THE ULTIMATE RUIN PROBABILITY OF A DEPENDENT DELAYED-CLAIM RISK MODEL PERTURBED BY DIFFUSION WITH CONSTANT FORCE OF INTEREST

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ABSTRACT. Recently, Li [12] gave an asymptotic formula for the ultimate ruin probability in a delayed-claim risk model with constant force of interest and pairwise quasi-asymptotically independent and extended-regularly-varying-tailed claims. This paper extends Li's result to the case in which the risk model is perturbed by diffusion, the claims are consistently-varying-tailed and the main-claim interarrival times are widely lower orthant dependent.

1. Introduction

Consider a delayed-claim risk model perturbed by diffusion with constant force of interest, which involves two kinds of insurance claims, namely the main claims and the by-claims, such that each main claim may cause a by-claim occurring after a period of delay. In this model, the main claims $\{X_i, i \geq 1\}$, by-claims $\{Y_i, i \geq 1\}$, inter-arrival times of main claims $\{\theta_i, i \geq 1\}$ are three sequences of nonnegative and identically distributed, but not necessarily independent, random variables (r.v.s) with common distributions F, G and K, respectively. Denote by $\tau_i = \sum_{k=1}^{i} \theta_k$, $i \geq 1$, the arrival times of successive main claims, which constitute a counting process

$$N(t) = \sup\{i \geq 1 : \tau_i \leq t\}, \quad t \geq 0;$$

and by $\{T_i, i \geq 1\}$ the delay times of by-claims, which are nonnegative (but possibly degenerated at 0), arbitrarily dependent and identically distributed r.v.s with common distribution H.

Assume that the total amount of premiums accumulated before time $t \geq 0$, denoted by $C(t)$, is a nonnegative and nondecreasing stochastic process with $C(0) = 0$ and $C(t) < \infty$ almost surely for every $0 \leq t < \infty$; and that the diffusion process, as a perturbed term, $\{B(t), t \geq 0\}$ is a standard Brownian

motion with volatility parameter $\sigma \geq 0$ and independent of the other sources of randomness. In practice, the diffusion-perturbed term can be interpreted as an additional uncertainty of the aggregate claims or the premium income of an insurance company. Let $r > 0$ be the constant force of interest and $x \geq 0$ be the insurer's initial reserve. Then the total reserve up to time $t \geq 0$, denoted by $R(t)$, satisfies

$$R(t) = xe^{rt} + \int_{0-}^{t} e^{r(t-s)}dC(s) - \sum_{i=1}^{\infty} X_i e^{r(t-\tau_i)}\mathbf{1}_{\{\tau_i \leq t\}}$$

$$(1.1) \qquad - \sum_{i=1}^{\infty} Y_i e^{r(t-\tau_i-T_i)}\mathbf{1}_{\{\tau_i+T_i \leq t\}} + \sigma \int_{0-}^{t} e^{r(t-s)}dB(s),$$

where $\mathbf{1}_E$ is the indicator function of an event E. Hence the ultimate ruin probability is defined by

$$(1.2) \qquad \psi_r(x) = P\left(R(t) < 0 \text{ for some } t \geq 0\right).$$

It is well-known that the delayed-claim risk model was firstly introduced by Waters and Papatriandafylou [22] so that the independence assumption between claim sizes and their inter-arrival times can be relaxed, and since then it has been extensively investigated by many researchers. See, for example, Yuen and Guo [26], Xiao and Guo [23], Li and Wu [13], among others. For the continuous-time counterparts, the readers are referred to Yuen et al. [27], Xie and Zou [24, 25], Meng and Wang [15], Zou and Xie [28], and references therein. We notice that all the references above only discussed the case when the claims are light-tailed and mutually independent. But recently, Li [12] considered the delayed-claim risk model with heavy-tailed and dependent claims, and an asymptotic formula for the ultimate ruin probability was reached. So in the following, we introduce some dependence structures and some classes of heavy-tailed distributions.

Wang et al. [20] introduced the widely lower orthant dependent (WLOD) structure. Say that r.v.s $\{\xi_i, i \geq 1\}$ are WLOD, if there exists a sequence of finite positive numbers $\{g_L(n), n \geq 1\}$ such that for each $n \geq 1$ and all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i \leq x_i\}\right) \leq g_L(n)\prod_{i=1}^{n} P(\xi_i \leq x_i).$$

Clearly, if $\{\xi_i, i \geq 1\}$ are WLOD r.v.s, then for each $n \geq 1$ and any $s > 0$,

$$(1.3) \qquad E\exp\left\{-s\sum_{i=1}^{n}\xi_i\right\} \leq g_L(n)\prod_{i=1}^{n} Ee^{-s\xi_i}.$$

Also, Chen and Yuen [2] proposed a more general dependence structure below. Say that r.v.s $\{\xi_i, i \geq 1\}$ are pairwise quasi-asymptotically independent

(PQAI), if for any $1 \leq i \neq j < \infty$,

$$\lim_{x \to \infty} P\left(|\xi_i| \wedge \xi_j > x \mid \xi_i \vee \xi_j > x\right) = 0,$$

where we denote $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. For further study on the above dependence structures and their analogues, we refer to Geluk and Tang [11], Wang and Cheng [21], Liu et al. [14], Gao and Jin [6], Gao and Liu [8], Gao and Yang [9, 10], Gao et al. [7], Gao and Bao [5], and others.

Henceforth, all limit relationships are for $x \to \infty$ unless mentioned otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(1)b(x)$ if $\limsup a(x)/b(x) = C < \infty$, write $a(x) = o(1)b(x)$ if $C = 0$, write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $C \leq 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$, write $a(x) \asymp b(x)$ if $a(x) = O(1)b(x)$ and $b(x) = O(1)a(x)$.

In the paper, we are concerned with some classes of heavy-tailed distributions, which have no finite exponential moments. Say that a r.v. ξ or its distribution V belongs to the ERV class of extended-regularly-varying-tailed distributions if there exist some $0 < \alpha \leq \beta < \infty$ such that $y^{-\beta} \leq \overline{V}_*(y) \leq \overline{V}^*(y) \leq y^{-\alpha}$ for all $y \geq 1$, where $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$; belongs to the consistently-varying-tailed class, denoted by $V \in \mathcal{C}$, if $\lim_{y \searrow 1} \overline{V}_*(y) = 1$ or $\lim_{y \nearrow 1} \overline{V}^*(y) = 1$; belongs to the dominatedly-varying-tailed class, denoted by $V \in \mathcal{D}$, if $\overline{V}^*(y) < \infty$ for all $y > 0$; belongs to the long-tailed class, denoted by $V \in \mathcal{L}$, if

$$(1.4) \qquad\qquad \overline{V}(x + y) \sim \overline{V}(x) \quad \text{for all } y > 0.$$

In conclusion,

$$ERV \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D}.$$

More details of heavy-tailed distributions and their applications can be found in Bingham et al. [1] and Embrechts et al. [4].

Recently, Li [12] showed that in the delayed-claim risk model (1.1) with $\sigma = 0$ and $C(\cdot)$ a deterministic linear function, if the claim sizes $\{X_i, Y_i, i \geq 1\}$ are PQAI, random pairs $\{(X_i, Y_i), i \geq 1\}$ are identically distributed with marginal distributions $F \in ERV$ and $G \in ERV$, the inter-arrival times of main claims $\{\theta_i, i \geq 1\}$ are independent, and $\{X_i, Y_i, i \geq 1\}$, $\{\theta_i, i \geq 1\}$ and $\{T_i, i \geq 1\}$ are mutually independent, then

$$(1.5) \quad \psi_r(x) \sim \int_{0-}^{\infty} \overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty} \int_{0-}^{\infty} \overline{G}(xe^{r(u+t)})dH(u)dEN(t).$$

Motivated by Li's result in [12], in the paper we further consider the delayed-claim risk model (1.1) perturbed by diffusion, in which the claim sizes (including main claims and by-claims) are PQAI and consistently-varying-tailed, the inter-arrival times of main claims satisfy WLOD structure, and the premium income follows a general stochastic process. In our main result, we will discuss two cases, one is that the premium process $\{C(t), t \geq 0\}$ is independent of the other sources of randomness, and the other is that $\{C(t), t \geq 0\}$ is not necessarily so.

In the rest part of this paper, we will present the main result in Section 2, and prove it in Section 3 after preparing some lemmas.

2. Main result

In this section, we state our main result. For a distribution V and any $y > 0$, we set

$$J_V^+ = -\lim_{y \to \infty} \log \overline{V}_*(y)/\log y \quad \text{and} \quad J_V^- = -\lim_{y \to \infty} \log \overline{V}^*(y)/\log y.$$

Assume that the total discounted amount of premiums is finite, namely,

$$(2.1) \qquad 0 \le \widetilde{C} = \int_{0-}^{\infty} e^{-rs} dC(s) < \infty \quad \text{almost surely.}$$

Our main result is given below.

Theorem 2.1. *Consider the delayed-claim risk model* (1.1), *in which the claim sizes* $\{X_i, Y_i, i \ge 1\}$ *are PQAI r.v.s, random pairs* $\{(X_i, Y_i), i \ge 1\}$ *are identically distributed by marginal distributions* $F \in \mathcal{C}$ *and* $G \in \mathcal{C}$ *with* $J_F^- > 0$ *and* $J_G^- > 0$, *respectively, and* $\{X_i, Y_i, i \ge 1\}$, $\{\theta_i, i \ge 1\}$ *and* $\{T_i, i \ge 1\}$ *are mutually independent. Assume that the inter-arrival times of main claims* $\{\theta_i, i \ge 1\}$ *are WLOD r.v.s such that*

$$(2.2) \qquad \lim_{n \to \infty} g_L(n) e^{-\epsilon n} = 0$$

holds for every $\epsilon > 0$, *depending on* F, G, K *and* H. *Then relation* (1.5) *holds, if one of the following conditions is true*:

  1. *the premium process* $\{C(t), t \ge 0\}$ *is independent of the other sources of randomness*;

  2. *the total discounted amount of premiums satisfies*

$$(2.3) \qquad P(\widetilde{C} > x) = o(1)\big(\overline{F}(x) + \overline{G}(x)\big).$$

Remark 2.1. As was pointed out by Tang [16], condition (2.3), which does not require the independence between the premium process and the other sources of randomness, allows for a more realistic case that the premium income varies as a deterministic or stochastic function of the insurer's current reserve.

3. Proof of Theorem 2.1

Before proving Theorem 2.1, we prepare the following lemmas. The first one is a direct result of Proposition 2.2.1 of Bingham et al. [1] or Lemma 3.5 of Tang and Tsitsiashvili [17].

Lemma 3.1. *If a distribution* $V \in \mathcal{D}$ *with* $J_V^- > 0$, *then for any* $0 < \hat{p} < J_V^- \le J_V^+ < p < \infty$, *there exist some* $C > 1$ *and* $D > 0$ *such that*

$$(3.1) \qquad C^{-1}\Big(\frac{x}{y}\Big)^{\hat{p}} \le \frac{\overline{V}(y)}{\overline{V}(x)} \le C\Big(\frac{x}{y}\Big)^{p} \quad \text{for all } x \ge y \ge D.$$

The second lemma is from Theorem 3.3(iv) of Cline and Samorodnitsky [3] and Lemma 2.5 of Wang et al. [19].

Lemma 3.2. *Let ξ and η be two independent r.v.s, where ξ is distributed by V, and η is nonnegative, not degenerate at 0 and satisfying $E\eta^p < \infty$ for some $p > J_V^+$.*
  (1) *If $V \in \mathcal{D}$, then $P(\xi\eta > x) \asymp \overline{V}(x)$.*
  (2) *If $V \in \mathcal{C}$, then the distribution of $\xi\eta$ still belongs to the class $\mathcal{C}$.*

The third lemma is a restatement of Theorem 2.1 of Li [12]. Also, see Lemma 3.2 of Gao and Yang [10]. It should be mentioned that the asymptotic formula in the lemma was first developed by Tang and Tsitsiashvili [18].

Lemma 3.3. *If $\{\xi_i, 1 \le i \le n\}$ are n PQAI and real-valued r.v.s with distributions $V_i \in \mathcal{C}$, $1 \le i \le n$, respectively, then for any fixed $0 < a \le b < \infty$,*

$$P\left(\sum_{i=1}^n c_i\xi_i > x\right) \sim \sum_{i=1}^n \overline{V}_i\left(\frac{x}{c_i}\right)$$

holds uniformly for all $(c_1, c_2, \ldots, c_n) \in [a, b]^n$.

The lemma below is due to Lemma 3.1 of Chen and Yuen [2] or Theorem 2.2 of Li [12].

Lemma 3.4. *Let $\{\xi_i, 1 \le i \le n\}$ be n PQAI and real-valued r.v.s with distributions $V_i \in \mathcal{D}$, $1 \le i \le n$, respectively, and $\{\eta_i, 1 \le i \le n\}$ be n nonnegative r.v.s, independent of $\{\xi_i, 1 \le i \le n\}$ and satisfying $E\eta_i^p < \infty$ for some $p > \bigvee_{i=1}^n J_{V_i}^+$, $1 \le i \le n$. Then $\{\xi_i\eta_i, 1 \le i \le n\}$ are still PQAI.*

Lemma 3.5. *If $\{\xi_i, \eta_i, i \ge 1\}$ are PQAI and real-valued r.v.s with ξ_i and η_i distributed by $V_i \in \mathcal{C}$ and $W_i \in \mathcal{C}$, $i \ge 1$, respectively, then $\{\zeta_i = \xi_i + \eta_i, i \ge 1\}$ are still PQAI r.v.s with distributions $U_i \in \mathcal{C}$, $i \ge 1$.*

Proof. By Theorem 2.5(ii) of Li ([12]), we know that $\{\zeta_i, i \ge 1\}$ are PQAI. By Lemma 3.3, it holds that

$$\overline{U}_i(x) \sim \overline{V}_i(x) + \overline{W}_i(x), \quad i \ge 1$$

which, along with $V_i \in \mathcal{C}$ and $W_i \in \mathcal{C}$, $i \ge 1$, leads to $U_i \in \mathcal{C}$, $i \ge 1$. □

The last lemma comes from Lemma 3.3 of Gao et al. [7].

Lemma 3.6. *Consider the main-claim arrival process $\{N(t), t \ge 0\}$ with WLOD inter-arrival times $\{\theta_i, i \ge 1\}$ satisfying (2.2) for every $\epsilon > 0$. Then for any fixed $t > 0$ and any $p > 0$,*

$$E(N(t))^p < \infty.$$

Now we proceed to prove Theorem 2.1.

Proof of Theorem 2.1. By the reserve process (1.1), we get its discounted value as

$$(3.2) \qquad \widetilde{R}(t) = x + \int_{0-}^{t} e^{-rs} dC(s) - \sum_{i=1}^{\infty} X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t\}}$$

$$- \sum_{i=1}^{\infty} Y_i e^{-r(\tau_i + T_i)} \mathbf{1}_{\{\tau_i + T_i \le t\}} + \sigma \int_{0-}^{t} e^{-rs} dB(s), \quad t \ge 0.$$

Then by (1.2), it follows that

$$(3.3) \qquad \psi_r(x) = P\left(\widetilde{R}(t) < 0 \ \text{ for some } t \ge 0\right).$$

Let $\widetilde{B} = \sigma \sup_{t \in [0,\infty]} |\int_{0-}^{t} e^{-rs} dB(s)|$. It is well-known that the stochastic integral $\int_{0-}^{t} e^{-rs} dB(s)$, $0 < t \le \infty$, follows a normal distribution with mean 0 and variance $\int_0^t e^{-2rs} ds$, then by many classic martingale inequalities, $\widetilde{B}$ has finite moments of arbitrary orders. So if a distribution $U \in \mathcal{D}$, then

$$(3.4) \qquad P(\widetilde{B} > x) = o(1)\overline{U}(x).$$

Clearly, by (3.2) we get

$$\widetilde{R}(t) \ge x - \sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} - \widetilde{B}, \quad t \ge 0,$$

which, along with (3.3), implies that

$$(3.5) \qquad \psi_r(x) \le P\left(\sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} + \widetilde{B} > x \right).$$

On the other hand, again by (3.2) we have

$$\widetilde{R}(t) \le x + \widetilde{C} - D_r(t) + \widetilde{B}, \quad t \ge 0,$$

where $D_r(t) = \sum_{i=1}^{\infty} X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t\}} + \sum_{i=1}^{\infty} Y_i e^{-r(\tau_i + T_i)} \mathbf{1}_{\{\tau_i + T_i \le t\}}$, and $\widetilde{C}$ is defined in (2.1). Hence by (3.3), we derive that

$$(3.6) \qquad \psi_r(x) \ge P\left(D_r(t) - \widetilde{B} > x + \widetilde{C} \ \text{ for some } t \ge 0 \right)$$

$$= P\left(\bigcup_{t \ge 0} \left\{ D_r(t) - \widetilde{B} > x + \widetilde{C} \right\} \right)$$

$$\ge P\left(D_r(\infty) - \widetilde{B} > x + \widetilde{C} \right)$$

$$= P\left(\sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} - \widetilde{B} > x + \widetilde{C} \right).$$

Firstly, we deal with the asymptotic upper-bound of $\psi_r(x)$. By Lemmas 3.2(2) and 3.4, one can easily see that the common distribution of $Y_i e^{-rT_i}$, $i \ge 1$, belongs to the class $\mathcal{C}$, and $\{X_i, Y_i e^{-rT_i}, i \ge 1\}$ are PQAI. Then by

Lemma 3.5, we know that $X_i + Y_i e^{-rT_i}$, $i \geq 1$, are also PQAI and identically distributed by $U \in \mathcal{C}$. So, following the proof of Lemma 3.5 of Gao and Liu [8] or going along the similar lines of the proof of Lemma 3.5 of Gao and Jin [6], there exists a positive integer n_0 such that for any $0 < v < 1$,

$$(3.7) \qquad P\left(\sum_{i=n_0+1}^{\infty} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > \frac{vx}{2} \right)$$

$$= o(1) P\left(\left(X_1 + Y_1 e^{-rT_1} \right) e^{-r\tau_1} > x \right).$$

Note that the common distribution of $X_i + Y_i e^{-rT_i}$, $i \geq 1$, belongs to $\mathcal{C}$, then again by Lemma 3.2(2), the distributions of $(X_i + Y_i e^{-rT_i}) e^{-r\tau_i}$, $i \geq 1$, also belong to $\mathcal{C}$. Hence for any given $\varepsilon > 0$, there exists a number v_0, $0 < v_0 < 1$, such that for all large x,

$$(3.8) \qquad \sum_{i=1}^{n_0} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > (1 - v_0)x \right)$$

$$\leq (1 + \varepsilon) \sum_{i=1}^{n_0} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > x \right).$$

Let n_0 and v_0 be fixed as above. By (3.5), we have

$$(3.9) \quad \psi_r(x) \leq P\left(\sum_{i=1}^{n_0} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > (1 - v_0)x \right)$$

$$+ P\left(\sum_{i=n_0+1}^{\infty} \left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > \frac{v_0 x}{2} \right) + P\left(\widetilde{B} > \frac{v_0 x}{2} \right)$$

$$= \sum_{i=1}^{3} H_i.$$

For H_1, by Theorem 3.2 of Chen and Yuen [2] and (3.8), we show that for all large x,

$$H_1 \sim \sum_{i=1}^{n_0} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > (1 - v_0)x \right)$$

$$\leq (1 + \varepsilon) \sum_{i=1}^{n_0} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > x \right).$$

For H_2, by (3.7) with $v = v_0$, we get

$$H_2 = o(1) P\left(\left(X_1 + Y_1 e^{-rT_1} \right) e^{-r\tau_1} > x \right).$$

For H_3, by (3.4), $U \in \mathcal{C} \subset \mathcal{D}$ and Lemma 3.2(1), we obtain

$$H_3 = o(1) \left(\left(X_1 + Y_1 e^{-rT_1} \right) e^{-r\tau_1} > x \right).$$

Therefore, substituting the derivations for H_i, $i = 1, 2, 3$, into (3.9) and considering the arbitrariness of $\varepsilon > 0$ can imply that

$$\psi_r(x) \lesssim \sum_{i=1}^{\infty} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x\Big)$$

$$(3.10) \qquad \sim \int_{0-}^{\infty} \overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}(xe^{r(u+t)})dH(u)dEN(t),$$

where in the second step we used Lemma 3.3.

Subsequently, we turn to the asymptotic lower-bound of $\psi_r(x)$. As mentioned above, $X_i + Y_i e^{-rT_i}$, $i \geq 1$, are identically distributed by $U \in \mathcal{C} \subset \mathcal{D}$. Thus by (3.1), it holds that for all $x \geq D$ and any $0 < t < \infty$,

$$(3.11)$$
$$\frac{\int_t^{\infty} \overline{U}(xe^{rs})dEN(s)}{\int_{0-}^{\infty} \overline{U}(xe^{rs})dEN(s)} = \frac{\int_t^{\infty} \overline{U}(xe^{rs})/\overline{U}(x)dEN(s)}{\int_{0-}^{\infty} \overline{U}(xe^{rs})/\overline{U}(x)dEN(s)} \leq C^2 \frac{\int_t^{\infty} e^{-r\hat{p}s}dEN(s)}{\int_{0-}^{\infty} e^{-rps}dEN(s)}.$$

By (1.3), we get

$$\int_{0-}^{\infty} e^{-rps}dEN(s) = \sum_{n=1}^{\infty} E(e^{-rp\tau_n}) \leq \sum_{n=1}^{\infty} g_L(n)(Ee^{-rp\theta_1})^n.$$

Take $\epsilon = -\log(Ee^{-rp\theta_1}) - c$ for some $c > 0$ in (2.2), then there exists a positive integer n_1 such that for all $n \geq n_1$,

$$g_L(n) \leq e^{-cn}\exp\{-n\log(Ee^{-rp\theta_1})\}.$$

Thus, we have

$$\int_{0-}^{\infty} e^{-rps}dEN(s) \leq \sum_{n=1}^{n_1-1} g_L(n)\big(Ee^{-rp\theta_1}\big)^n + \sum_{n=n_1}^{\infty} e^{-cn} < \infty.$$

Similarly,

$$\int_{0-}^{\infty} e^{-r\hat{p}s}dEN(s) < \infty.$$

Hence, the third item of (3.11) tends to 0 as $t \to \infty$, which yields that for any given $\varepsilon > 0$, there exists a number t_0, $0 < t_0 < \infty$, such that for all $x \geq D$,

$$(3.12) \qquad \int_{t_0}^{\infty} \overline{U}(xe^{rt})dEN(t) \leq \varepsilon \int_{0-}^{\infty} \overline{U}(xe^{rt})dEN(t).$$

Write $\widetilde{D}_r(t_0) = \sum_{i=1}^{\infty}(X_i + Y_i e^{-rT_i})e^{-r\tau_i}\mathbf{1}_{\{\tau_i \leq t_0\}}$, where t_0 is fixed as above. Now we estimate the asymptotic lower-bound of $P(\widetilde{D}_r(t_0) > x)$, which will be used in the proof below. For an arbitrarily fixed integer m, we derive from Lemma 3.3 that

$$(3.13) \quad P(\widetilde{D}_r(t_0) > x)$$

$$\geq \sum_{n=1}^{m} P\left(\sum_{i=1}^{n}\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x, N(t_0) = n\right)$$

$$= \sum_{n=1}^{m} \int_{\{0 < t_1 \le t_2 \le \cdots \le t_n \le t_0, \ t_{n+1} > t_0\}} P\left(\sum_{i=1}^{n} \left(X_i + Y_i e^{-rT_i} \right) e^{-rt_i} > x \right)$$
$$dG(t_1, t_2, \ldots, t_{n+1})$$

$$\sim \left(\sum_{n=1}^{\infty} - \sum_{n=m+1}^{\infty} \right) \sum_{i=1}^{n} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} > x, N(t_0) = n \right)$$

$$= \sum_{i=1}^{\infty} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t_0\}} > x \right) - H_4,$$

where $G(t_1, t_2, \ldots, t_{n+1})$ is the joint distribution of $(\tau_1, \tau_2, \ldots, \tau_{n+1})$, $1 \le n \le m$. For H_4, it follows from Lemma 3.6 that

$$H_4 \le \overline{U}(x) \sum_{n=m+1}^{\infty} nP(N(t_0) = n)$$
$$= \overline{U}(x) EN(t_0)\mathbf{1}_{\{N(t_0) > m\}} = o(1)\overline{U}(x) \quad \text{as } m \to \infty,$$

which, along with $U \in \mathcal{C} \subset \mathcal{D}$ and Lemma 3.2(1), yields that

$$(3.14) \qquad H_4 = o(1)P\left(\left(X_1 + Y_1 e^{-rT_1} \right) e^{-r\tau_1} \mathbf{1}_{\{\tau_1 \le t_0\}} > x \right).$$

So, we substitute (3.14) into (3.13) to obtain that

$$P\left(\widetilde{D}_r(t_0) > x \right) \gtrsim \sum_{i=1}^{\infty} P\left(\left(X_i + Y_i e^{-rT_i} \right) e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t_0\}} > x \right)$$
$$(3.15) \qquad\qquad = \int_{0-}^{t_0} \overline{U}(xe^{rt}) dEN(t).$$

Under condition 1 of Theorem 2.1, we show from (3.6) and (3.15) that for all large $x \ge D$,

$$(3.16) \quad \psi_r(x)$$

$$\ge P\left(\widetilde{D}_r(t_0) > x + \widetilde{C} + \widetilde{B} \right)$$

$$\gtrsim \int_{0-}^{\infty} \int_{0-}^{t_0} \overline{U}((x+y)e^{rt}) dEN(t) P\left(\widetilde{C} + \widetilde{B} \in dy \right)$$

$$\sim \int_{0-}^{t_0} \overline{U}(xe^{rt}) dEN(t)$$

$$= \left(\int_{0-}^{\infty} - \int_{t_0}^{\infty} \right) \overline{U}(xe^{rt}) dEN(t)$$

$$\ge (1-\varepsilon)\left(\int_{0-}^{\infty} \overline{F}(xe^{rt}) dEN(t) + \int_{0-}^{\infty} \int_{0-}^{\infty} \overline{G}(xe^{r(u+t)}) dH(u) dEN(t) \right),$$

where the third step is due to $U \in \mathcal{C} \subset \mathcal{L}$ and the local uniformity of (1.4), and the last step is due to (3.12) and Lemma 3.3. Therefore, by (3.10), (3.16) and

the arbitrariness of $\varepsilon > 0$, we obtain that relation (1.5) holds under condition 1 of this theorem.

Under condition 2 of Theorem 2.1, we know from $U \in \mathcal{C}$ that for the given $\varepsilon > 0$, there exists a number $\delta_0 > 0$ such that for all large x,

$$(3.17) \qquad \overline{U}((1 + \delta_0)x) \geq (1 - \varepsilon)\overline{U}(x).$$

By (3.6), we see that for $\delta_0 > 0$ as above,

$$(3.18) \qquad \psi_r(x) \geq P\left(\widetilde{D}_r(t_0) - \widetilde{B} > (1 + \delta_0)x\right) - P\left(\widetilde{C} > \delta_0 x\right) = H_5 - H_6.$$

For H_5, arguing as (3.16) and using (3.17) leads to that for all large x,

(3.19)

$$H_5 \gtrsim \int_{0-}^{t_0} \overline{U}((1 + \delta_0)x e^{rt})dEN(t)$$

$$\geq (1 - \varepsilon)\left(\int_{0-}^{\infty} - \int_{t_0}^{\infty}\right)\overline{U}(x e^{rt})dEN(t)$$

$$\geq (1 - \varepsilon)^2 \left(\int_{0-}^{\infty} \overline{F}(x e^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}(x e^{r(u+t)})dH(u)dEN(t)\right),$$

where the last step is due to (3.12) and Lemma 3.3. For H_6, by (2.3), $F \in \mathcal{C} \subset \mathcal{D}$, $G \in \mathcal{C} \subset \mathcal{D}$ and Lemma 3.2(1), we have

(3.20)

$$H_6 = o(1)\left(\overline{F}(x) + \overline{G}(x)\right)$$

$$= o(1)\left(P(X_1 e^{-r\tau_1} > x) + P(Y_1 e^{-r(\tau_1 + T_1)} > x)\right)$$

$$= o(1)\sum_{i=1}^{\infty}\left(P(X_i e^{-r\tau_i} > x) + P(Y_i e^{-r(\tau_i + T_i)} > x)\right)$$

$$= o(1)\left(\int_{0-}^{\infty} \overline{F}(x e^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}(x e^{r(u+t)})dH(u)dEN(t)\right).$$

Consequently, by (3.10), (3.18)-(3.20) and the arbitrariness of $\varepsilon > 0$, we arrive at relation (1.5) under condition 2 of the theorem, and hence the proof is completed. $\square$

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