S|}$, $\operatorname{multideg}(b(p;S)) = X_p Y_{[n] \setminus \wedge\{r : r \in S\}}$ and so $\deg(b(p;S)) = |p| + n - | \wedge \{r : r \in S\}|$. For $S \subseteq N(p)$ from Lemma 1.1 we deduce that $|N(p)| - |S| \leq | \wedge \{r : r \in S\}| - | \wedge \{r : r \in N(p)\}|$ and then $| \wedge \{r : r \in N(p)\}| + |N(p)| \leq | \wedge \{r : r \in S\}| + |S|$. Therefore by Lemma 2.2

$$\operatorname{reg}\ (H_{\mathcal{L}_G}) = n + \max_{p \in A_G}\{|p| - |N(p)| - | \wedge \{r : r \in N(p)\}|\}.$$

Since $\operatorname{reg}(H_{\mathcal{L}_G}) = \operatorname{pd}(R/I(G))$, from the Auslander-Buchsbaum formula one has (i). The other statement is clear from Theorem 1.2 and using the equality $\operatorname{reg}(R/I(G)) = \operatorname{pd}(H_{\mathcal{L}_G})$ (see [14, Theorem 2.1]). $\square$

In a connected graph G, two edges $\{i, j\}$ and $\{k, \ell\}$ are called 3-disjoint if the induced subgraph on the vertices i, j, k, ℓ consists of exactly two disjoint edges.

Corollary 2.4. *Let G be an unmixed bipartite graph on (X,Y) such that $|X| = |Y| = n$. Then* $\mathrm{reg}(R/I(G)) = a(G)$*, where $a(G)$ is the maximum number of pairwise* 3*-disjoint edges in G.*

Proof. By [10, Lemma 2.2] we have that $\mathrm{reg}(R/I(G)) \geq a(G)$ and by Corollary 2.3 we have $\mathrm{reg}(R/I(G)) = \max_{p\in\mathcal{L}_G} |N(p)|$. So it is enough to show that $|N(p)| \leq a(G)$ for any $p \in \mathcal{L}_G$. Let $p \in \mathcal{L}_G$. As we show in the proof of Proposition 2.1 the interval $[\wedge\{q : q \in N(p)\}, p]$ is isomorphic to $B_{|N(p)|}$. Let $[\wedge\{q : q \in N(p)\}, p]$ be a Boolean lattice on the elements $v_1, \ldots, v_{|N(p)|}$ and $p_i = v_1 \vee \cdots \vee v_{i-1} \vee v_{i+1} \vee \cdots \vee v_{|N(p)|}$ for any $1 \leq i \leq |N(p)|$. By the same argument given in the proof of Lemma 2.2, for any $1 \leq i \leq |N(p)|$ there exists an element $x_i \in v_i$ such that x_i is not in any other v_j with $1 \leq j \leq |N(p)|$. We claim that the set $A = \{\{x_i, y_i\} : 1 \leq i \leq |N(p)|\}$ is a set of pairwise 3-disjoint edges in G. For any distinct pair of edges $\{x_i, y_i\}$ and $\{x_j, y_j\}$ in A, it is enough to show that $\{x_i, y_j\}$ and $\{x_j, y_i\}$ are not edges in G. Since $p_i, p_j \in \mathcal{L}_G$, there are minimal vertex covers C_i and C_j of G, with $p_i = C_i \cap \{x_1, \ldots, x_n\}$ and $p_j = C_j \cap \{x_1, \ldots, x_n\}$. Since $x_i \notin p_i$ and $x_j \in p_i$, then $x_i, y_j \notin C_i$ which shows that $\{x_i, y_j\} \notin E(G)$. Similarly one can see that $x_j, y_i \notin C_j$ and then $\{x_j, y_i\} \notin E(G)$ which completes the proof. □

Recall that a multigraded Betti number $\beta_{i,\mathbf{b}}$ is called i-extremal if $\beta_{i,\mathbf{c}} = 0$ for all $\mathbf{c} > \mathbf{b}$, that is all multigraded entries below $\mathbf{b}$ on the i-th column vanish in the Betti diagram as a Macaulay output [2]. Define $\beta_{i,\mathbf{b}}$ to be extremal if $\beta_{j,\mathbf{c}} = 0$ for all $j \geq i$, and $\mathbf{c} > \mathbf{b}$ so $|\mathbf{c}| - |\mathbf{b}| \geq j - i$. In other words, $\beta_{i,\mathbf{b}}$ corresponds to the top left cornerof a box of zeroes in the multigraded Betti diagram. A graded Betti number $\beta_{i,r}$ is called extremal if $\beta_{j,l} = 0$ for any $j \geq i$, $l > r$ and $l - r \geq j - i$. In other words, $\beta_{i,r}$ corresponds to the top left cornerof a box of zeroes in the Betti diagram.

Corollary 2.5. *All multigraded Betti numbers in homological degree i are i-extremal. A multigraded Betti number $\beta_{i,\boldsymbol{b}}$ of $H_{\mathcal{L}_G}$ is extremal if and only if there exists $p \in A_G$ such that $i = |N(p)|$ and $\boldsymbol{b} = \mathrm{multideg}(b(p; N(p)))$.*

Proof. Let $b(p; S)$ and $b(p'; S')$ be two basis elements of $\mathbb{F}_i$ such that $\mathrm{multideg}(b(p; S))$ divides $\mathrm{multideg}(b(p'; S'))$. Then $p \leq p'$, $\wedge\{q : q \in S'\} \leq \wedge\{q : q \in S\}$ and $|S| = |S'| = i$. Therefore $[\wedge\{q : q \in S\}, p] \subseteq [\wedge\{q : q \in S'\}, p']$. Since both intervals are isomorphic to a Boolean lattice of rank i, one has $p = p'$ and $\wedge\{q : q \in S\} = \wedge\{q : q \in S'\}$. Since $S, S' \subseteq N(p)$ from Lemma 1.1 we have $S = S'$. Therefore all multigraded Betti numbers in homological degree i are i-extremal.

For any basis element $b(p; S)$ we have $\mathrm{multideg}(b(p; S)) = X_p Y_{[n]\setminus\wedge\{q:q\in S\}}$. Therefore $\mathrm{multideg}(b(p; S))$ divides $\mathrm{multideg}(b(p; N(p)))$ and $|N(p)| - |S| \leq |\wedge\{q : q \in S\}| - |\wedge\{q : q \in N(p)\}|$. Therefore $\beta_{i,\mathbf{b}}$ is extremal only if $\mathbf{b} = \mathrm{multideg}(p; N(p))$ for some $p \in \mathcal{L}_G$. Let p and q be two elements of $\mathcal{L}_G$ and $\mathrm{multideg}(b(p; N(p)))$ divides $\mathrm{multideg}(b(q; N(q)))$. Then $p \leq q$ and $\wedge\{r : r \in N(q)\} \leq \wedge\{r : r \in N(p)\}$. Therefore $[\wedge\{r : r \in N(p)\}, p] \subseteq [\wedge\{r : r \in$

$N(q)\}, q]$. Then $\beta_{|N(p)|,\mathbf{b}}$ is extremal precisely when $\mathbf{b} = \text{multideg}(b(p; N(p)))$, where $[\wedge\{q : q \in N(p)\}, p]$ is a maximal Boolean sublattice of $\mathcal{L}_G$. $\square$

Lemma 2.6. *The graded extremal Betti numbers of $R/I(G)$ can be seen from the lattice $\mathcal{L}_G$. Indeed, the Betti number $\beta_{i,i+j}$ for $R/I(G)$ is extremal if and only if there exists $p \in A_G$ with $|N(p)| = j$ and $n + |p| - |N(p)| - |\wedge\{r : r \in N(p)\}| = i$ such that:*

(a) *For any $q \in A_G$ with $|N(q)| > |N(p)|$, one has*

$$|q| - |N(q)| - |\wedge\{r : r \in N(q)\}| < |p| - |N(p)| - |\wedge\{r : r \in N(p)\}|.$$

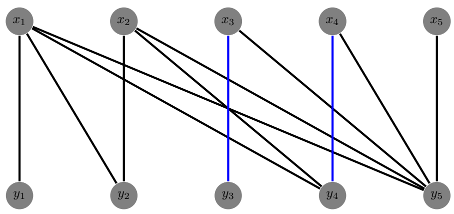
(b) *For any $q \in A_G$ with $|N(q)| = |N(p)|$, one has*

$$|q| - |\wedge\{r : r \in N(q)\}| \leq |p| - |\wedge\{r : r \in N(p)\}|.$$

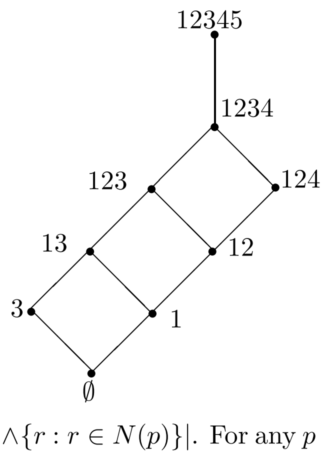
Proof. This statement follows immediately from the definition of graded extremal Betti numbers by using [1, Theorem 2.8]. $\square$

Example 2.7. Let G be the following unmixed bipartite graph on the vertex set $X = \{x_1, \ldots, x_5\}$ and $Y = \{y_1, \ldots, y_5\}$ with the edge ideal

$$I(G) = (x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_1y_2, x_1y_4, x_1y_5, x_2y_4, x_2y_5, x_3y_5, x_4y_5).$$



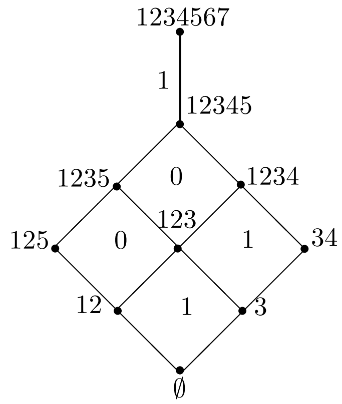
The lattice $\mathcal{L}_G$ is depicted in the following figure. For any $p \in \mathcal{L}_G$ let



$f(p) = |p| - |N(p)| - |\wedge\{r : r \in N(p)\}|$. For any $p = \{i_1, \ldots, i_k\} \in \mathcal{L}_G$ we denote

p by $i_1 \cdots i_k$. We have $f(p) = 0$ for any $p \in \mathcal{L}_G$. Therefore by Lemma 2.6, the extremal Betti number of the unmixed bipartite graph corresponding to this lattice is $\beta_{i,i+j}$ for $i = 5 + f(p) = 5 + 0 = 5$ and $j = 2$ and $\beta_{5,7} = 3$. Also from Corollary 2.3, we have $\operatorname{depth}(R/I(G)) = 5 - \max\{f(p) : p \in A_G\} = 5 - 0 = 5$ and $\operatorname{reg}(R/I(G)) = 2$. The edges $\{x_3, y_3\}$ and $\{x_4, y_4\}$ are two pairwise 3-disjoint edges in G and one can easily see that $a(G) = 2$ (see Corollary 2.4).

Example 2.8. Consider the unmixed bipartite graph on the vertex set $X = \{x_1, \ldots, x_7\}$ and $Y = \{y_1, \ldots, y_7\}$ with the edge ideal $I(G) = (x_1y_1,\ x_2y_2,\ x_3y_3,\ x_4y_4,\ x_5y_5,\ x_6y_6,\ x_7y_7,\ x_6y_7,\ x_5y_7,\ x_4y_7,\ x_3y_7,\ x_2y_7,\ x_1y_7,\ x_7y_6,\ x_5y_6,\ x_4y_6,\ x_3y_6,\ x_2y_6,\ x_1y_6,\ x_3y_4,\ x_2y_1,\ x_1y_2,\ x_1y_5,\ x_2y_5)$. The lattice $\mathcal{L}_G$ is depicted in the following picture. For any $p \in \mathcal{L}_G$ let $f(p) = |p| - |N(p)| - |\wedge\{r : r \in N(p)\}|$.



For any $p = \{i_1, \ldots, i_k\} \in \mathcal{L}_G$ we denote p by $i_1 \cdots i_k$. We have $f(p) = 1$ for $p = 123$, $p = 1234$ and $p = 1234567$ and $f(p) = 0$ for $p = 1235$ and $p = 12345$. Therefore by Lemma 2.6 the extremal Betti number of the unmixed bipartite graph corresponding to this lattice is $\beta_{i,i+j}$ for $i = 7 + f(1234) = 7 + f(123) = 8$ and $j = 2$ and $\beta_{8,10} = 2$. Also from Corollary 2.3, we have $\operatorname{depth}(R/I(G)) = 6$ and $\operatorname{reg}(R/I(G)) = 2$.

As a final application we give a lower bound for the last nonzero total Betti number of an unmixed bipartite graph. To describe the result, we introduce the set $B_G \subseteq A_G$ consisting of all elements q such that $|q| - |N(q)| - |\wedge\{r : r \in N(q)\}| = \max_{p \in A_G}\{|p| - |N(p)| - |\wedge\{r : r \in N(p)\}|\}$.

For an R-module M, let $t(M)$ denote the last nonzero total Betti number of M. Note that $t(M)$ is the Cohen-Macaulay type of M in the Cohen-Macaulay case. Then we have the following corollary.

Corollary 2.9. *Let G be an unmixed bipartite graph. Then $t(R/I(G)) \geq |B_G|$.*

Proof. Let $p \in B_G$ and $r = \operatorname{pd}(R/I(G))$. Then $r = n + |p| - |N(p)| - |\wedge\{r : r \in N(p)\}|$. Let $\mathbf{b} = \operatorname{multideg}(b(p; N(p)))$, from [1, Theorem 2.8] we have $\beta_{r,\mathbf{b}}(R/I(G)) = \beta_{|N(p)|,\mathbf{b}}(H_{\mathcal{L}_G}) = 1$. Therefore $T(R/I(G)) \geq |B_G|$. $\square$

We do not know of any example of a monomial ideal I of projective dimension r for which there exists a nonzero multigraded Betti number $\beta_{r,\mathbf{b}}$ which is not extremal. If such ideals don't exist, at least among the edge ideals of unmixed bipartite graphs, then we would have equality in Corollary 2.9.

In general the multigraded extremal Betti numbers of monomial ideal do not appear only in the last step of the resolution. However we have

Proposition 2.10. *Let $R = k[x_1, \dots, x_n]$ for a field k and I a monomial ideal of R such that R/I is a Cohen-Macaulay ring. Then for all multigraded extremal Betti numbers $\beta_{i,\boldsymbol{b}}(R/I)$ we have $i = \mathrm{pd}(R/I)$.*

Proof. Assume that

$$0 \to F_r \to \cdots \to F_{i+1} \to F_i \to \cdots \to F_0 \to R/I \to 0$$

is a minimal graded free resolution of R/I, where $r = \mathrm{pd}(R/I)$ and φ is the function $F_{i+1} \to F_i$ in the resolution. Let $\beta_{i,\mathbf{b}}(R/I)$ be a multigraded extremal Betti number with $i < \mathrm{pd}(R/I)$ and e be the basis element in F_i with multidegree $\mathbf{b}$. Then for any basis element g of F_{i+1} the coefficient of e in $\varphi(g)$ is zero. Otherwise $\mathrm{multideg}(e) < \mathrm{multideg}(g)$ and $g \in F_{i+1}$ but then $\beta_{i,\mathbf{b}}$ is not extremal, a contradiction. This means that e^* is a cycle in the resolution

$$0 \to (R/I)^* \to F_0^* \to \cdots \to F_i^* \to F_{i+1}^* \to \cdots \to F_r^* \to 0,$$

where $F_i^* = \mathrm{Hom}_R(F_i, R)$. Therefore $\mathrm{Ext}_R^i(R/I, R) \neq 0$. But we know that $\mathrm{Ext}_R^j(R/I, R) = 0$ for any $j < r$, a contradiction. $\square$

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References

[1] D. Bayer, H. Charalambous, and S. Popescu, *Extremal Betti numbers and applications to monomial ideals*, J. Algebra **221** (1999), no. 2, 497–512.

[2] D. Bayer and M. Stillman, *Macaulay*, A system for computation in algebraic geometry and commutative algebra.

[3] H. T. Há and A. Van Tuyl, *Resolutions of square-free monomial ideals via facet ideals*: *a survey*, Algebra, geometry and their interactions, 91–117, Contemp. Math., 448, Amer. Math. Soc., Providence, RI, 2007.

[4] ———, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin. **27** (2008), no. 2, 215–245.

[5] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*, J. Algebraic Combin. **22** (2005), no. 3, 289–302.

[6] J. Herzog, T. Hibi, and H. Ohsugi, *Unmixed bipartite graphs and sublattices of the Boolean lattices*, J. Algebraic Combin. **30** (2009), no. 4, 415–420.

[7] J. Herzog, T. Hibi, and X. Zheng, *Cohen-Macaulay chordal graphs*, J. Combin. Theory Ser. A **113** (2006), no. 5, 911–916.

[8] ———, *The monomial ideal of a finite meet-semilattice*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 4119–4134.

[9] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in Commutative algebra and combinatorics (Kyoto, 1985), 93–109, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.
[10] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A **113** (2006), no. 3, 435–454.
[11] F. Khosh-Ahang and S. Moradi, *Regularity and projective dimension of edge ideal of C_5-free vertex decomposable graphs*, Proc. Amer. Math. Soc. **142** (2014), no. 5, 1567–1576.
[12] K. Kimura, *Non-vanishingness of Betti numbers of edge ideals*, Harmony of Gröbner bases and the modern industrial society, 153–168, World Sci. Publ., Hackensack, NJ, 2012.
[13] M. Kummini, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin. **30** (2009), 429–445.
[14] N. Terai, *Alexander duality theorem and Stanley-Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics*, (Japanese) (Kyoto, 1998), Sürikaisekikenkyüsho Kökyüroku (1999), no. 1078, 174–184.
[15] A. Van Tuyl, *Sequentially Cohen-Macaulay bipartite graphs*: *vertex decomposability and regularity*, Arch. Math. (Basel) **93** (2009), no. 5, 451–459.
[16] R. H. Villarreal, *Unmixed bipartite graphs*, Rev. Colombiana Mat. **41** (2007), no. 2, 393–395.
[17] X. Zheng, *Resolutions of facet ideals*, Comm. Algebra **32** (2004), no. 6, 2301–2324.

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