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REGULARITY OF A DEGENERATE PARABOLIC EQUATION APPEARING IN VECER'S UNIFIED PRICING OF ASIAN OPTIONS

Hongjie Dong and Seick Kim

Abstract. Vecer derived a degenerate parabolic equation characterizing the price of Asian options with generally sampled average. It is well understood that there exists a unique probabilistic solution to Vecer's PDE but it remained unclear whether the probabilistic solution is a classical solution. We prove that the probabilistic solution to Vecer's PDE is indeed regular.

1. Introduction and main result

An Asian option is a specialized form of an option where the payoff is not determined by the underlying price at maturity, but is connected to the average value of the underlying security over certain time interval. In an interesting article [4], Vecer presented a unifying PDE approach for pricing Asian options that works for generally sampled average including both discrete and continuous arithmetic average types. By using a dimension reduction technique, he derived a simple degenerate parabolic equation in two variables $(t,x) \in [0,T)\times\mathbb{R}$

$$u_t + \frac{1}{2}\left(x - e^{-\int_0^t d\nu(s)} q(t)\right)^2 \sigma^2 u_{xx} = 0 \tag{1.1}$$

supplemented by a terminal condition

$$u(T,x) = (x-K)_+ := \max(x-K, 0), \tag{1.2}$$

which gives the price of the Asian option. Here, $\nu(t)$ is the measure representing the dividend yield, σ is the volatility of the underlying asset, $q(t)$ is the trading strategy given by

$$q(t) = \exp\left\{-\int_t^T d\nu(s)\right\}\cdot\int_t^T \exp\left\{-r(T-s)+\int_s^T d\nu(\tau)\right\} d\mu(s),$$

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where r is the interest rate and $\mu(t)$ represents a general weighting factor. We ask readers to refer to [4] for the derivation of the equation (1.1). It should be noted that the function b given by

$$\begin{aligned} b(t) &:= e^{-\int_0^t d\nu(s)} q(t) \\ &= \exp\left\{ -\int_0^T d\nu(s) \right\} \cdot \int_t^T \exp\left\{ -r(T-s) + \int_s^T d\nu(\tau) \right\} d\mu(s) \end{aligned} \tag{1.3}$$

is a nonnegative monotone decreasing function defined on $[0,T]$, and the problem now reads as follows:

$$u_t + \tfrac{1}{2}\left(x - b(t)\right)^2 \sigma^2 u_{xx} = 0, \tag{1.4}$$

$$u(T,x) = (x-K)_+. \tag{1.5}$$

It is mathematically natural to investigate existence, uniqueness, and regularity of a solution to the degenerate parabolic problem (1.4), (1.5). The existence and uniqueness question can be easily answered by using a probabilistic argument. Indeed, the problem (1.4), (1.5) admits a unique probabilistic solution

$$u(t,x) := \mathbb{E} f(X_T(t,x)), \tag{1.6}$$

where $f(y) := (y-K)_+$ and $X_s = X_s(t,x)$ is the stochastic process that satisfies, for $t \in [0,T]$ and $x \in \mathbb{R}$, the following SDE:

$$\begin{cases} dX_s = (X_s - b_s)\sigma\, dw_s, \quad s \geq t, \quad (\, b_s = b(s)\,) \\ X_t = x. \end{cases} \tag{1.7}$$

On the other hand, regularity of the probabilistic solution u defined in (1.6) is a more subtle issue. There is a classical result, originally due to Freidlin, saying that if f in (1.6) is twice continuously differentiable and satisfies a certain growth condition, then $u(t,x)$ defined by (1.6) is meaningful and twice differentiable with respect to x continuously in (t,x), etc.; see e.g. [2, Theorem V.7.4]. However, in our case, f is only Lipschitz continuous and thus the above mentioned result is not applicable. As a matter of fact, it is not trivial whether or not the problem (1.1), (1.2) admits any classical or strong solution. This regularity question was studied by one of the authors in [1]. It is shown in [1] that if $K = 0$ (in this case we have the fixed strike Asian call option) and if $b(t)$ is a monotone decreasing Lipschitz continuous function, then the probabilistic solution u defined in (1.6) is indeed a classical solution. We note that $b(t)$ satisfies such an assumption if $d\mu(t) = \rho(t)\, dt$ for some $\rho \in L^\infty([0,T])$ satisfying $\rho(t) \geq \rho_0 > 0$; i.e., the measure $\mu(t)$ has the density $\rho(t)$ that is strictly positive and bounded. This excludes the cases when the density $\rho(t)$ is a nonnegative step function that vanishes on some intervals or when $\mu(t)$ is a linear combination of Dirac delta functions, which corresponds to discretely sampled Asian options. These two cases are especially important in practice but have been left out in [1].

The goal of this article is, roughly speaking, to show that even in those cases, the probabilistic solution of problem (1.1), (1.2) is still a classical solution. As a matter of fact, we even give an improvement to the main result of [1]. To be precise, we will assume that the function $b(t)$ has the following properties.

i) $b(t)$ is nonnegative and monotone decreasing on $[0, T]$.
ii) $b(t)$ is discontinuous at most finitely many points $t_1 < \cdots < t_n$ in $[0, T]$.
iii) $b(T) = 0$ and there is an $\varepsilon > 0$ such that
  (a) $b \equiv 0$ on $[T - \varepsilon, T]$ if $K \neq 0$.
  (b) $m_1 \leq -b'(t) \leq m_2$ for all $t \in [T - \varepsilon, T]$ for some $m_1, m_2 > 0$ if $K = 0$.

It is clear that condition ii) allows us to treat the discrete sampling case. We point out that in [1] it is assumed that $K = 0 = b(T)$ and b is a Lipschitz function satisfying $m_1 \leq -b'(t) \leq m_2$ on $[0, T]$, so that $b(t)$ in [1] satisfies the above properties. We also note that the condition iii) is technical but is a generic one in the sense that it can be always realized in practice by perturbing sampling strategy. In particular, note that by (1.3), we have $b(T) = 0$ unless the measure $\mu(t)$ has a point mass at T. We use the notation

$$\mathbb{H}_T := [0, T) \times \mathbb{R}, \quad \bar{\mathbb{H}}_T := [0, T] \times \mathbb{R}, \quad \mathring{\mathbb{H}}_T := \mathbb{H}_T \setminus \left(\bigcup_{i=1}^{n} \{t_i\} \times \mathbb{R} \right)$$

in our main result stated below.

Theorem 1.8. *Let $b(t)$ satisfy the conditions* i) - iii) *above. Suppose u is the probabilistic solution of the problem* (1.4), (1.5); *i.e., $u(t, x)$ is defined by* (1.6). *Then $u(t, x)$ is continuous in $\bar{\mathbb{H}}_T$ and satisfies the terminal condition* (1.5). *Moreover, $u(t, x)$ is twice differentiable with respect to x continuously in $\mathbb{H}_T$, differentiable with respect to t continuously in $\mathring{\mathbb{H}}_T$, and satisfies the equation* (1.4) *in $\mathring{\mathbb{H}}_T$.*

It is clear from (1.4) that u_t cannot be continuous where u_{xx} is continuous but $b(t)$ is not continuous. Therefore, if the measure $\mu(t)$ contains a pure point mass (i.e., a discrete sampling), then u_t cannot be continuous in the entire $\mathbb{H}_T$.

Conclusion. The case $K = 0$ corresponds to the fixed strike Asian call option. In that case, it is recommended to employ a continuous sampling near the terminal time T. Except for the case $K = 0$, it is recommended not to sample near the terminal time to ensure that the solution becomes a classical one.

A couple of further remarks are in order.

Remark 1.9. Suppose $K = b(T) \neq 0$ and let $T' = \inf \{t \in [0, T] : b(t) = K\}$. It is a matter of computation to verify that probabilistic solution u of the problem (1.4), (1.5) in $[T', T] \times \mathbb{R}$ is given by $u(t, x) = (x - K)_+$. Of course, u is not twice continuously differentiable with respect to x there.

Remark 1.10. In Vecer's PDE method, the price of Asian option is determined by $u(0, \cdot)$. Theorem 1.8 suggests that to minimize numerical error in computing

$u(0,\cdot)$ by finite difference methods, one should include in time grids all the points where $b(t)$ is discontinuous (i.e., discrete sampling points).

2. Proof of Theorem 1.8

For $(t,x) \in \bar{\mathbb{H}}_T$, let $X_s = X_s(t,x)$ be the stochastic process which satisfies (1.7). It is well known that such a process X_s exists; see e.g., [2, Theorem V.1.1]. The probabilistic solution u of the problem (1.4), (1.5) is then given by the formula (1.6). It is then evident that u is continuous in $\bar{\mathbb{H}}_T$ and satisfies the terminal condition (1.5). If f in (1.6) were twice continuously differentiable, then, as it was mentioned in the introduction, the theorem would follow from [2, Theorem V.7.4]. But this is clearly not the case since $f(y) = (y-K)_+$ is merely a Lipschitz continuous function. However, it should be pointed out that u also has the representation

$$u(t,x) = \mathbb{E}u(X_{\tilde{T}}(t,x)), \quad \forall \tilde{T} \in [t,T].$$

Therefore, if $u(\tilde{T},\cdot)$ is twice continuously differentiable for some $\tilde{T} \in (0,T)$, then we would have the second part of the theorem with T replaced by $\tilde{T}$; i.e., $u(t,x)$ is twice differentiable with respect to x continuously in $\mathbb{H}_{\tilde{T}}$, differentiable with respect to t continuously in $\mathring{\mathbb{H}}_{\tilde{T}}$, and satisfies the equation (1.4) in $\mathring{\mathbb{H}}_{\tilde{T}}$.

In the case when $K=0$, we have $m_1 \le -b'(t) \le m_2$ on $[T-\varepsilon, T]$ for some $m_1, m_2 > 0$, and thus by [1, Theorem 1.10], we are done. It only remains to consider the case when $K \neq 0$ and $b(t) = 0$ on $[T-\varepsilon, T]$. By the above observation, we have the theorem if we show that $u \in \mathcal{C}^{1,2}_{t,x;\,loc}([T-\varepsilon,T)\times\mathbb{R})$ and satisfies the equation (1.4) there. Therefore, it will be enough for us to prove that the probabilistic solution u of the problem

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} = 0 \quad \text{in} \quad \mathbb{H}_T, \tag{2.1}$$

$$u(T,x) = (x-K)_+ \tag{2.2}$$

is a classical solution. By the classical Black-Scholes-Merton's options pricing formula, we know its solution is C^∞ in both x and t in $\mathbb{H}_T$ and thus we are done. □

3. Appendix

We give a self-contained proof that the probabilistic solution $u(t,x)$ of the problem (2.1), (2.2) is a classical solution without invoking Black-Scholes-Merton's formula. Let $Y_s = Y_s(x)$ be the process satisfying the stochastic equation

$$dY_s = \sigma Y_s\, dw_s, \quad Y_0 = x. \tag{3.1}$$

It is easy to verify that Y_s is given by $Y_s = xe^{\sigma w_s - \sigma^2 s/2}$. Then $u(t,x)$ is given by

$$u(t,x) = \mathbb{E}(Y_{T-t}(x)-K)_+ = \mathbb{E}\left(xe^{\sigma w_{T-t} - \sigma^2(T-t)/2} - K\right)_+ .$$

Since Y_s is a martingale and $(y-K)_+ = (y-K) + (K-y)_+$, we also have

$$u(t,x) = x - K + \mathbb{E}\left(K - xe^{\sigma w_{T-t} - \sigma^2(T-t)/2}\right)_+.$$

The above computations lead us to define

$$v(t,x) := \begin{cases} \mathbb{E}\left(xe^{\sigma w_{T-t} - \sigma^2(T-t)/2} - K\right)_+ & \text{if} \quad K > 0, \\ \mathbb{E}\left(K - xe^{\sigma w_{T-t} - \sigma^2(T-t)/2}\right)_+ & \text{if} \quad K < 0. \end{cases} \tag{3.2}$$

Note that it is enough to show that $v \in \mathcal{C}^{1,2}_{t,x;\,loc}(\mathbb{H}_T)$ and satisfies (2.1). Denote

$$\mathbb{H}_T^+ := [0,T) \times (0,\infty), \quad \mathbb{H}_T^- := [0,T) \times (-\infty, 0).$$

It is easy to check that $v \equiv 0$ in $\mathbb{H}_T \setminus \mathbb{H}_T^+$ if $K > 0$ and $v \equiv 0$ in $\mathbb{H}_T \setminus \mathbb{H}_T^-$ if $K < 0$. Also, similar to [1], we find that v belongs to $\mathcal{C}^{1+\alpha/2,2+\alpha}_{t,x,loc}(\mathbb{H}_T^+ \cup \mathbb{H}_T^-)$ and satisfies the equation (2.1) in $\mathbb{H}_T^+ \cup \mathbb{H}_T^-$ regardless the sign of K. Therefore, the proof is complete once we show

$$\lim_{x\to 0} (|v_x| + |v_{xx}| + |v_t|)(t,x) = 0 \quad \text{locally uniformly in } t \in [0,T). \tag{3.3}$$

Lemma 3.4. *Let v be defined by* (3.2) *and denote*

$$Q := \begin{cases} [0,T) \times (0,K) & \textit{if} \quad K > 0, \\ [0,T) \times (K,0) & \textit{if} \quad K < 0. \end{cases}$$

Then we have the following estimate for $v(t,x)$ in Q :

$$0 \le v(t,x) \le \sqrt{\frac{2}{\pi}} \frac{\sigma K \sqrt{T}}{\ln |K/x|} \exp\left\{-\frac{(\ln |K/x|)^2}{\sigma^2 T}\right\}. \tag{3.5}$$

Proof. We shall assume that $K > 0$. The proof for the case when $K < 0$ is parallel. First of all, it is clear from (3.2) that $v \ge 0$. For any $(t,x) \in Q$, we define the process $Z_s = Z_s(t,x) = (t+s, Y_s(x))$, where $Y_s = Y_s(x) = xe^{\sigma w_s - \sigma^2 s/2}$ satisfies the stochastic equation (3.1) above. Let $\tau = \tau(t,x)$ be the first exit time of $Z_s(t,x)$ from Q. We define $\tilde{v}(t,x)$ by

$$\tilde{v}(t,x) = \mathbb{E}g(Z_\tau(t,x)) = \mathbb{E}g(t+\tau, Y_\tau(x)),$$

where the values of $g = g(s,y)$ on the parabolic boundary $\partial_p Q$ of Q are given by

$$\begin{cases} g(T,y) = 0 \quad \text{for} \quad 0 < y < K, \\ g(s,0) = 0 \quad \text{and} \quad g(s,K) = K \quad \text{for} \quad 0 \le s \le T. \end{cases}$$

We claim that $v \le \tilde{v}$ in Q. To see this, first note that

$$v(t,x) = \mathbb{E}v(Z_\tau(t,x)) = \mathbb{E}v(t+\tau, Y_\tau(x)).$$

Thus, it is enough to show that $v \le g$ on $\partial_p Q$. It is obvious that $v(T,y) = 0$ for any $y \in (0,K)$ and $v(t,0) = 0$ for any $t \in [0,T]$. Also, since $(y-K)_+ \le y$ for any $y \ge 0$ and $e^{\sigma w_s - \sigma^2 s/2}$ is a martingale, we have

$$v(t,K) \le \mathbb{E}\left(Ke^{\sigma w_{T-t} - \sigma^2(T-t)/2}\right) = K, \quad \forall t \in [0,T].$$

It thus follows that $v \le \tilde{v}$ in Q. Therefore, we have

$$\begin{aligned} v(t,x) &\le \tilde{v}(t,x) = K\mathbb{P}\{xe^{\sigma w_\tau - \sigma^2\tau/2} = K\} \\ &\le K\mathbb{P}\left\{\sup_{0\le s<T-t}(\sigma w_s - \sigma^2 s/2) \ge \ln(K/x)\right\} \\ &\le K\mathbb{P}\left\{\sup_{0\le s<T}(\sigma w_s - \sigma^2 s/2) \ge \ln(K/x)\right\} \\ &\le K\mathbb{P}\left\{\sup_{0\le s<T} w_s \ge \sigma^{-1}\ln(K/x)\right\} = 2K\mathbb{P}\left\{w_T \ge \sigma^{-1}\ln(K/x)\right\} \\ &\le K\sqrt{\frac{2}{\pi}}\frac{\sigma\sqrt{T}}{\ln(K/x)}\exp\left\{-\frac{(\ln(K/x))^2}{\sigma^2 T}\right\}, \end{aligned}$$

where, in the last step, we have used an inequality

$$\int_\alpha^\infty e^{-x^2/2}\,dx \le \frac{1}{\alpha}\int_\alpha^\infty xe^{-x^2/2}\,dx = \alpha^{-1}e^{-\alpha^2/2}, \quad \forall \alpha > 0.$$

The lemma is proved. □

Now, we prove the statement (3.3). We shall assume that $K > 0$ since the case when $K < 0$ can be treated in a similar way. We extend v to zero on $(T,\infty)\times(0,K)$. Then, it is easy to see that v belongs to $\mathcal{C}^{1+\alpha/2,2+\alpha}_{t,x;\,loc}([0,\infty)\times(0,K))$ and satisfies both the equation (2.1) and the estimate (3.5) in $[0,\infty)\times(0,K)$. Denote

$$Q_r(t_0,x_0) := (t_0,t_0+1)\times(x_0-r,x_0+r), \quad \Pi_\rho := (0,\rho^2)\times(-\rho,\rho).$$

Let $(t_0,x_0)\in[0,T)\times(0,2K/3)$ and set $r = r(x_0) := x_0/2$. Define

$$V(t,x) = v(t_0+t, x_0+rx).$$

It is easy to verify that $V(t,x)$ satisfies the equation

$$V_t + \frac{1}{2}a(x)V_{xx} = 0 \quad \text{in} \quad \Pi_1,$$

where $a(x) := \sigma^2 r^{-2}(x_0+rx)^2$ satisfies

$$\sigma^2 \le a(x) \le 9\sigma^2 \quad \text{and} \quad |a'(x)| \le 6\sigma^2, \quad \forall x\in(-1,1).$$

By the Schauder estimates (see e.g., [3]), there is some $C = C(\sigma)$ such that

$$|V_x(0,0)| + |V_{xx}(0,0)| + |V_t(0,0)| \le C\sup_{\Pi_1}|V|,$$

while by the estimate (3.5) we have

$$\sup_{\Pi_1}|V| = \sup_{Q_r(t_0,x_0)}|v| \le \frac{\sigma K\sqrt{2T}}{\sqrt{\pi}\ln|K/3r|}\exp\left\{-\frac{(\ln|K/3r|)^2}{\sigma^2 T}\right\}.$$

Hence, there is some $r_0 = r_0(K)$ and $N = N(\sigma,T,K)$ such that

$$|V_x(0,0)| + |V_{xx}(0,0)| + |V_t(0,0)| \le Ne^{-(\ln r)^2/N}, \tag{3.6}$$

provided that $r < r_0$. Note that (3.6) translates to as follows: There is some $r_0 = r_0(K)$ and $N = N(\sigma, T, K)$ such that if $x_0 < r_0$, then

$$x_0|v_x(t_0,x_0)| + x_0^2|v_{xx}(t_0,x_0)| + |v_t(t_0,x_0)| \le Ne^{-(\ln x_0)^2/N}.$$

The above estimate obviously implies (3.3). □

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Hongjie Dong
Division of Applied Mathematics
Brown University
182 George Street, Box F, Providence, RI 02912, USA
E-mail address: Hongjie_Dong@brown.edu

Seick Kim
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail address: kimseick@yonsei.ac.kr