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ON ϕ-PSEUDO ALMOST VALUATION RINGS

Afsaneh Esmaeelnezhad and Parviz Sahandi

Abstract. The purpose of this paper is to introduce a new class of rings that is closely related to the classes of pseudo valuation rings (PVRs) and pseudo-almost valuation domains (PAVDs). A commutative ring R is said to be a ϕ-ring if its nilradical $\mathrm{Nil}(R)$ is both prime and comparable with each principal ideal. The name is derived from the natural map ϕ from the total quotient ring $\mathrm{T}(R)$ to R localized at $\mathrm{Nil}(R)$. A prime ideal P of a ϕ-ring R is said to be a *ϕ-pseudo-strongly prime ideal* if, whenever $x, y \in R_{\mathrm{Nil}(R)}$ and $(xy)\phi(P) \subseteq \phi(P)$, then there exists an integer $m \geqslant 1$ such that either $x^m \in \phi(R)$ or $y^m\phi(P) \subseteq \phi(P)$. If each prime ideal of R is a ϕ-pseudo strongly prime ideal, then we say that R is a *ϕ-pseudo-almost valuation ring* (ϕ-PAVR). Among the properties of ϕ-PAVRs, we show that a quasilocal ϕ-ring R with regular maximal ideal M is a ϕ-PAVR if and only if $V = (M : M)$ is a ϕ-almost chained ring with maximal ideal $\sqrt{MV}$. We also investigate the overrings of a ϕ-PAVR.

1. Introduction

Throughout R is a commutative ring with $1 \neq 0$. The nilradical of R is denoted by $\mathrm{Nil}(R)$, and $\mathrm{T}(R)$ denotes the total quotient ring of R. We use R' to denote the integral closure of R in $\mathrm{T}(R)$. Also we use $\mathrm{Z}(R)$ to denote the set of zero divisors of R. If I is an ideal of R, then $\sqrt{I}$ is the radical ideal of I and $(I : I) = \{x \in \mathrm{T}(R) \mid xI \subseteq I\}$. The elements of $R\setminus \mathrm{Z}(R)$ is called regular elements of R. An ideal of R is said to be a regular ideal if, it contains at least one regular element. Recall from [8] and [15] that a prime ideal P of R is called a *divided prime ideal* if $P \subseteq (x)$ for every $x \in R\setminus P$; thus a divided prime ideal is comparable to every ideal of R. Recently, Badawi in [6], [7], [9], [10] and [11], has studied the following class of rings: $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. It is easy to see that every integral domain is a ϕ-ring. An ideal I of R is said to be a nonnil ideal if, $I \nsubseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ϕ-ring R, then $\mathrm{Nil}(R) \subseteq I$. Recall from [7] that for a ring $R \in \mathcal{H}$ with total quotient ring $\mathrm{T}(R)$, the map $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ such that

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$\phi(\frac{a}{b}) = \frac{a}{b}$ for $a \in R$ and $b \in R\setminus \mathrm{Z}(R)$ is a ring homomorphism, and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. It is easy to see that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $\mathrm{Ker}(\phi) \subseteq \mathrm{Nil}(R)$, $\mathrm{Nil}(\mathrm{T}(R)) = \mathrm{Nil}(R)$, $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$, $\mathrm{T}(\phi(R)) = R_{\mathrm{Nil}(R)}$. If B is an overring of R (that is a ring between R and $\mathrm{T}(R)$), then $\mathrm{T}(R) = \mathrm{T}(B)$ and $\mathrm{Nil}(R) = \mathrm{Nil}(B)$. For a subset S of a ring R we use $\mathrm{E}(S)$ to denote the subset $\{x \in \mathrm{T}(R) \mid x^n \notin S$ for every integer $n \geqslant 1\}$ of $\mathrm{T}(R)$.

Let D be an integral domain with quotient field K. Then D is said to be an *almost valuation domain* if for every nonzero $x \in K$, there exists an integer $n \geqslant 1$ such that either $x^n \in D$ or $x^{-n} \in D$ [4]. Also D is said to be a *pseudo-valuation domain* in case, each prime ideal P of D is a *strongly prime ideal*, in the sense that $xy \in P$, $x, y \in K$ implies that either $x \in P$ or $y \in P$ [16]. It is known in [16, Theorem 1.5], that an integral domain D is a pseudo valuation domain if and only if for every nonzero $x \in K$, either $x \in D$ or $ax^{-1} \in D$ for every nonunit $a \in R$. It is easy to see that a valuation domain is a pseudo-valuation domain. Also, it is known in [17, Example 3.1], that for each integer $n \geqslant 1$, there is a pseudo-valuation domain with Krull dimension n which is not a valuation domain.

Recently Badawi [12] introduced a new class of integral domains, that is closely related to pseudo-valuation domain. An integral domain D is said to be a *pseudo-almost valuation domain* (PAVD), in case each prime ideal P of D is a *pseudo-strongly prime ideal*, in the sense that $xyP \subseteq P$, $x, y \in K$ implies that either $x^n \in R$ or $y^n P \subseteq P$ for some integer $n \geqslant 1$. It is known in [12, Theorem 2.8], that an integral domain D is a PAVD if and only if for every nonzero $x \in K$, there is an integer $n \geqslant 1$ such that either $x^n \in D$ or $ax^{-n} \in D$ for every nonunit $a \in R$. Therefore an almost valuation domain is a PAVD; however Badawi in [12, Example 3.6], shows that for each $n \geqslant 1$ there is a PAVD with Krull dimension n which is not an almost valuation domain. A ring $R \in \mathcal{H}$ is said to be ϕ-chained ring (ϕ-CR) if for each $x \in R_{\mathrm{Nil}(R)}\setminus\phi(R)$ we have $x^{-1} \in \phi(R)$ [10]. Also a ring $R \in \mathcal{H}$ is said to be a *ϕ-pseudo-valuation ring* (ϕ-PVR) if every nonnil prime ideal of R is a ϕ-strongly prime ideal of $\phi(R)$, in the sense that $xy \in \phi(P)$, $x \in R_{\mathrm{Nil}(R)}$, $y \in R_{\mathrm{Nil}(R)}$ implies that either $x \in \phi(P)$ or $y \in \phi(P)$ [7] (and [13]). It is known in [10, Proposition 3.3], that if $R \in \mathcal{H}$ is a quasilocal ring with the regular maximal ideal M, then R is a ϕ-PVR if and only if $(M : M)$ is a ϕ-CR with maximal ideal M.

In this article, we introduced a new closely related class of ϕ-rings. We define a prime ideal P of R to be a *ϕ-pseudo-strongly prime ideal* of R if, whenever $x, y \in R_{\mathrm{Nil}(R)}$ (observe that $\mathrm{T}(\phi(R)) = R_{\mathrm{Nil}(R)}$) and $(xy)\phi(P) \subseteq \phi(P)$, then there exists an integer $m \geqslant 1$ such that either $x^m \in \phi(R)$ or $y^m\phi(P) \subseteq \phi(P)$. If each prime ideal of R is a ϕ-pseudo strongly prime ideal, then we say that R is a *ϕ-pseudo-almost valuation ring* (ϕ-PAVR). In Section 2 we investigate the properties of ϕ-PAVRs. We show in Corollary 2.8 that $R \in \mathcal{H}$ is a ϕ-PAVR if and only if for every $x \in R_{\mathrm{Nil}(R)}$ there is an integer $n \geqslant 1$ such that either $x^n \in \phi(R)$ or $ax^{-n} \in \phi(R)$ for every nonunit $a \in \phi(R)$. We generalized the concept of ϕ-CR to ϕ-almost chained ring in the sense that a ring $R \in \mathcal{H}$ is called a ϕ-almost chained ring (ϕ-ACR) if for each $x \in R_{\mathrm{Nil}(R)}$, there exists an integer $n \geqslant 1$ such that either $x^n \in \phi(R)$ or $x^{-n} \in \phi(R)$. In Theorem 2.17 we show that a quasilocal $R \in \mathcal{H}$ with the regular maximal ideal M is ϕ-PAVR if and only if $V = (M : M)$ is a ϕ-ACR with the maximal ideal $\sqrt{MV}$.

In Section 3 we study the overrings of ϕ-PAVRs and prove the following equivalent conditions for a ϕ-PAVR, R with maximal ideal M, and $V := (M : M)$.

(1) Every overring of R is a ϕ-PAVR;
(2) $R[u]$ is a ϕ-PAVR for each $u \in V' \backslash R$, and every integral overring of R is a ϕ-PAVR;
(3) $R[u]$ is quasilocal for each $u \in V' \backslash R$, and every integral overring of R is a ϕ-PAVR;
(4) If B is an overring of R such that $B \subseteq V'$, then B is a ϕ-PAVR;
(5) $R' = V'$ is a ϕ-CR and every integral overring of R is a ϕ-PAVR.

2. Main properties of ϕ-PAVRs

In this section, we introduce the ϕ-pseudo-almost valuation rings and prove some properties of these rings.

Definition 2.1. Let $R \in \mathcal{H}$. A nonnil prime ideal P of R is called a ϕ-pseudo-strongly prime ideal, if for each $x, y \in \mathrm{T}(\phi(R))$ whenever $(xy)\phi(P) \subseteq \phi(P)$, then there exists an integer $m \geqslant 1$ such that either $x^m \in \phi(R)$ or $y^m\phi(P) \subseteq \phi(P)$. A ring $R \in \mathcal{H}$ is said to be a ϕ-pseudo-almost valuation ring (ϕ-PAVR) if every nonnil prime ideal of R is a ϕ-pseudo-strongly prime ideal.

Lemma 2.2. *Let P be a nonnil prime ideal of R. Then P is a ϕ-pseudo-strongly prime ideal if and only if for every $x \in \mathrm{T}(\phi(R))$ there exists an integer $n \geqslant 1$ such that either $x^n \in \phi(R)$ or $x^{-n}\phi(P) \subseteq \phi(P)$.*

Proof. Let P be a ϕ-pseudo-strongly prime ideal of R, $x \in \mathrm{E}(\phi(R))$, and set $x := \frac{a}{b}$ for some $a \in R$ and $b \in R\backslash \mathrm{Z}(R)$. Note that if $a \in \mathrm{Nil}(R)$, then there exists an integer $t \geqslant 1$ such that $x^t = 0 \in \phi(R)$, which is a contradiction. Hence $x^{-1} = \frac{b}{a} \in R_{\mathrm{Nil}(R)}$. Since $xx^{-1}\phi(P) \subseteq \phi(P)$, using the definition, we have $x^{-n}\phi(P) \subseteq \phi(P)$ for some integer $n \geqslant 1$. Conversely assume that $x, y \in \mathrm{T}(\phi(R))$ such that $xy\phi(P) \subseteq \phi(P)$ and $x \in \mathrm{E}(\phi(R))$. Then by the hypothesis there is an integer $n \geqslant 1$ such that $x^{-n}\phi(P) \subseteq \phi(P)$. Therefore $y^n\phi(P) = x^{-n}(x^ny^n\phi(P)) \subseteq x^{-n}\phi(P) \subseteq \phi(P)$. □

The following theorem is one of the main results on ϕ-pseudo valuation rings.

Theorem 2.3 ([6, Proposition 2.9] and [14, Theorem 3.1]). *Let $R \in \mathcal{H}$. Then R is a ϕ-PVR if and only if $\frac{R}{\mathrm{Nil}(R)}$ is a PVD.*

In the following theorem we prove analogous result for ϕ-PAVRs.

Theorem 2.4. *Let $R \in \mathcal{H}$. Then R is a ϕ-PAVR if and only if $\frac{R}{\mathrm{Nil}(R)}$ is a PAVD.*

Proof. By definition it is enough to show that, if P is a nonnil prime ideal of R, then P is a ϕ-pseudo-strongly prime ideal of R if and only if $\frac{P}{\mathrm{Nil}(R)}$ is a pseudo-strongly prime ideal of $\frac{R}{\mathrm{Nil}(R)}$. So let P be a ϕ-pseudo-strongly prime ideal of R. Let $x \in E(\frac{R}{\mathrm{Nil}(R)})$ and assume that $x = \frac{a+\mathrm{Nil}(R)}{b+\mathrm{Nil}(R)}$ for some $a, b \in R \setminus \mathrm{Nil}(R)$. It is easy to see that $\frac{a}{b} \in E(\phi(R))$. Thus using Lemma 2.2, there exists an integer $n \geqslant 1$ such that $\frac{b^n}{a^n}\phi(P) \subseteq \phi(P)$. Hence one can easily see that $x^{-n}\frac{P}{\mathrm{Nil}(R)} \subseteq \frac{P}{\mathrm{Nil}(R)}$. Therefore $\frac{P}{\mathrm{Nil}(R)}$ is a pseudo-strongly prime ideal of $\frac{R}{\mathrm{Nil}(R)}$ by [12, Lemma 2.1]. The converse follows by similar reasoning. □

Corollary 2.5. *Let R be a ϕ-PAVR. Then the prime ideals of R are linearly ordered. In particular R is quasilocal.*

Proof. Using Theorem 2.4, $\frac{R}{\mathrm{Nil}(R)}$ is a PAVD. Hence by [12, Proposition 2.2] the prime ideals of $\frac{R}{\mathrm{Nil}(R)}$ are linearly ordered. In particular $\frac{R}{\mathrm{Nil}(R)}$ is quasilocal. Thus the prime ideals of R are linearly ordered and R is quasilocal. □

Remark 2.6. Let R be a ϕ-PAVR. Since $\mathrm{Z}(R)$ is a union of prime ideals, then by Corollary 2.5, $\mathrm{Z}(R)$ is a prime ideal of R.

Corollary 2.7. *A ring R is a ϕ-PAVR if and only if some maximal ideal of R is a ϕ-pseudo-strongly prime ideal.*

Proof. Use Theorem 2.4 and [12, Theorem 2.5]. □

Corollary 2.8. *A ring $R \in \mathcal{H}$ is a ϕ-PAVR if and only if for every $x \in \mathrm{E}(\phi(R))$ there exists an integer $n \geqslant 1$ such that $ax^{-n} \in \phi(R)$ for every nonunit $a \in \phi(R)$.*

Proof. Use Theorem 2.4 and [12, Theorem 2.8]. □

Proposition 2.9. *A ring $R \in \mathcal{H}$ is a ϕ-PAVR if and only if for every $a, b \in R \setminus \mathrm{Nil}(R)$ either $a^n \mid b^n$ in R for some integer $n \geqslant 1$, or there exists an integer $t \geqslant 1$ such that $b^t \mid ca^t$ in R for every nonunit c of R.*

Proof. Assume that R is a ϕ-PAVR and $a, b \in R \setminus \mathrm{Nil}(R)$. Set $x = \frac{b}{a} \in R_{\mathrm{Nil}(R)}$. If $x^n \in \phi(R)$ for some $n \geqslant 1$, then $\frac{b^n}{a^n} = \frac{r}{1}$ for some $r \in R$. Hence there exists an element $u \in R \setminus \mathrm{Nil}(R)$ such that $u(b^n - a^n r) = 0$. Thus $b^n - a^n r \in \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R)$ is a divided prime ideal of R and $a \notin \mathrm{Nil}(R)$, we have $\mathrm{Nil}(R) \subseteq (a^n)$. Therefore $b^n - a^n r = sa^n$ for some $s \in R$. Consequently $a^n \mid b^n$ in R. Now if $x \in \mathrm{E}(\phi(R))$, by Corollary 2.8, there exists an integer $t \geqslant 1$ such that $cx^{-t} \in \phi(R)$ for every nonunit $c \in \phi(R)$. Thus $\frac{ca^t}{b^t} = \frac{r'}{1}$ for some $r' \in R$. Hence there exists an element $w \in R \setminus \mathrm{Nil}(R)$ such that $w(ca^t - b^t r') = 0 \in \mathrm{Nil}(R) \subseteq (b^t)$, which implies that $b^t \mid ca^t$. The converse follows by similar reasoning. □

Theorem 2.10 ([1, Theorem 2.7]). *Let $R \in \mathcal{H}$. Then R is a ϕ-CR if and only if $\frac{R}{\mathrm{Nil}(R)}$ is a valuation domain.*

Now we introduce the class of ϕ-almost chained rings which is a generalization of almost valuation domains and ϕ-CRs.

Definition 2.11. A ϕ-ring is said to be a ϕ-almost chained ring (ϕ-ACR) if, for each $x \in \mathrm{E}(\phi(R))$, there exists an integer $n \geqslant 1$ such that $x^{-n} \in \phi(R)$.

It is clear that a ϕ-ring is a ϕ-ACR if and only if for each $a, b \in R \setminus \mathrm{Nil}(R)$ there exists an integer $n \geqslant 1$ such that either $a^n \mid b^n$ or $b^n \mid a^n$ in R.

Lemma 2.12. *Let $R \in \mathcal{H}$. Then R is a ϕ-ACR if and only if $\frac{R}{\mathrm{Nil}(R)}$ is an almost valuation domain.*

Proof. Let R be a ϕ-ACR. Let x be a nonzero element of K, the quotient field of $\frac{R}{\mathrm{Nil}(R)}$. Then there exist $a, b \in R\setminus \mathrm{Nil}(R)$ such that $x = \frac{a+\mathrm{Nil}(R)}{b+\mathrm{Nil}(R)}$. Hence by assumption there exists an integer $n \geqslant 1$ such that either $a^n \mid b^n$ in R or $b^n \mid a^n$ in R. It is easy to see that either $\bar{a}^n \mid \bar{b}^n$ in $\frac{R}{\mathrm{Nil}(R)}$ or $\bar{b}^n \mid \bar{a}^n$ in $\frac{R}{\mathrm{Nil}(R)}$. That is $\frac{R}{\mathrm{Nil}(R)}$ is an almost valuation domain. Conversely suppose that $\frac{R}{\mathrm{Nil}(R)}$ is an almost valuation domain and $a, b \in R\setminus \mathrm{Nil}(R)$. Set $x = \frac{a+\mathrm{Nil}(R)}{b+\mathrm{Nil}(R)}$. If $x^n \in \frac{R}{\mathrm{Nil}(R)}$ for some integer $n \geqslant 1$, then $\frac{a^n+\mathrm{Nil}(R)}{b^n+\mathrm{Nil}(R)} = r + \mathrm{Nil}(R)$ for some $r \in R$. Hence $a^n - b^n r \in \mathrm{Nil}(R)$. Since $\mathrm{Nil}(R)$ is a divide prime ideal of R and $b \notin \mathrm{Nil}(R)$, we have $\mathrm{Nil}(R) \subseteq (b^n)$. Hence $a^n - b^n r \in (b^n)$ which implies that $b^n \mid a^n$. On the other hand if $x^{-t} \in \frac{R}{\mathrm{Nil}(R)}$ for some integer $t \geqslant 1$, with the similar argument we get $a^t \mid b^t$. Therefore R is a ϕ-ACR. □

The following proposition holds easily by Theorem 2.4 and Lemma 2.12 and [12, Proposition 2.12].

Proposition 2.13. *Suppose that R is a ϕ-ACR. Then R is a ϕ-PAVR.*

The following result is an analog of [12, Corollary 4.2].

Proposition 2.14. *Let R be a ϕ-PAVR and P be a non maximal prime ideal of R. Then R_P is a ϕ-ACR.*

Proof. Let R be a ϕ-PAVR. Then by Theorem 2.4, $R/\mathrm{Nil}(R)$ is a PAVD, and by [12, Corollary 4.2], $R_P/\mathrm{Nil}(R)R_P$ is an almost valuation domain. Since $\mathrm{Nil}(R_P) = \mathrm{Nil}(R)R_P$, Lemma 2.12 implies that R_P is a ϕ-ACR. □

Now we are looking for examples of the rings which are ϕ-PAVR but are not ϕ-ACR. Recall that if M is a unitary R-module, then $R(+)M$ with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ is a commutative ring with 1 called the *idealization* of M or the *trivial extension* of R by M. Note that if N is a submodule of M, then $0(+)N$ is an ideal of $R(+)M$, and for every ideal J of R, $R(+)M/J(+)M \cong R/J$. Thus $P(+)M$ is a prime ideal of $R(+)M$ for every prime ideal P of R. Also note that a module

M over a ring R is called *divisible* if, for all r in R which are not zero divisors, every element m of M can be *divided* by r, in the sense that there is an element m' in M such that $m = rm'$. This condition can be reformulated by saying that the multiplication by r defines a surjective map from M to M. Recall that $\mathrm{Nil}(R(+)M) = \mathrm{Nil}(R)(+)M$ [3, Theorem 3.2(3)], and that for an integral domain D and a D-module M, every ideal of $D(+)M$ is comparable to $0(+)M$ if and only if M is divisible by [3, Corollary 3.4].

Example 2.15. Let D be a PAVD which is not an almost valuation domain (cf. [12, Example 2.20]). Assume that K is the quotient field of D. Then $D(+)K \in \mathcal{H}$. Thus using Theorem 2.4 and Lemma 2.12 it is easy to see that $D(+)K$ is a ϕ-PAVR which is not a ϕ-ACR.

Recall that R is called *root closed* if, whenever $x \in \mathrm{T}(R)$ and $x^n \in R$ for some integer $n \geqslant 1$, then $x \in R$.

Proposition 2.16. *Let R be a ϕ-PAVR such that $\phi(R)$ is root closed. Then R is a ϕ-PVR.*

Proof. It is easy to check that if $\phi(R)$ is root closed, then $\frac{R}{\mathrm{Nil}(R)}$ is also root closed. Therefore the assertion is clear by [12, Theorem 2.13], Theorems 2.3 and 2.4. □

It is known that a quasilocal domain D with the maximal ideal M is a pseudo-valuation domain if and only if $(M : M)$ is a valuation domain with maximal ideal M [2, Proposition 2.5]. Also a quasilocal $R \in \mathcal{H}$ with the regular maximal ideal M is a ϕ-PVR if and only if $(M : M)$ is a ϕ-CR with maximal ideal M and a quasilocal domain D with the maximal ideal M is a PAVD if and only if $V = (M : M)$ is an almost valuation domain with maximal ideal $\sqrt{MV}$, [10, Proposition 3.3] and [12, Theorem 2.15].

Inspired by those facts, we are willing to prove the following theorem.

Theorem 2.17. *A quasilocal $R \in \mathcal{H}$ with the regular maximal ideal M is a ϕ-PAVR if and only if $V := (M : M)$ is a ϕ-ACR with the maximal ideal $\sqrt{MV}$.*

Proof. We follow the technique of [10, Proposition 3.3]. Assume that R is a ϕ-PAVR and $0 \neq x \in \mathrm{T}(\phi(V)) = R_{\mathrm{Nil}(R)}$ be such that $x \in \mathrm{E}(\phi(V))$. Set $x = \frac{a}{b}$. Note that if $a \in \mathrm{Nil}(R)$, then $x^n = 0$ for some integer $n \geqslant 1$. So that $x \notin \mathrm{E}(\phi(V))$, which is a contradiction. Hence $a, b \in R \setminus \mathrm{Nil}(R)$. Assume that $b^n \mid a^n$ in R for some integer $n \geqslant 1$. Then $a^n = b^n r$ for some $r \in R$, and hence $x^n = \frac{a^n}{b^n} = \frac{r}{1} \in \phi(R) \subseteq \phi(V)$ which is absurd. Therefore by Proposition 2.9, there exists an integer $t \geqslant 1$ such that $a^t \mid b^t c$ in R for every $c \in M$. Let $s \in M \setminus \mathrm{Z}(R)$. Thus $b^t s = a^t d$ for some $d \in R$. If $t' \geqslant 1$ is an integer such that $d^{t'} \mid s^{t'}$ in R, then $s^{t'} = d^{t'} r'$ for some $r' \in R$. Thus from $b^{tt'} s^{t'} = a^{tt'} d^{t'}$ we get $b^{tt'} d^{t'} r' = a^{tt'} d^{t'}$. Since $d \in R \setminus \mathrm{Z}(R)$ we get $b^{tt'} r' = a^{tt'}$ which is a contradiction. Hence $d^{t'} \nmid s^{t'}$ for every integer $t' \geqslant 1$. Once again by Proposition 2.9 we have $s^l \mid d^l m$ for some integer $l \geqslant 1$ and all $m \in M$. For $m \in M$, there is some $y \in R$ such that $d^l m = s^l y$. Thus $\frac{d^l}{s^l} m = y \in R$. If $\frac{d^l}{s^l} m$ is a unit element, we get $d^l m y^{-1} = s^l$, then $d^l \mid s^l$ which is a contradiction. Therefore $\frac{d^l}{s^l} m \in M$, that is $\frac{d^l}{s^l} \in (M : M) = V$. On the other hand the equality $b^{tl} s^l = a^{tl} d^l$ implies that $x^{-tl} = \frac{b^{tl}}{a^{tl}} = \frac{d^l}{s^l} \in (M : M) = V$, then $x^{-tl} = \phi(x^{-tl}) \in \phi(V)$. Thus V is a ϕ-ACR. Now let y be a nonunit element of V which is not in $\sqrt{MV}$. Assume that $\phi(y^n) \in \phi(R)$ for some positive integer n. Then $\phi(y^n) \in \phi(M)$, since $\phi(y)$ is a nonunit element of $\phi(R)$. Therefore $y^n \in M = MV$, which implies that $y \in \sqrt{MV}$. Hence $\phi(y) \in \mathrm{E}(\phi(R))$. Then $\phi(y)^{-t}\phi(M) \subseteq \phi(M)$ for some positive integer t, and thus $y^{-t}M \subseteq M$. Therefore $y^{-t} \in V$. Since $y \in V$ we conclude that y is a unit element of V which is a contradiction. Hence $\sqrt{MV}$ is the maximal ideal of V. Conversely suppose that $V = (M : M)$ is a ϕ-ACR with the maximal ideal $\sqrt{MV}$. Assume that $a, b \in R \setminus \mathrm{Nil}(R)$ be such that $b^n \nmid a^n$ in R for every integer $n \geqslant 1$. Thus $x = \frac{a}{b} \notin \phi(\sqrt{MV})$. If $x^n \in \phi(V)$ for some integer $n \geqslant 1$, then x^n is a unit of $\phi(V)$ and $x^{-n}\phi(M) \subseteq \phi(M)$. Thus $\frac{b^n}{a^n}\frac{m}{1} \in \phi(M) \subseteq \phi(R)$ for every $m \in M$. So that $a^n \mid b^n m$ in R for every nonunit $m \in R$. On the other hand if $x \in \mathrm{E}(\phi(V))$, then $x^{-t} \in \phi(V)$ for some integer $t \geqslant 1$, since V is a ϕ-ACR. Thus $x^{-t}\phi(M) \subseteq \phi(M)$, i.e., $\frac{b^t}{a^t}\frac{m}{1} \in \phi(M)$ for every $m \in M$, that is $a^t \mid b^t m$ in R for every nonunit $m \in R$. Hence by Proposition 2.9, R is a ϕ-PAVR. $\square$

Now we are looking to find a quasilocal ring R with maximal ideal N such that R is a ϕ-PAVR which is not a ϕ-ACR and $(N : N)$ is a ϕ-ACR which is not a ϕ-CR.

Example 2.18. Let D be a PAVD with maximal ideal M such that D is not an almost valuation domain and $V = (M : M)$ is an almost valuation domain that is not a valuation domain (cf. [12, Example 3.7]). Consider $R = D(+)K$. Then $N = M(+)K$ is the maximal ideal of R. Set $\bar{V} = (N : N)$. It is easy to see that $\bar{V} = V(+)K$. Note that $\mathrm{Nil}(R) = \mathrm{Nil}(\bar{V}) = 0(+)K$. Therefore using the isomorphisms $D(+)K/0(+)K \cong D$ and $V(+)K/0(+)K \cong V$, Theorem 2.4, and Lemma 2.12, one can see that R is a ϕ-PAVR which is not a ϕ-ACR and $\bar{V} = (N : N)$ is a ϕ-ACR which is not a ϕ-CR.

Recall that an overring S of R is said to be a root extension, if for every $x \in S$, there is an integer $n \geqslant 1$ such that $x^n \in R$.

Theorem 2.19. *Let R be a quasilocal ring with the maximal ideal M. Suppose that V is an overring of R which is a ϕ-ACR, such that M is an ideal of V and $\sqrt{M}$ (in V) is the maximal ideal of V. Then R is a ϕ-ACR if and only if V is a root extension of R.*

Proof. First of all note that V is a root extension of R if and only if $\phi(V)$ is a root extension of $\phi(R)$. If $\phi(V) = \phi(R)$ there is nothing to prove. Hence assume that $\phi(R) \subsetneq \phi(V)$ and R is a ϕ-ACR. Let $x \in \phi(V)$. If $x \in \phi(\sqrt{M})$

(in V), then $x^k \in \phi(M) \subseteq \phi(R)$ for some integer $k \geqslant 1$. On the other hand if $x \notin \phi(\sqrt{M})$ (in V), then x is an unit of $\phi(V)$. If $x^{-1} \in \mathrm{E}(\phi(R))$, then $x^n \in \phi(R)$ for some integer $n \geqslant 1$, since R is a ϕ-ACR. Now assume that $x^{-t} \in \phi(R)$ for some integer $t \geqslant 1$. If x^{-t} is a nonunit of $\phi(R)$ we get $x^{-t} \in \phi(M)$. So that $x^{-1} \in \phi(\sqrt{M})$ (in V) which is a contradiction, since x^{-1} is a unit element of $\phi(V)$. Thus x^{-t} is a unit element of $\phi(R)$ and hence $x^t \in \phi(R)$. Hence V is a root extension of R. Conversely assume that V is a root extension of R. Then $\phi(V)$ is a root extension of $\phi(R)$. Then $\mathrm{E}(\phi(R)) = \mathrm{E}(\phi(V))$. Let $x \in \mathrm{E}(\phi(R))$. Since V is a ϕ-ACR we get $x^{-n} \in \phi(V)$ for some integer $n \geqslant 1$ and since $\phi(V)$ is a root extension of $\phi(R)$ we have $x^{-nt} \in \phi(R)$ for some integer $t \geqslant 1$. Hence R is a ϕ-ACR. □

We conclude this section by showing that every ϕ-PAVR is a pullback of a ϕ-ACR.

Theorem 2.20. *Let V be a ϕ-ACR with nonzero maximal ideal N and let M be an ideal of V such that $\sqrt{M} = N$, $F = V/M$, $\alpha : V \to F$ the canonical epimorphism, H be a field contained in F, and $R = \alpha^{-1}(H)$. Then the pullback $R = V \times_F H$ is a ϕ-PAVR with maximal ideal M. In particular, if H is properly contained in F and V is not a root extension of R, then R is a ϕ-PAVR which is not a ϕ-ACR.*

Proof. In view of the hypothesis we are deal with the following commutative diagram:

$$\begin{array}{ccc} R & \twoheadrightarrow & H \\ \downarrow & & \downarrow \\ V & \xrightarrow{\alpha} & F \end{array}$$

By construction it is clear that M is the maximal ideal of R, R is a ϕ-ring and $\mathrm{Nil}(R) = \mathrm{Nil}(V)$. Therefore an obvious result of the above diagram is the following diagram:

$$\begin{array}{ccc} R/\mathrm{Nil}(R) & \twoheadrightarrow & H \\ \downarrow & & \downarrow \\ V/\mathrm{Nil}(V) & \xrightarrow{\bar{\alpha}} & F \end{array}$$

where $\bar{\alpha}(v + \mathrm{Nil}(V)) = \alpha(v)$ for every $v \in V$. Since V is a ϕ-ACR, by Lemma 2.12 $V/\mathrm{Nil}(V)$ is an almost valuation domain, and $R/\mathrm{Nil}(R)$ is the pullback, $V/\mathrm{Nil}(V) \times_F H$. Therefore [12, Theorem 2.19] implies that $R/\mathrm{Nil}(R)$ is a PAVD. Hence by Theorem 2.4 R is a ϕ-PAVR. The proof of the in particular case is obvious by Theorem 2.19. □

3. Overrings

The purpose of this section is to characterize when each overring of a ϕ-ring is a ϕ-PAVR.

Lemma 3.1. *Let R be a ϕ-PAVR and let P be a prime ideal of R. Then for each $x \in \mathrm{E}(R)$, there exists an integer $m \geqslant 1$ such that $x^{-m}P \subseteq P$.*

Proof. Let $x = \frac{a}{b} \in \mathrm{E}(R)$ for some $a \in R$ and $b \in R\setminus \mathrm{Z}(R)$. Then $b^n \nmid a^n$ in R for every positive integer n. Note that $\mathrm{Z}(R)$ is a ϕ-pseudo-strongly prime by Remark 2.6. Since $x = \phi(x) \in \mathrm{E}(\phi(R))$, by Lemma 2.2, There exists an integer $t \geqslant 1$ such that $x^{-t}\phi(\mathrm{Z}(R)) \subseteq \phi(\mathrm{Z}(R))$. Now if $a \in \mathrm{Z}(R)$, then $(b^t/a^t)(a^t/1) = b^t/1 \in \phi(\mathrm{Z}(R))$, which is a contradiction. Thus $a \notin \mathrm{Z}(R)$, and $x^{-1} = \frac{b}{a} \in \mathrm{T}(R)$. Let P be a prime ideal of R. Thus from $\phi(x)\phi(x^{-1})\phi(P) \subseteq \phi(P)$ we get $\phi(x)^n \in \phi(R)$ for some integer $n \geqslant 1$ or there exists an integer $m \geqslant 1$ such that $\phi(x)^{-m}\phi(P) \subseteq \phi(P)$. If $\phi(x)^n \in \phi(R)$ for some integer $n \geqslant 1$, then $\frac{a^n}{b^n} = \frac{s}{1}$ for some $s \in R$. Thus $a^n - sb^n \in \mathrm{Nil}(R) \subset (b^n)$, since $\mathrm{Nil}(R)$ is a divided prime ideal of R and $b \notin \mathrm{Nil}(R)$. Therefore $b^n \mid a^n$ in R; hence $x^n \in R$, which is a contradiction. Thus there exists an integer $m \geqslant 1$ such that $\phi(x)^{-m}\phi(P) \subseteq \phi(P)$. Now let $p \in P$. Then $\phi(x)^{-m}\phi(p) \in \phi(P)$. Thus $x^{-m}p - q \in \mathrm{Nil}(R) \subseteq P$ for some $q \in P$. Therefore $x^{-m}p \in P$. □

Proposition 3.2. *Let R be a ϕ-PAVR with the maximal ideal M and $z \in \mathrm{E}(R)$ be integral over R. Then there is a monic polynomial $f(x) \in R[x]$ such that $f(z) = 0$ and $f(0)$ is a unit in R.*

Proof. By Lemma 3.1 there exists an integer $n \geqslant 1$ such that $z^{-n}M \subseteq M$. Let $f(x) \in R[x]$ be a minimal monic polynomial such that $f(z^n) = 0$. Set $a_0 = f(0)$. We claim that a_0 is unit. Otherwise $a_0 \in M$, and hence $z^{-n}a_0 = m \in M$ is a nonunit element of R. Thus $a_0 = mz^n$. Hence we can replace the constant term a_0 in $f(x)$ with mx to construct a monic polynomial $g(x)$. Thus if we factor x from $g(x)$, we get a monic polynomial $h(x)$ of less degree than $f(x)$ such that $h(z^n) = 0$, a contradiction. □

By [6, Proposition 3.3], it is well known that, the integral closure of a ϕ-PVR is a ϕ-PVR. In the following proposition we show the similar result for ϕ-PAVRs.

Proposition 3.3. *Let R be a ϕ-PAVR with the maximal ideal M and let B be an overring of R such that $B \subseteq R'$. Then B is quasilocal with maximal ideal $\sqrt{MB}$. Furthermore R' is a ϕ-PAVR with the maximal ideal $\sqrt{MR'}$.*

Proof. First we show that B is quasilocal with the maximal ideal $\sqrt{MB}$. Let $z \in B$. If $z \in \mathrm{E}(R)$, then by Proposition 3.2 there exists a monic polynomial $f(x) = x^t + a_{t-1}x^{t-1} + \cdots + a_1x + a_0$ such that $f(z) = 0$ and a_0 is a unit element of R. Hence $-(a_0^{-1}z^t + a_0^{-1}a_{t-1}z^{t-1} + \cdots + a_0^{-1}a_1z) = 1$. Therefore $z(-a_0^{-1}z^{t-1} - a_0^{-1}a_{t-1}z^{t-2} - \cdots - a_0^{-1}a_1) = 1$. That is z is unit in B. Now assume that there exists an integer $n \geqslant 1$ such that $z^n \in R$. If z is nonunit in B, then z^n is nonunit in B. Hence z^n is nonunit in R. Thus $z^n \in M$, and hence $z \in \sqrt{MB}$. On the other hand using [5, Theorem 5.10], we see that $\sqrt{MB}$ is a proper ideal of B. Therefore $\sqrt{MB}$ is the maximal ideal of B. To complete the proof we have to show that $\sqrt{MR'}$ is a ϕ-pseudo-strongly prime ideal of R'. Let $x \in \mathrm{E}(\phi(R'))$ and $a \in \sqrt{MR'}$. Then there is an integer $t \geqslant 1$ such that $a^t \in MR'$. Hence there exist $m_1, \ldots, m_s \in M$ and $r_1, \ldots, r_s \in R'$ such that $a^t = \Sigma_{i=1}^{i=s} m_i r_i$. Note that $x \in \mathrm{E}(\phi(R))$, and M is a ϕ-pseudo-strongly prime ideal of R. Thus there is an integer $n \geqslant 1$ such that $x^{-n}\phi(M) \subseteq \phi(M)$ by Lemma 2.2. Hence $x^{-nt}\phi(M) \subseteq \phi(M)$. Therefore $x^{-nt}\frac{a^t}{1} = x^{-nt}\frac{\Sigma_{i=1}^{i=s} m_i r_i}{1} = \Sigma_{i=1}^{i=s} x^{-nt}\frac{m_i r_i}{1} \in \phi(MR')$. Since $\phi(R')$ is root closed $x^{-n}\frac{a}{1} \in \phi(R')$, so that $x^{-n}\frac{a}{1} \in \sqrt{\phi(MR')}$. Now as $MR' \subseteq \sqrt{MR'}$, we have $\phi(MR') \subseteq \phi(\sqrt{MR'})$, hence $\sqrt{\phi(MR')} \subseteq \sqrt{\phi(\sqrt{MR'})} = \phi(\sqrt{MR'})$, since $\phi(\sqrt{MR'})$ is the maximal ideal of $\phi(R')$. Therefore $x^{-n}\frac{a}{1} \in \sqrt{\phi(MR')} \subseteq \phi(\sqrt{MR'})$. Thus $x^{-n}\phi(\sqrt{MR'}) \subseteq \phi(\sqrt{MR'})$ and R' is a ϕ-PAVR by Corollary 2.7. $\square$

By using Propositions 2.16 and 3.3 we get the following result.

Corollary 3.4. *If R is a ϕ-PAVR, then R' is a ϕ-PVR.*

Proposition 3.5. *Let R be a ϕ-PAVR with the maximal ideal M and $u \in V' \backslash R$, where $V := (M : M)$. Then $R[u] \subseteq R'$ if and only if $R[u]$ is quasilocal.*

Proof. Let $u \in V' \backslash R$. If $R[u] \subseteq R'$, then $R[u]$ is quasilocal by Proposition 3.3. Conversely assume that $R[u]$ is quasilocal. If u is a nonunit element of $R[u]$, then $1+u$ is a unit element of $R[u]$, since $R[u]$ is quasilocal. Thus $1+u$ is a unit element of $R[u+1]$. Thus $(1+u)^{-1} \in R[u+1]$. Hence $(1+u)^{-1} \in R'$ by [18, Theorem 15]. On the other hand by Proposition 3.3, R' is a ϕ-PAVR with maximal ideal $\sqrt{MR'}$. We claim that $(1+u)^{-1} \in R' \backslash \sqrt{MR'}$. Indeed by Theorem 2.17, V is a ϕ-ACR with the maximal ideal $\sqrt{MV}$. Thus by Proposition 3.3, V' is a ϕ-PAVR with the maximal ideal $\sqrt{\sqrt{MV}V'} = \sqrt{MV'}$. If $(1+u)^{-1} \in \sqrt{MR'}$, then we have $(1+u)^{-1} \in \sqrt{MV'}$. But $1+u \in V'$, which implies that $1 \in \sqrt{MV'}$, which is a contradiction. Hence $(1+u)^{-1} \in R' \backslash \sqrt{MR'}$. Thus $1+u \in R'$, whence $u = u+1-1 \in R'$. Now if u is a unit element of $R[u]$, then $u^{-1} \in R[u]$. Hence again by [18, Theorem 15], $u^{-1} \in R'$. As above V' is a ϕ-PAVR with the maximal ideal $\sqrt{MV'}$. Thus $u^{-1} \notin \sqrt{MV'}$, so $u^{-1} \in R' \backslash \sqrt{MR'}$. Hence $u \in R'$. Therefore $R[u] \subseteq R'$. $\square$

Proposition 3.6. *Let R be a ϕ-PAVR with the maximal ideal M. If B is an overring of R such that B does not contain an element of the form $\frac{1}{s}$ for some nonzero divisor $s \in M$, then $B \subseteq V'$, where $V := (M : M)$.*

Proof. Let $x \in B$. If $x^{-n} \in R$ for some integer $n \geqslant 1$, then x^n is a unit element of B. Hence x^{-n} must be a nonzero divisor of R. So that by the assumption $x^n \notin B$, which is a contradiction. Thus assume that $x^{-1} \in \mathrm{E}(R)$. Hence by Lemma 3.1, there exists an integer $t \geqslant 1$ such that $x^t M \subseteq M$. Therefore $x^t \in V$ and hence $x \in V'$. $\square$

Proposition 3.7. *Let R be a ϕ-PAVR with the maximal ideal M such that every integral overring of R is a ϕ-PAVR. Then every overring of R is a ϕ-PAVR if and only if $R[u]$ is quasilocal for each $u \in V' \backslash R$, where $V := (M : M)$.*

Proof. Suppose that $R[u]$ is quasilocal for each $u \in V' \backslash R$. Let B be an overring of R. Assume that B contains an element of the form $\frac{1}{s}$ for some nonzero divisor $s \in M$ and $x \in \mathrm{E}(\phi(B)) \subseteq \mathrm{E}(\phi(R))$. So there exists an integer $n \geqslant 1$ such that $x^{-n}\phi(a) \in \phi(M)$ for each $a \in M$. In particular $x^{-n}\frac{s}{1} = \frac{m}{1}$ for some $m \in M$. Thus $x^{-n} = \frac{m}{s} = \phi(m)\phi(\frac{1}{s}) \in \phi(B)$. Hence B is a ϕ-ACR. Then B is a ϕ-PAVR by Proposition 2.13. Now assume that B does not contains an element of the form $\frac{1}{s}$ for some $s \in M$. Hence $B \subseteq V'$ by Proposition 3.6. Let $u \in B \backslash R$. Then $R[u]$ is quasilocal by hypothesis and so by Proposition 3.5 we get $u \in R'$. Thus $B \subseteq R'$. Therefore by assumption, B is a ϕ-PAVR. □

Corollary 3.8. *Let R be a ϕ-PAVR. Then every overring of R is a ϕ-PAVR if and only if $R' = V'$, and every integral overring of R is a ϕ-PAVR, where $V := (M : M)$.*

Proof. Assume that every overring of R is a ϕ-PAVR, then $R[u]$ is quasilocal for every $u \in V' \backslash R$. Hence by Proposition 3.5, $u \in R'$. Thus $V' \subseteq R'$. Conversely assume that $V' = R'$ and every integral overring of R is a ϕ-PAVR. By Proposition 3.7, it is enough to show that $R[u]$ is quasilocal for every $u \in R' \backslash R$, and this holds by Proposition 3.3. □

Combining Propositions 3.5, 3.6, and 3.7 and Corollary 3.8, we get the following result that is a generalization of [6, Corollary 3.17] and [12, Corollary 4.12].

Corollary 3.9. *Let R be a ϕ-PAVR with maximal ideal M and $V := (M : M)$. Then the following statement are equivalent:*

(1) *Every overring of R is a ϕ-PAVR;*
(2) *$R[u]$ is a ϕ-PAVR for each $u \in V' \backslash R$, and every integral overring of R is a ϕ-PAVR;*
(3) *$R[u]$ is quasilocal for each $u \in V' \backslash R$, and every integral overring of R is a ϕ-PAVR;*
(4) *If B is an overring of R such that $B \subseteq V'$, then B is a ϕ-PAVR;*
(5) *$R' = V'$ is ϕ-CR and every integral overring of R is a ϕ-PAVR.*

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Afsaneh Esmaeelnezhad
Faculty of Mathematical Sciences and Computer
Kharazmi University
Tehran, Iran
E-mail address: esmaeilnejad@gmail.com

Parviz Sahandi
Department of Mathematics
University of Tabriz
Tabriz, Iran
E-mail address: sahandi@ipm.ir, sahandi@tabrizu.ac.ir