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ON TRIANGLES ASSOCIATED WITH A CURVE

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Abstract. It is well-known that the area of parabolic region between a parabola and any chord P_1P_2 on the parabola is four thirds of the area of triangle ΔP_1P_2P. Here we denote by P the point on the parabola where the tangent is parallel to the chord P_1P_2. In the previous works, the first and third authors of the present paper proved that this property is a characteristic one of parabolas. In this paper, with respect to triangles ΔP_1P_2Q where Q is the intersection point of two tangents to X at P_1 and P_2 we establish some characterization theorems for parabolas.

1. Introduction

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, usually we say that f is *strictly convex* if the graph of f has positive curvature κ with respect to the upward unit normal N. This condition is equivalent to $f''(x) > 0$ on the interval I.

In this article, we study strictly locally convex plane curves. A regular plane curve $X : I \to \mathbb{R}^2$ defined on an open interval I, is called *convex* if, for all $s \in I$ the trace $X(I)$ of X lies entirely on one side of the closed half-plane determined by the tangent line to X at s ([4]). A regular plane curve $X : I \to \mathbb{R}^2$ is called *locally convex* if, for each $s \in I$ there exists an open subinterval $J \subset I$ with $s \in J$ such that the curve $X|_J$ restricted to the subinterval J is a convex curve.

Hereafter, we will say that a locally convex curve X in the plane $\mathbb{R}^2$ is *strictly locally convex* if the curve is smooth (that is, of class $C^{(3)}$) and has positive curvature κ with respect to the unit normal N pointing to the convex side. Therefore, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $X(s)$ is a unit speed parametrization of X.

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Let us denote by X a strictly locally convex curve in the plane $\mathbb{R}^2$ and by N the unit normal pointing to the convex side. For a fixed point $P \in X$, and for a sufficiently small $h > 0$, we consider the line m passing through $P + hN(P)$ which is parallel to the tangent ℓ to X at P and the points P_1 and P_2 where the line m intersects the curve X. We denote by ℓ_1, ℓ_2 the tangent lines to X at the points P_1, P_2 and by Q the intersection point $\ell_1 \cap \ell_2$, respectively. We let $L_P(h)$ and $H_P(h)$ denote the length $|P_1P_2|$ and the height of the triangle $\triangle QP_1P_2$ from the vertex Q to the edge P_1P_2, respectively.

We also consider $S_P(h)$ and $T_P(h)$ defined by the area of the region bounded by the curve X and chord P_1P_2, the area $|\triangle PP_1P_2|$ of triangle $\triangle PP_1P_2$, respectively. Then, we have

$$T_P(h) = \frac{1}{2}hL_P(h). \tag{1.1}$$

From [10], we also obtain

$$\frac{d}{dh}S_P(h) = L_P(h). \tag{1.2}$$

We have the following well known properties of parabolas ([16]).

Proposition 1. *Suppose that X denotes an open arc of a parabola. Then we have the following.*

1) *For an arbitrary point $P \in X$ and a sufficiently small $h > 0$, X satisfies*

$$S_P(h) = \frac{4}{3}T_P(h). \tag{1.3}$$

2) *The tangent lines to X at the end points of a chord P_1P_2 of X always meet at a point Q on the line parallel to the axis and passing through the point $P \in X$ where the tangent to X is parallel to the chord P_1P_2.*

3) *For an arbitrary point $P \in X$ and a sufficiently small $h > 0$, X satisfies*

$$H_P(h) = 2h. \tag{1.4}$$

Actually, Archimedes proved that parabolas satisfy (1.3) ([16]). Recently, in [10] the first and third authors of the present paper proved that (1.3) is a characteristic property of parabolas and established some characterization theorems for parabolas. Some properties and characteristic ones of parabolas with respect to the area of triangles associated with a curve were given in [3, 11, 13, 14]. For the higher dimensional analogues of some results in [10], we refer [8] and [9].

In this article, we study whether the properties 2) and 3) of parabolas in Proposition 1 characterize parabolas.

First of all, in Section 2 we prove the following:

Theorem 2. *Suppose that X denote the graph of a strictly convex $C^{(3)}$ function $f : I \to \mathbb{R}$ defined on an open interval I. Then the following are equivalent.*

1) *The tangent lines to X at the end points of a chord P_1P_2 of X always meet on the line parallel to the y-axis and passing through the point $P \in X$ where the tangent line to X is parallel to the chord P_1P_2.*

2) *X is an open arc of a parabola with axis parallel to the y-axis.*

Next, in Section 3 we establish some lemmas on properties of the height function $H_P(h)$ for a strictly locally convex $C^{(3)}$ curve.

Finally, using the characterization theorem for parabolas (Theorem 3 in [10]), in Section 4 we establish the following.

Theorem 3. *We denote by X a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then the following are equivalent.*

1) *For an arbitrary point $P \in X$ and a sufficiently small $h > 0$, X satisfies*

$$H_P(h) = \lambda(P)h^{\mu(P)}, \tag{1.5}$$

where $\lambda(P)$ and $\mu(P)$ are some functions of P.

2) *For an arbitrary point $P \in X$ and a sufficiently small $h > 0$, X satisfies*

$$H_P(h) = 2h. \tag{1.6}$$

3) *X is an open arc of a parabola.*

Some characterizations of conic sections (especially parabolas) by properties of tangent lines were established in [6] and [12]. With respect to the curvature function κ and support function h of a plane curve, the first and third authors of the present paper established a characterization theorem for ellipses and hyperbolas centered at the origin ([7]). For a higher dimensional analogues, we refer a recent paper ([5]).

When a curve is the graph of a function, Á. Bényi et al. established some characterization theorems for parabolas ([1, 2]). B. Richmond and T. Richmond also gave a dozen conditions for the graph of a function to be a parabola by using elementary techniques ([15]).

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise stated.

2. Proof of Theorem 2

In this section, we prove Theorem 2 stated in Section 1.

It is obvious that in Theorem 2, 2) implies 1).

Conversely, suppose that X denote the graph of a strictly convex $C^{(3)}$ function $f : I \to \mathbb{R}$ defined on an open interval I which satisfies 1) of Theorem 2. For distinct points $s, t \in I$, we put $P_1 = (s, f(s))$ and $P_2 = (t, f(t))$. If we denote by $P = (x, f(x))$ with $x = x(s,t)$ the point where the tangent line to the graph is parallel to the chord P_1P_2, then we have

$$(s - t)f'(x(s,t)) = f(s) - f(t). \tag{2.1}$$

By the assumption, we also obtain

$$x(s,t)(f'(s)-f'(t)) = sf'(s) - tf'(t) - f(s) + f(t). \tag{2.2}$$

Differentiating (2.1) with respect to s and t respectively, we get

$$x_s(s,t) = \frac{f'(s) - f'(x(s,t))}{(s-t)f''(x(s,t))} \tag{2.3}$$

and

$$x_t(s,t) = \frac{f'(x(s,t)) - f'(t)}{(s-t)f''(x(s,t))}. \tag{2.4}$$

Differentiating (2.2) with respect to s and t respectively also yields

$$x_s(s,t) = \frac{(s - x(s,t))f''(s)}{f'(s) - f'(t)} \tag{2.5}$$

and

$$x_t(s,t) = \frac{(x(s,t) - t)f''(t)}{f'(s) - f'(t)}. \tag{2.6}$$

From (2.3) and (2.5) one obtains

$$(t-s)(x(s,t)-s)f''(s)f''(x(s,t)) = (f'(s)-f'(t))(f'(s)-f'(x(s,t))). \tag{2.7}$$

Eliminating $x_t(s,t)$ from (2.4) and (2.6) also gives

$$(t-s)(t-x(s,t))f''(t)f''(x(s,t)) = (f'(x(s,t))-f'(t))(f'(s)-f'(t)). \tag{2.8}$$

Note that on the whole interval I, we have $f''(t) > 0$. Then, on I we get from (2.7) and (2.8)

$$\begin{aligned} &f''(t)(x(s,t)-t)(f'(x(s,t))-f'(s)) \\ &= f''(s)(x(s,t)-s)(f'(x(s,t))-f'(t)). \end{aligned} \tag{2.9}$$

Substituting $f'(x(s,t))$ in (2.1) into (2.9) implies

$$\begin{aligned} &f''(t)(x(s,t)-t)\{f(t)-f(s)-(t-s)f'(s)\} \\ &= f''(s)(x(s,t)-s)\{f(t)-f(s)-(t-s)f'(t)\}. \end{aligned} \tag{2.10}$$

Replacing $x(s,t)$ in (2.10) with that in (2.2), we see that for all distinct s and t in I the function f satisfies

$$f''(t)\{f(t)- f'(s)(t-s) - f(s)\}^2 = f''(s)\{f(s)- f'(t)(s-t) - f(t)\}^2. \tag{2.11}$$

Since $f''(t) > 0$ on I, it follows that for all distinct $s, t \in I$ we have

$$f(t) > f'(s)(t-s) + f(s). \tag{2.12}$$

Thus, from (2.11) we get

$$\sqrt{f''(t)}\{f(t)-f'(s)(t-s)-f(s)\} = \sqrt{f''(s)}\{f(s)-f'(t)(s-t)-f(t)\}. \tag{2.13}$$

By differentiating (2.13) with respect to t, we obtain

$$k'(t)\{f(t)-f'(s)(t-s)-f(s)\} + k(t)\{f'(t)-f'(s)\} = k(s)k(t)^2(t-s), \tag{2.14}$$

where we put $k(t) = \sqrt{f''(t)}$. Differentiating (2.14) with respect to s gives

$$\{k'(t)k(s)^2 + k'(s)k(t)^2\}(s - t) = k(s)k(t)\{k(s) - k(t)\}. \tag{2.15}$$

It follows from (2.15) that for some ξ between s and t we have

$$\frac{k'(t)k(s)^2 + k'(s)k(t)^2}{k(s)k(t)} = \frac{k(s) - k(t)}{s - t} = k'(\xi). \tag{2.16}$$

By letting $s \to t$ in (2.16), we get

$$\frac{2k'(t)k(t)^2}{k(t)^2} = k'(t), \tag{2.17}$$

which yields $k'(t)k(t)^2 = 0$ on the interval I, and hence $k'(t) = 0$ on the interval I.

This shows that $k(t) = \sqrt{f''(t)}$ is a positive constant on the whole interval I, which completes the proof of Theorem 2.

3. Some lemmas

In this section, we give some lemmas which are useful in the proof of Theorem 3.

First of all, we need the following lemma ([10]).

Lemma 4. *Suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h \to 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{3.1}$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N pointing to the convex side.

Now, we prove the following lemma.

Lemma 5. *Let us denote by X a strictly locally convex $C^{(3)}$ curve in the Euclidean plane $\mathbb{R}^2$. Then we have*

$$\lim_{h \to 0} \frac{H_P(h)}{h} = 2. \tag{3.2}$$

Proof. We fix an arbitrary point P on X. Then, we may take a coordinate system (x, y) of $\mathbb{R}^2$ such that P is the origin $(0, 0)$ and x-axis is the tangent line ℓ of X at P. Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$. Then N is the upward unit normal.

Since the curve X is of class $C^{(3)}$, the Taylor's formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x), \tag{3.3}$$

where $2a = f''(0)$ and $f_3(x)$ is an $O(|x|^3)$ function. Noting that the curvature κ of X at P is given by $\kappa(P) = f''(0) > 0$, we see that a is positive.

For a sufficiently small $h > 0$, the line m through $P + hN(P)$ and orthogonal to $N(P)$ is given by $y = h$. We denote by $P_1(s, f(s))$ and $P_2(t, f(t))$ the points where the line $m : y = h$ meets the curve X with $s < 0 < t$. Then we have $f(s) = f(t) = h$ and the tangent lines to X at P_i, $i = 1, 2$, intersect at the point $Q = (x_0(h), y_0(h))$ with

$$x_0(h) = \frac{tf'(t) - sf'(s)}{f'(t) - f'(s)} \tag{3.4}$$

and

$$y_0(h) = h + \frac{(t - s)f'(t)f'(s)}{f'(t) - f'(s)}. \tag{3.5}$$

Noting $L_P(h) = t - s$, one gets

$$H_P(h) = h - y_0(h) = \frac{-L_P(h)f'(t)f'(s)}{f'(t) - f'(s)}. \tag{3.6}$$

Hence we obtain

$$\frac{H_P(h)}{h} = \frac{L_P(h)}{\sqrt{h}} \frac{1}{\alpha_P(h)}, \tag{3.7}$$

where we use

$$\alpha_P(h) = \frac{(f'(t) - f'(s))}{-f'(s)f'(t)} \sqrt{h}. \tag{3.8}$$

On the other hand, it follows from Lemma 5 in [13] that

$$\lim_{h \to 0} \alpha_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}. \tag{3.9}$$

Together with Lemma 4 and (3.9), (3.7) completes the proof. □

Finally, we prove:

Lemma 6. *Suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$H_P(h) \frac{d}{dh} L_P(h) = L_P(h). \tag{3.10}$$

Proof. As in the proof of Lemma 5, for an arbitrary point P on X we take a coordinate system (x, y) of $\mathbb{R}^2$ so that (3.3) holds with $f(0) = f'(0) = 0$ and $2a = f''(0) > 0$. Then, for sufficiently small $h > 0$, we put $f(t) = h$ with $t > 0$ and we denote by $P_1(s(t), h)$ and $P_2(t, h)$ the points where the line $m : y = h$ meets the curve X with $s = s(t) < 0 < t$. Then we have

$$f(s(t)) = f(t) = h \tag{3.11}$$

and

$$L_P(h) = t - s(t). \tag{3.12}$$

It follows from (3.6) that

$$H_P(h) = \frac{-L_P(h)f'(t)f'(s)}{f'(t)-f'(s)}. \tag{3.13}$$

Noting $h = f(t)$, one obtains from (3.12) that

$$\frac{d}{dh}L_P(h) = (1-s'(t))\frac{dt}{dh} = \frac{1-s'(t)}{f'(t)}. \tag{3.14}$$

Therefore, it follows from (3.11) that

$$\frac{d}{dh}L_P(h) = \frac{1}{f'(t)} - \frac{1}{f'(s(t))} = \frac{f'(s)-f'(t)}{f'(t)f'(s)}. \tag{3.15}$$

Together with (3.13), this completes the proof. □

4. Proof of Theorem 3

In this section, we use the main result of [10] (Theorem 3 in [10]) and lemmas in Section 3 in order to prove Theorem 3.

It is obvious that any open arc of parabolas satisfy 1) and 2) in Theorem 3.

Conversely, suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies for all $P \in X$ and sufficiently small $h > 0$

$$H_P(h) = \lambda(P)h^{\mu(P)}, \tag{1.5}$$

where $\lambda(P)$ and $\mu(P)$ are some functions. Using Lemma 5, by letting $h \to 0$ we see that

$$\lim_{h\to 0} h^{\mu(P)-1} = \frac{2}{\lambda(P)}. \tag{4.1}$$

Hence, (4.1) shows that $\mu(P) = 1$ and $\lambda(P) = 2$. Therefore, the curve X satisfies for all $P \in X$ and sufficiently small $h > 0$

$$H_P(h) = 2h. \tag{1.6}$$

Now, using Lemma 6 we get the following.

Lemma 7. *Suppose that X denote a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $H_P(h) = 2h$ for all $P \in X$ and sufficiently small $h > 0$. Then for all $P \in X$ and sufficiently small $h > 0$ we have*

$$L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \tag{4.2}$$

Proof. It follows from Lemma 6 that

$$2h\frac{d}{dh}L_P(h) = L_P(h). \tag{4.3}$$

By integrating (4.3), we get for some constant $C = C(P)$

$$L_P(h) = C\sqrt{h}. \tag{4.4}$$

Thus, Lemma 4 completes the proof. □

Finally, we prove Theorem 3 as follows.

Since $\frac{d}{dh}S_P(h) = L_P(h)$ ([10]) and $S_P(0) = 0$, by integrating (4.2) we get

$$S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}} h\sqrt{h}. \tag{4.5}$$

Hence, it follows from (1.1), (4.2) and (4.5) that for all $P \in X$ and sufficiently small $h > 0$

$$S_P(h) = \frac{4}{3} T_P(h). \tag{4.6}$$

Theorem 3 of [10] states that (4.6) implies X is an open arc of a parabola. Therefore, (4.6) completes the proof of Theorem 3.

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