

STABILITY FOR A VISCOELASTIC PLATE EQUATION WITH p -LAPLACIAN

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ABSTRACT. In this paper, we consider a viscoelastic plate equation with p -Laplacian

$$u'' + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) + \sigma(t) \int_0^t g(t-s) \Delta u(s) ds - \Delta u' = 0.$$

By introducing suitable energy and Lyapunov functionals, we establish a general decay estimate for the energy, which depends on the behavior of both σ and g .

1. Introduction

This work is concerned with the asymptotic stability of solutions for a viscoelastic plate equation with p -Laplacian of the form

$$(1.1) \quad u'' + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) + \sigma(t) \int_0^t g(t-s) \Delta u(s) ds - \Delta u' = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$(1.2) \quad u = \Delta u = 0 \quad \text{on } \Gamma \times (0, \infty),$$

$$(1.3) \quad u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega,$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary Γ , σ and g are real functions.

This model can be regarded as a fourth order weak viscoelastic plate equation with a lower order perturbation of p -Laplacian type, and which is related to one-dimensional nonlinear equation of elastoplastic microstructure flows given by

$$(1.4) \quad u'' + u_{xxxx} - a(u_x^2)_x = 0.$$

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As a general form of (1.4), many authors [5, 9, 11, 12] studied the following equation with appropriate boundary and initial conditions:

$$(1.5) \quad u'' + \alpha \Delta^2 u - \operatorname{div}(\gamma(|\nabla u|^2)\nabla u) - \Delta u' + h(u') + g(u) = f(x).$$

Ma and Soriano [5] proved the global existence and decay of the solutions when $\alpha = h(u') = 0$ and $\gamma(|\nabla u|^2) = |\nabla u|^{p-2}$ in (1.5). Park et al. [9] improved the results in [5] by generalizing the assumptions on h , that is, the function h is assumed to be a discontinuous and nonlinear multi-valued function. The authors in [11, 12] proved the global existence and asymptotic behavior of weak and strong solutions of (1.5). These works are concerned with exponential decay rates.

On the other hand, Andrade et al. [1] discussed the energy decay of solutions for problem (1.1)-(1.3) when $\sigma(t) = 1$. The interaction of the memory term with p -Laplacian operator was first considered by them. They showed the existence of solutions and exponential decay rates of energy of solutions under the condition $g'(t) \leq -c_1 g(t)$ for some $c_1 > 0$. The aim of this work is to investigate the general decay of solutions for problem (1.1)-(1.3). Concerning the study of viscoelastic equations with memory, there is a substantial number of papers (see [2, 3, 4, 6, 8, 10] and references therein). Cavalcanti et al. [2] proved exponential and polynomial decay for the viscoelastic wave equation under the usual condition

$$-c_1 g(t) \leq g'(t) \leq -c_2 g(t) \quad \text{and} \quad 0 \leq g''(t) \leq c_3 g(t)$$

for some positive constants c_1, c_2, c_3 . Since then, several authors are interested in weakening the condition of relaxation function g . Messaoudi and Tatar [8] introduced conditions on g such as $g'(t) \leq -\zeta g^p(t)$ for $1 \leq p < \frac{3}{2}$, and get the exponential and polynomial decay rates. Moreover, the authors in [4] obtained general stability for the Timoshenko system under weaker condition on g such as $g'(t) \leq -\zeta(t)g(t)$.

Recently, Messaoudi [7] considered the viscoelastic equation of the form

$$u'' - \Delta u + \sigma(t) \int_0^t g(t-s)\Delta u(s)ds = 0,$$

and proved a general decay result which depends both on the behavior of σ and g . Inspired by [1, 7], we investigate the general decay estimate of solutions for the weak viscoelastic beam equation with p -Laplacian. The results of this article extend those of [1] with respect to some aspects.

The plan of this paper is as follows. In Section 2, we give some notations and material needed for our work. In Section 3, we derive general decay of the energy of solutions for problem (1.1)-(1.3).

2. Statement of main results

In this section, we present some material needed in the proof of our results. Throughout this paper, $(u, v) = \int_{\Omega} u(x)v(x)dx$. For a Banach space X , $\|\cdot\|_X$

denotes the norm of X . For simplicity, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$. Let λ_0, λ_1 and λ_2 be positive constants satisfying

$$(2.1) \quad \lambda_0 \|u\|^2 \leq \|\Delta u\|^2, \quad \lambda_1 \|\nabla u\|^2 \leq \|\Delta u\|^2, \quad \lambda_2 \|u\|^2 \leq \|\nabla u\|^2.$$

Now, we give assumptions on the system (1.1)-(1.3).

For $n \in \mathbb{N}$, we assume that p satisfies

$$2 \leq p \leq \frac{2n-2}{n-2} \text{ if } n \geq 3 \text{ or } p \geq 2 \text{ if } n = 1, 2.$$

Then, it holds

$$H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

For the relaxation function g and potential σ , as in [7], we assume that $g, \sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nonincreasing differentiable functions satisfying

$$(2.2) \quad g(0) > 0, \quad k := \int_0^\infty g(s)ds < \infty,$$

$$(2.3) \quad \sigma(t) > 0, \quad \frac{1}{2} - \frac{2\sigma(t)}{\lambda_1} \int_0^t g(s)ds \geq l > 0 \text{ for } t \geq 0,$$

and there exist a nonincreasing differentiable function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with

$$(2.4) \quad \zeta(t) > 0, \quad g'(t) \leq -\zeta(t)g(t), \quad \text{for } t \geq 0, \quad \lim_{t \rightarrow \infty} \frac{-\sigma'(t)}{\zeta(t)\sigma(t)} = 0.$$

Theorem 2.1. *If $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, then problem (1.1)-(1.3) has a unique weak solution*

$$u \in C(0, \infty; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(0, \infty; L^2(\Omega)).$$

Moreover, if $(u_0, u_1) \in H_\Gamma^3(\Omega) \times H_0^1(\Omega)$, where $H_\Gamma^2(\Omega) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \Gamma\}$, then the problem (1.1)-(1.3) has a unique solution u satisfying

$$u \in L^\infty(0, \infty; H_\Gamma^3(\Omega)), \quad u' \in L^\infty(0, T; H_0^1(\Omega)), \quad u'' \in L^2(0, T; H^{-1}(\Omega)).$$

Proof. See Andrade, Jorge Silva and Ma [1]. □

3. General decay of solutions

For simplicity of notations, we shall omit x and t in all functions of x and t if there is no ambiguity, and c denotes a generic positive constant different from line to line or even in the same line, and let

$$g * v = \int_0^t g(t-s)v(s)ds,$$

$$g \square v = \int_0^t g(t-s) \|v(t) - v(s)\|^2 ds.$$

We introduce the energy functional $E(t)$ of the problem (1.1)-(1.3) as

$$E(t) = \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|\Delta u(t)\|^2 + \frac{1}{p} \|\nabla u(t)\|_p^p,$$

and a modified energy by

$$(3.1) \quad \mathcal{E}(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{\sigma(t)}{2}g\Box\nabla u - \frac{\sigma(t)}{2}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2.$$

Remark 3.1. From definitions of $E(t)$ and $\mathcal{E}(t)$, it is noted that $E(t) \leq \frac{1}{2l}\mathcal{E}(t)$ for all $t \geq 0$. Therefore it is enough to obtain the desired decay for the modified energy $\mathcal{E}(t)$.

To demonstrate the stability of the system (1.1)-(1.3), the lemmas below are essential.

Lemma 3.1. *The modified energy $\mathcal{E}(t)$ satisfies*

$$(3.2) \quad \mathcal{E}'(t) = -\|\nabla u'(t)\|^2 + \frac{\sigma(t)}{2}g'\Box\nabla u - \frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - \frac{\sigma(t)}{2}g(t)\|\nabla u(t)\|^2 + \frac{\sigma'(t)}{2}g\Box\nabla u.$$

Proof. Multiplying (1.1) by $u'(t)$, we have

$$(3.3) \quad \frac{d}{dt}E(t) = -\|\nabla u'(t)\|^2 + \sigma(t)(g * \nabla u, \nabla u'(t)).$$

A direct calculation gives

$$(3.4) \quad \begin{aligned} &\sigma(t)(g * \nabla u, \nabla u') \\ &= -\frac{\sigma(t)}{2}g(t)\|\nabla u\|^2 + \frac{\sigma(t)}{2}g'\Box\nabla u - \frac{\sigma(t)}{2}\frac{d}{dt}\left(g\Box\nabla u - \int_0^t g(s)ds\|\nabla u\|^2\right) \\ &= -\frac{d}{dt}\left(\frac{\sigma(t)}{2}g\Box\nabla u - \frac{\sigma(t)}{2}\int_0^t g(s)ds\|\nabla u\|^2\right) - \frac{\sigma(t)}{2}g(t)\|\nabla u\|^2 \\ &\quad + \frac{\sigma(t)}{2}g'\Box\nabla u + \frac{\sigma'(t)}{2}g\Box\nabla u - \frac{\sigma'(t)}{2}\int_0^t g(s)ds\|\nabla u\|^2. \end{aligned}$$

Applying this to the second term in the right hand side of (3.3), the proof is completed. □

Remark 3.2. Since $-\frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 \geq 0$, $\mathcal{E}(t)$ may not be nonincreasing.

Now, let us define the perturbed modified energy by

$$(3.5) \quad L(t) = N\mathcal{E}(t) + \sigma(t)\Psi(t),$$

where $\Psi(t) = (u'(t), u(t))$ and $N > 0$.

Lemma 3.2. *There exist $c_1 > 0$ and $c_2 > 0$ such that for appropriately large $N > 0$*

$$c_1\mathcal{E}(t) \leq L(t) \leq c_2\mathcal{E}(t) \quad \text{for all } t \geq 0.$$

Proof. Young's inequality gives that

$$|\Psi(t)| \leq \frac{1}{2}(\|u'\|^2 + \|u\|^2) \leq c(\|u'\|^2 + \|\Delta u\|^2) \leq c\mathcal{E}(t).$$

Since σ is non-increasing, we get

$$|L(t) - N\mathcal{E}(t)| \leq \sigma(t)|\Psi(t)| \leq c\sigma(0)\mathcal{E}(t).$$

Choosing suitably $N > 0$, we complete the proof. \square

Lemma 3.3. *There exist positive constants c_3, c_4 and t_0 satisfying*

$$(3.6) \quad \frac{d}{dt}L(t) \leq -c_3\sigma(t)\mathcal{E}(t) + c_4\sigma(t)g\Box\nabla u \quad \text{for } t \geq t_0.$$

Proof. Using the problem (1.1), we have

$$\Psi'(t) = \|u'\|^2 - \|\Delta u\|^2 - \|\nabla u\|_p^p + \sigma(t)(g * \nabla u, \nabla u) - (\nabla u, \nabla u').$$

Young's inequality and (2.1) give that

$$\begin{aligned} (g * \nabla u, \nabla u) &= \int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))ds + \int_0^t g(s)ds\|\nabla u(t)\|^2 \\ &\leq 2 \int_0^t g(s)ds\|\nabla u\|^2 + \frac{1}{4}g\Box\nabla u \\ &\leq \frac{2\sigma(t)}{\lambda_1} \int_0^t g(s)ds\|\Delta u\|^2 + \frac{1}{4}g\Box\nabla u, \end{aligned}$$

and

$$-(\nabla u, \nabla u') \leq \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2\lambda_1}\|\nabla u'\|^2.$$

Thus, we obtain

$$(3.7) \quad \begin{aligned} \Psi'(t) &\leq \left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1}\right)\|\nabla u'(t)\|^2 - \left(\frac{1}{2} - \frac{2\sigma(t)}{\lambda_1} \int_0^t g(s)ds\right)\|\Delta u(t)\|^2 \\ &\quad - \|\nabla u(t)\|_p^p + \frac{\sigma(t)}{4}g\Box\nabla u. \end{aligned}$$

Combining (3.2), (3.5) and (3.7), we have

$$\begin{aligned} \frac{d}{dt}L(t) &= N\mathcal{E}'(t) + \sigma(t)\Psi'(t) + \sigma'(t)\Psi(t) \\ &\leq -N\|\nabla u'\|^2 - \frac{N\sigma'(t)}{2} \int_0^t g(s)ds\|\nabla u\|^2 + \sigma(t)\left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1}\right)\|\nabla u'\|^2 \\ &\quad - \sigma(t)\left(\frac{1}{2} - \frac{2\sigma(t)}{\lambda_1} \int_0^t g(s)ds\right)\|\Delta u\|^2 - \sigma(t)\|\nabla u\|_p^p \\ &\quad + \frac{\sigma^2(t)}{4}g\Box\nabla u + \sigma'(t)(u, u'). \end{aligned}$$

Using

$$\sigma'(t)(u, u') \leq -\frac{\sigma'(t)}{2\lambda_0} \|\Delta u\|^2 - \frac{\sigma'(t)}{2\lambda_2} \|\nabla u'\|^2,$$

we get

$$\begin{aligned} & \frac{d}{dt} L(t) \\ & \leq -\left(N - \sigma(t)\left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1}\right) + \frac{\sigma'(t)}{2\lambda_2}\right) \|\nabla u'\|^2 - \sigma(t) \|\nabla u\|_p^p + \frac{\sigma^2(t)}{4} g \square \nabla u \\ & \quad - \left(\sigma(t)\left(\frac{1}{2} - \frac{2\sigma(t)}{\lambda_1} \int_0^t g(s) ds\right) + \frac{N\sigma'(t)}{2\lambda_1} \int_0^t g(s) ds + \frac{\sigma'(t)}{2\lambda_0}\right) \|\Delta u\|^2. \end{aligned}$$

Noting that (2.3) and the relations

$$\frac{\sigma(t)}{\sigma(0)} \leq 1 \quad \text{and} \quad \int_0^t g(s) ds \leq k,$$

we arrive

$$\begin{aligned} \frac{d}{dt} L(t) & \leq -\sigma(t) \left(\frac{N}{\sigma(0)} - \left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1} \right) + \frac{\sigma'(t)}{2\lambda_2\sigma(t)} \right) \|\nabla u'\|^2 - \sigma(t) \|\nabla u\|_p^p \\ & \quad + \frac{\sigma^2(t)}{4} g \square \nabla u - \sigma(t) \left(l + \frac{Nk\sigma'(t)}{2\lambda_1\sigma(t)} + \frac{\sigma'(t)}{2\lambda_0\sigma(t)} \right) \|\Delta u\|^2. \end{aligned}$$

First, we take $N > 0$ such that

$$\frac{N}{\sigma(0)} - \left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1} \right) > 0.$$

Since

$$\lim_{t \rightarrow \infty} \frac{-\sigma'(t)}{\sigma(t)} = 0,$$

we can choose $t_0 > 0$ sufficiently large so that, for $t \geq t_0$,

$$\frac{N}{\sigma(0)} - \left(\frac{1}{\lambda_2} + \frac{1}{2\lambda_1} \right) + \frac{\sigma'(t)}{2\lambda_2\sigma(t)} > 0$$

and

$$l + \frac{Nk\sigma'(t)}{2\lambda_1\sigma(t)} + \frac{\sigma'(t)}{2\lambda_0\sigma(t)} > 0.$$

This completes the proof. \square

Our main theorem is the following.

Theorem 3.1. *There exist positive constants C_0 , ω and t_1 such that*

$$E(t) \leq C_0 e^{-\omega \int_0^t \sigma(s) \zeta(s) ds} \quad \text{for } t \geq t_1.$$

Proof. Multiplying (3.6) by $\zeta(t)$, using (2.4) and (3.2), we get

$$\begin{aligned} \zeta(t)L'(t) &\leq -c_3\sigma(t)\zeta(t)\mathcal{E}(t) + c_4\sigma(t)\zeta(t)g\Box\nabla u \\ &\leq -c_3\sigma(t)\zeta(t)\mathcal{E}(t) - c_4\sigma(t)g'\Box\nabla u \\ &\leq -c_3\sigma(t)\zeta(t)\mathcal{E}(t) + c_4\left(-2\mathcal{E}'(t) - \sigma'(t)\int_0^t g(s)ds\|\nabla u(t)\|^2\right). \end{aligned}$$

Since ζ is nonincreasing, by (2.1) and (2.2), one sees that

$$\begin{aligned} \left(\zeta(t)L(t) + 2c_4\mathcal{E}(t)\right)' &\leq -c_3\sigma(t)\zeta(t)\mathcal{E}(t) - \frac{c_4\sigma'(t)}{\lambda_1}\int_0^t g(s)ds\|\Delta u(t)\|^2 \\ &\leq -\sigma(t)\zeta(t)\left(c_3 + \frac{c_4k\sigma'(t)}{\lambda_1l\sigma(t)\zeta(t)}\right)\mathcal{E}(t) \quad \text{for } t \geq t_0. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \frac{-\sigma'(t)}{\zeta(t)\sigma(t)} = 0$, we can choose $t_1 \geq t_0$ such that

$$\left(\zeta(t)L(t) + 2c_4\mathcal{E}(t)\right)' \leq -\frac{c_3\sigma(t)\zeta(t)}{2}\mathcal{E}(t) \quad \text{for } t \geq t_1.$$

Thus, by letting $\mathcal{L}(t) = \zeta(t)L(t) + 2c_4\mathcal{E}(t)$ and $c_5 = \frac{c_3}{2}$, we get

$$\mathcal{L}'(t) \leq -c_5\sigma(t)\zeta(t)\mathcal{E}(t) \quad \text{for } t \geq t_1.$$

Since $\zeta(t)$ is a nonincreasing positive function, we can easily observe that $\mathcal{L}(t)$ is equivalent to $\mathcal{E}(t)$. Subsequently, it follows that

$$\mathcal{L}'(t) \leq -c\sigma(t)\zeta(t)\mathcal{L}(t) \quad \text{for } t \geq t_1.$$

Integrating this over (t_1, t) , we conclude that

$$\mathcal{L}(t) \leq \mathcal{L}(t_1)e^{-c\int_{t_1}^t \sigma(s)\zeta(s)ds} \quad \text{for all } t \geq t_1.$$

The equivalent relations of \mathcal{L} , L , and \mathcal{E} yield

$$E(t) \leq c_6e^{-c\int_{t_1}^t \sigma(s)\zeta(s)ds} \quad \text{for all } t \geq t_1,$$

for some positive constant c_6 . By the virtue of the continuity and boundedness of $E(t)$ in the interval $[0, t_1]$, we complete the proof. \square

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