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EXPANDING THE APPLICABILITY OF SECANT METHOD WITH APPLICATIONS

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Abstract. We present new sufficient convergence criteria for the convergence of the secant-method to a locally unique solution of a nonlinear equation in a Banach space. Our idea uses Lipschitz and center–Lipschitz instead of just Lipschitz conditions in the convergence analysis. The new convergence criteria can always be weaker than the corresponding ones in earlier studies. Numerical examples are also provided in this study to solve equations in cases not possible before.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A vast number of problems from applied science including engineering can be solved by means of finding the solutions equations in a form like (1.1) using mathematical modelling [7, 11, 16, 19]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations cannot be found in closed form. This is the main reason why the most commonly used solution methods are iterative. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in

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the local analysis one finds estimates of the radii of convergence balls from the information around a solution.

We consider the secant method in the form

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \quad (n \geq 0), \quad (x_{-1}, x_0 \in \mathcal{D}) \tag{1.2}$$

where $\delta F(x,y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x, y \in \mathcal{D})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ of the Fréchet–derivative of F [16, 19].

The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$. These are our objectives in this paper.

The secant method, also known under the name of Regula Falsi or the method of chords, is one of the most used iterative procedures for solving nonlinear equations. According to A. N. Ostrowski [20], this method is known from the time of early Italian algebraists. In the case of equations defined on the real line, the secant method is better than Newton's method from the point of view of the efficiency index [7]. The secant method was extended for the solution of nonlinear equations in Banach Spaces by A. S. Sergeev [25] and J. W. Schmidt [24].

The simplified secant method

$$x_{n+1} = x_n - \delta F(x_{-1}, x_0)^{-1} F(x_n) \qquad (n \geq 0), \quad (x_{-1}, x_0 \in D)$$

was first studied by S. Ulm [26]. The first semilocal convergence analysis was given by P. Laasonen [22]. His results was improved by F. A. Potra and V. Pták [21–23]. A semilocal convergence analysis for general secant-type methods was given in general by J. E. Dennis [15]. Bosarge and Falb [10], Dennis [11], Potra [21–23], Argyros [5–9], Hernández et al. [15] and others [14], [19], [27], have provided sufficient convergence conditions for the secant method based on Lipschitz–type conditions on δF.

The conditions usually associated with the semilocal convergence of secant method (1.2) are:

- F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;
- x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\| x_0 - x_{-1} \| \leq c;$$

- F is Fréchet–differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F \colon \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that:

the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded, and:

$$\| A^{-1}F(x_0) \| \leq \eta;$$

$$\| A[\delta F(x,y) - F'(z)] \| \leq \ell(\| x-z \| + \| y-z \|);$$

for all $x, y, z \in \mathcal{D}$;

$$\ell c + 2\sqrt{\ell \eta} \leq 1. \tag{1.3}$$

The sufficient convergence condition (1.3) is easily violated (see the Numerical Examples). Hence, there is no guarantee in these cases that equation (1.1) under the information (ℓ, c, η) has a solution that can be found using secant method (1.2). In this study we are motivated by optimization considerations, and the above observation.

The use of Lipschitz and center–Lipschitz conditions is one way used to enlarge the convergence domain of different methods. This technique consists of using both conditions together instead of using only the Lipschitz one which allows us to find a finer majorizing sequence, that is, a larger convergence domain. It has been used in order to find weaker convergence criteria for Newton's method by Argyros in [8]. Gutiérrez et al. in [13] give sufficient conditions for Newton's method using both Lipschitz and center-Lipschitz conditions, Magreñán in [18] for the damped Newton's methods and Amat et al. in [2, 4] or García-Olivo [12] for other methods.

Here using Lipschitz and center–Lipschitz conditions, we provide a new semilocal convergence analysis for (1.2). It turns out that our new convergence criteria can always be weaker than the old ones given in earlier studies such as [1, 14, 17, 19, 21–24, 27, 28]. The paper is organized as follows: The semilocal convergence analysis of the secant method is presented in Section 2. Numerical examples are provided in Section 3.

2. Semilocal convergence analysis of the secant method

In this Section, we present the semilocal convergence analysis of the secant-method (1.2). First, we present two auxiliary results concerning convergence criteria and majorizing sequences.

Lemma 2.1. *Let $\ell_0 > 0$, $\ell > 0$, $c > 0$ and $\eta > 0$ be constants with $\ell_0 \leq \ell$. Then, the following items hold*

(i)

$$0 < \frac{\ell(c+\eta)}{1-\ell_0(c+\eta)} \leq \frac{2\ell}{\ell+\sqrt{\ell^2+4\ell_0\ell}} < \frac{1-\ell_0(c+\eta)}{1-\ell_0 c} \Leftrightarrow c+\eta \leq \frac{4\ell^2}{\left(\ell+\sqrt{\ell^2+4\ell_0\ell}\right)^2}; \tag{2.1}$$

(ii)

$$\ell c \leq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}} \Leftrightarrow \frac{(1-\ell c)^2}{4} \leq b^2 - \ell c; \tag{2.2}$$

(iii)

$$\ell c \geq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}} \Leftrightarrow \frac{(1-\ell c)^2}{4} \geq b^2 - \ell c; \tag{2.3}$$

(iv)

$$\ell c \leq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}} \textit{ and } \ell c + \sqrt{\ell \eta} \leq 1 \Rightarrow c + \eta \leq \frac{4\ell}{\left(\ell+\sqrt{\ell^2+4\ell_0\ell}\right)^2} c; \tag{2.4}$$

(v)

$$\ell c \geq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}} \textit{ and } c + \eta \leq \frac{4\ell}{\left(\ell+\sqrt{\ell^2+4\ell_0\ell}\right)^2} \Rightarrow \ell c + \sqrt{\ell \eta} \leq 1. \tag{2.5}$$

Proof. Let $x = 1 - \ell c$, $y = \ell \eta$, $a = \frac{\ell_0}{\ell}$ and $b = \frac{2}{1+\sqrt{1+4a}}$. Then, we have that $ab^2 + b - 1 = 0$ and $ab + 1 = \frac{1}{b}$.

(i) The triple inequality in (2.1) holds, if

$$\frac{\ell c + \ell \eta}{1 - a\ell(c+\eta)} \leq \frac{2\ell}{\ell + \sqrt{\ell^2 + 4a\ell^2}} = b, \tag{2.6}$$

$$b < \frac{1 - a\ell(c+\eta)}{1 - a\ell c} \tag{2.7}$$

and

$$\ell(c+\eta) < \frac{1}{a} \tag{2.8}$$

or, if

$$y \leq b^2 - (1-x), \tag{2.9}$$

$$y < \frac{1-b}{a} - (1-b)(1-x) = b^2 - (1-b)(1-x), \tag{2.10}$$

and

$$y \leq \frac{1}{a} - (1-x), \tag{2.11}$$

respectively. We have that $ab^2 = 1 - b < 1$ by the definition of a and b. It follows that

$$b^2 - (1-x) < \frac{1}{a} - (1-x) \tag{2.12}$$

and from $(1-b)(1-x) < (1-x)$ we get that

$$b^2 - (1-x) < b^2 - (1-b)(1-x). \tag{2.13}$$

Hence, it follows from (2.12) and (2.13) that (2.6)–(2.8) are satisfied if (2.9) holds. But (2.9) is equivalent to the right hand side inequality in (2.1). Conversely, if the right hand side inequality in (2.1) holds, then (2.9), (2.12) and

(2.13) imply (2.10) and (2.11) imply (2.6)–(2.8) which imply the triple inequality in (2.1).

(ii)

$$\begin{aligned}\ell c &\leq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}} \Leftrightarrow 2(1-b) < x < 2(1+b) \Leftrightarrow x^2-4x+4(1-b^2) \leq 0\\ &\Leftrightarrow \frac{x^2}{4} \leq b^2-(1-x) \Leftrightarrow \frac{(\ell\eta)^2}{4} \leq b^2-\ell c.\end{aligned}$$

(iii)

$$\frac{(\ell\eta)^2}{4} \geq b^2-\ell c \Leftrightarrow \frac{x^2}{4} \geq b^2-(1-x) \Leftrightarrow x^2-4x+4(1-b^2) \geq 0$$

$$\Rightarrow x \leq 2(1-b) \Leftrightarrow \ell c \geq \frac{3-\sqrt{1+4\frac{\ell_0}{\ell}}}{1+\sqrt{1+4\frac{\ell}{\ell_0}}}$$

(since $x \geq 2(1+b)$ cannot hold).

(iv) The hypotheses in (2.4) and (2.2) imply $\ell\eta \leq b^2-\ell c$ which is

$$c+\eta \leq \frac{4\ell}{\left(\ell+\sqrt{\ell^2+4\ell_0\ell}\right)^2}.$$

(v) The hypothesis in (2.5) and (2.3) imply

$$\ell c+\sqrt{\ell\eta} \leq 1. \qquad \square$$

We need the following result on majorizing sequences for the secant method (1.2).

Lemma 2.2. *Let $\ell_0>0$, $\ell>0$, $c>0$, and $\eta>0$ be constants with $\ell_0 \leq \ell$. Suppose*

$$c+\eta \leq \frac{4\ell^2}{\ell+\sqrt{\ell^2+4\ell_0\ell}}. \tag{2.14}$$

Then, scalar sequence $\{t_n\}$ $(n \geq -1)$ given by

$$t_{-1}=0,\ t_0=c,\ t_1=c+\eta,\ t_{n+2}=t_{n+1}+\frac{\ell(t_{n+1}-t_{n-1})(t_{n+1}-t_n)}{1-\ell_0(t_{n+1}-t_0+t_n)} \tag{2.15}$$

is increasing, bounded from above by

$$t^{\star\star}=\frac{\eta}{1-b}+c, \tag{2.16}$$

and converges to its unique least upper bound $t^\star$ such that

$$c+\eta \leq t^\star \leq t^{\star\star}, \tag{2.17}$$

Moreover, the following estimates hold for all $n \geq 0$:

$$0 \leq t_{n+2}-t_{n+1} \leq b(t_{n+1}-t_n) \leq b^{n+1}\eta, \tag{2.18}$$

where b is given in Lemma 2.1.

Proof. We shall show using induction on $k \geq 0$ that

$$0 \leq t_{k+2} - t_{k+1} \leq b(t_{k+1} - t_k). \tag{2.19}$$

Using (2.15) for $k = 0$, we must show

$$0 < \frac{\ell(t_1 - t_{-1})}{1 - \ell_0 t_1} \leq b$$

or

$$0 < \frac{\ell(c + \eta)}{1 - \ell_0(c + \eta)} \leq b,$$

which is true by (2.1) and (2.14). Let assume that (2.19) holds for $k \leq n + 1$.
It then follows from the induction hypotheses that

$$\begin{aligned} t_{k+2} &\leq t_{k+1} + b(t_{k+1} - t_k) \\ &\leq t_k + b(t_k - t_{k-1}) + b(t_{k+1} - t_k) \\ &\leq t_1 + b(t_1 - t_0) + \cdots + b(t_{k+1} - t_k) \\ &\leq c + \eta + b\eta + \cdots + b^{k+1}\eta \\ &= c + \frac{1 - b^{k+2}}{1 - b}\eta < \frac{\eta}{1 - b} + c = t^{\star\star}. \end{aligned} \tag{2.20}$$

Moreover, we can have:

$$\begin{aligned} &\ell(t_{k+2} - t_{k+1}) + b\ell_0(t_{k+2} - t_0 + t_{k+1}) \\ &\leq \ell\Big((t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)\Big) + b\ell_0\left(\frac{1 - b^{k+2}}{1 - b} + \frac{1 - b^{k+1}}{1 - b}\right)\eta + b\ell_0 c \\ &\leq \ell(b^k + b^{k+1})\eta + \frac{b\ell_0}{1 - b}(2 - b^{k+1} - b^{k+2})\eta + b\ell_0 c. \end{aligned} \tag{2.21}$$

In view of (2.21), inequality (2.19) holds, if

$$\ell(b^k + b^{k+1})\eta + \frac{b\ell_0}{1 - b}(2 - b^{k+1} - b^{k+2})\eta + b\ell_0 c \leq b \tag{2.22}$$

or

$$\ell(b^{k-1} + b^k)\eta + \ell_0\Big((1 + b + \cdots + b^k) + (1 + b + \cdots + b^{k+1})\Big)\eta + \ell_0 c - 1 \leq 0. \tag{2.23}$$

In view of (2.23), we are motivated to define recurrent functions for $k \geq 1$ on $[0, 1)$ by

$$f_k(t) = \ell(t^{k-1} + t^k)\eta + \ell_0\Big(2(1 + t + \cdots + t^k) + t^{k+1}\Big)\eta + \ell_0 c - 1. \tag{2.24}$$

We need the relationship between two consecutive functions f_k. Using (2.24), we obtain

(2.25)

$$\begin{aligned} f_{k+1}(t) &= \ell(t^k + t^{k+1})\eta + \ell_0\bigg(2(1+t+\cdots+t^{k+1}) + t^{k+2}\bigg)\eta + \ell_0 c - 1 \\ &= \ell(t^{k-1} + t^k)\eta + \ell(t^k + t^{k+1})\eta - \ell(t^{k-1} + t^k)\eta \\ &\quad + \ell_0\bigg(2(1+t+\cdots+t^k) + t^{k+1}\bigg)\eta + \ell_0(2t^{k+1} + t^{k+2})\eta \\ &\quad - \ell_0 t^{k+1}\eta + \ell_0 c - 1 \\ &= f_k(t) + \ell(t^{k+1} - t^{k-1})\eta + \ell_0(t^{k+1} + t^{k+2})\eta \\ &= p(t)t^{k-1}\eta + f_k(t), \end{aligned}$$

where $p(t) = \ell_0 t^3 + (\ell_0 + \ell)t^2 - \ell$. Notice that by Descarte's rule of signs, b is the only positive root of polynomial p. We can show instead of (2.23)

$$f_k(b) \le 0, \quad k \ge 1. \tag{2.26}$$

Define functions f_∞ on interval $[0,1)$ by $f_\infty(t) = \lim_{k\to\infty} f_k(t)$. Then, in view of (2.24) we get that

$$f_\infty(t) = \frac{2\ell_0\eta}{1-t} + \ell_0 c - 1. \tag{2.27}$$

We have that $f_k(b) = f_{k+1}(b) = f_\infty(b)$. Hence, we can show instead of (2.26) that $f_\infty(b) \le 0$, which is true by (2.1), (2.14) and (2.27). Hence, we showed sequence $\{t_n\}$ $(n \ge -1)$ is increasing and bounded from above by $t^{\star\star}$, so that (2.18) holds. It follows that there exists $t^\star \in [c+\eta, t^{\star\star}]$, so that $\lim_{n\to\infty} t_n = t^\star$. □

We denote by $U(z,\varrho)$ the open ball centered ar $z \in \mathcal{X}$ and of radius $\varrho > 0$. We also denote by $\bar{U}(z,\varrho)$ the closure of $U(z,\varrho)$. We shall study the secant method (1.2) for triplets (F, x_{-1}, x_0) belonging to the class $\mathcal{C}(\ell, \ell_0, \eta, c)$ defined as follows:

Definition 2.3. Let ℓ, ℓ_0, η, c be positive constants satisfying the hypotheses of Lemma 2.2.

We say that a triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(\ell, \ell_0, \eta, c)$ if:

(c_1) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

(c_2) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\| x_0 - x_{-1} \| \le c;$$

(c_3) F is Fréchet–differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F\colon \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that:

the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded and:

$$\begin{aligned} \| A^{-1} F(x_0) \| &\le \eta; \\ \| A[\delta F(x,y) - F'(z)] \| &\le \ell(\| x - z \| + \| y - z \|); \\ \| A[\delta F(x,y) - F'(x_0)] \| &\le \ell_0(\| x - x_0 \| + \| y - x_0 \|) \end{aligned}$$

for all $x, y, z \in \mathcal{D}$.

(c$_4$) the set $\mathcal{D}_c = \{x \in \mathcal{D};\ F$ is continuous at $x\}$ contains the closed ball $\overline{U}(x_0, t^\star - t_0)$, where $t^\star$ is given in Lemma 2.2.

We present the following semilocal convergence theorem for secant method (1.2).

Theorem 2.4. *If* $(F, x_{-1}, x_0) \in \mathcal{C}(\ell, \ell_0, \eta, c)$, *then sequence* $\{x_n\}$ $(n \ge -1)$ *generated by secant method* (1.2) *is well defined, remains in* $\overline{U}(x_0, t^\star - t_0)$ *for all* $n \ge 0$ *and converges to a unique solution* $x^\star \in \overline{U}(x_0, t^\star - t_0)$ *of equation* $F(x) = 0$. *Moreover the following estimates hold for all* $n \ge 0$

$$\| x_{n+2} - x_{n+1} \| \le t_{n+2} - t_{n+1}, \tag{2.28}$$

and

$$\| x_n - x^\star \| \le t^\star - t_n, \tag{2.29}$$

where the sequence $\{t_n\}$ $(n \ge 0)$ *given by* (2.15). *Furthermore, if there exists* $R \ge t^\star - t_0$, *such that*

$$\ell_0 \ (c + \frac{\eta}{1-b} + R) \le 1, \tag{2.30}$$

and

$$U(x_0, R) \subseteq \mathcal{D}, \tag{2.31}$$

then the solution $x^\star$ *is unique in* $\overline{U}(x_0, R)$.

Proof. We first show operator $L = \delta F(u, v)$ is invertible for $u, v \in \overline{U}(x_0, t^\star - t_0)$. It follows from (2.1), (c$_2$) and (c$_3$) that:

$$\begin{aligned} \| I - A^{-1} L \| &= \| A^{-1}(L - A) \| \\ &\le \| A^{-1}(L - F'(x_0)) \| + \| A^{-1}(F'(x_0) - A) \| \\ &\le \ \ell_0(\| u - x_0 \| + \| v - x_0 \| + \| x_0 - x_{-1} \|) \\ &\le \ \ell_0(t^\star - t_0 + t^\star - t_0 + c) \\ &\le \ \ell_0\left(2\left(\frac{\eta}{1-b} + c\right) - c\right) < 1. \end{aligned} \tag{2.32}$$

According to the Banach Lemma on invertible operators [8], [16], and (2.32), L is invertible and

$$\| L^{-1} A \| \le \left(1 - \ell_0(\| x_k - x_0 \| + \| x_{k+1} - x_0 \| + c)\right)^{-1}. \tag{2.33}$$

The second condition in (c_3) implies the Lipschitz condition for F'

$$\| A^{-1}(F'(u) - F'(v)) \| \leq 2\ell \| u - v \|, \quad u, v \in \mathcal{D}^0. \tag{2.34}$$

By the identity,

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))dt(x - y) \tag{2.35}$$

we get

$$\| A_0^{-1}[F(x) - F(y) - F'(u)(x - y)] \| \leq \ell(\| x - u \| + \| y - u \|) \| x - y \| \tag{2.36}$$

and

$$\begin{aligned} &\| A_0^{-1}[F(x) - F(y) - \delta F(u, v)(x - y)] \| \\ &\leq \ell(\| x - v \| + \| y - v \| + \| u - v \|) \| x - y \| \end{aligned} \tag{2.37}$$

for all $x, y, u, v \in \mathcal{D}^0$. By a continuity argument (2.34)–(2.37) remain valid if x and/or y belong to $\mathcal{D}_c$. We first show (2.28). If (2.28) holds for all $n \leq k$ and if $\{x_n\}$ $(n \geq 0)$ is well defined for $n = 0, 1, 2, \ldots, k$ then

$$\| x_0 - x_n \| \leq t_n - t_0 < t^\star - t_0, \quad n \leq k. \tag{2.38}$$

That is (1.2) is well defined for $n = k + 1$. For $n = -1$, and $n = 0$, (2.28) reduces to $\| x_{-1} - x_0 \| \leq c$, and $\| x_0 - x_1 \| \leq \eta$. Suppose (2.28) holds for $n = -1, 0, 1, \ldots, k$ $(k \geq 0)$. Using (2.33), (2.37) and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)(x_{k+1} - x_k) \tag{2.39}$$

we obtain in turn:

$$\begin{aligned} \| A^{-1}F(x_{k+1}) \| &= \ell(\| x_{k+1} - x_k \| + \| x_k - x_{k-1} \|) \| x_{k+1} - x_k \| \\ &= \ell(t_{k+1} - t_k + t_k - t_{k-1})(t_{k+1} - t_k) \\ &= \ell(t_{k+1} - t_{k-1})(t_{k+1} - t_k) \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &= \| \delta F(x_k, x_{k+1})^{-1}F(x_{k+1}) \| \\ &\leq \| \delta F(x_k, x_{k+1})^{-1}A \| \| A^{-1}F(x_{k+1}) \| \\ &\leq \frac{\ell(t_{k+1} - t_k + t_k - t_{k-1})}{1 - \ell_0(t_{k+1} - t_0 + t_k - t_0 + t_0 - t_{-1})}(t_{k+1} - t_k) \\ &= t_{k+2} - t_{k+1}. \end{aligned} \tag{2.41}$$

The induction for (2.28) is completed. It follows from (2.28) and Lemma 2.2 that sequence $\{x_n\}$ $(n \geq -1)$ is complete in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star - t_0)$ (since $\overline{U}(x_0, t^\star - t_0)$ is a closed set). By letting $k \to \infty$ in (2.41), we obtain $F(x^\star) = 0$. Estimate (2.29) follows from (2.28) by using standard majoration techniques [7, 16, 19, 23]. We shall first show uniqueness in $\overline{U}(x_0, t^\star - t_0)$. Let $y^\star \in \overline{U}(x_0, t^\star - t_0)$ be a solution of equation (1.1).

Set

$$\mathcal{M} = \int_0^1 F'(y^\star + t(y^\star - x^\star))dt.$$

It then by (c_3):

$$\begin{aligned} \| A^{-1}(A - \mathcal{M}) \| &= \ell_0(\| y^\star - x_0 \| + \| x^\star - x_0 \| + \| x_0 - x_{-1} \|) \\ &\le \ell_0((t^\star - t_0) + (t^\star - t_0) + t_0) \\ &\le \ell_0\left(2\left(\frac{\eta}{1-b} + c\right) - c\right) \\ &= \ell_0\left(\frac{2\eta}{1-b} + c\right) < 1. \end{aligned} \tag{2.42}$$

It follows from (2.1), and the Banach lemma on invertible operators that $\mathcal{M}^{-1}$ exists on $\overline{U}(x_0, t^\star - t_0)$. Using the identity:

$$F(x^\star) - F(y^\star) = \mathcal{M}(x^\star - y^\star) \tag{2.43}$$

we deduce $x^\star = y^\star$. Finally, we shall show uniqueness in $U(x_0, R)$. As in (2.42), we arrive at

$$\| A^{-1}(A - \mathcal{M}) \| < \ell_0\left(\frac{\eta}{1-b} + c + R\right) \le 1,$$

by (2.30). □

Remark 2.5. (a) Let us define the majoring sequence $\{w_n\}$ used in earlier studies such as [1, 14, 17, 19, 21–24, 27, 28] (under condition (1.3)):

$$w_{-1} = 0,\ w_0 = c,\ w_1 = c + \eta,\ w_{n+2} = w_{n+1} + \frac{\ell(w_{n+1} - w_{n-1})(w_{n+1} - w_n)}{1 - \ell(w_{n+1} - w_0 + w_n)}. \tag{2.44}$$

Note that in general

$$\ell_0 \le \ell \tag{2.45}$$

holds, and $\frac{\ell}{\ell_0}$ can be arbitrarily large [5–8]. In the case $\ell_0 = \ell$, then $t_n = w_n$ ($n \ge -1$). Otherwise:

$$t_{n+1} - t_n \le w_{n+1} - w_n, \tag{2.46}$$

$$0 \le t^\star - t_n \le w^\star - w_n, \quad w^\star = \lim_{n\to\infty} w_n. \tag{2.47}$$

Note also that strict inequality holds in (2.46) for $n \ge 1$, if $\ell_0 < \ell$. It is worth noticing that the center-Lipschitz condition is not an additional hypothesis to the Lipschitz condition, since in practice the computation of constant ℓ requires the computation of ℓ_0. It follows from the proof of Theorem 2.4 that sequence $\{s_n\}$ defined by

$$s_{-1} = 0, \qquad s_0 = c, \qquad s_1 = c + \eta, \qquad s_2 = s_1 + \frac{\ell_0(s_1 - s_{-1})(s_1 - s_0)}{1 - \ell_0 s_1}$$

$$s_{n+2} = s_{n+1} + \frac{\ell(s_{n+1} - s_{n-1})(s_{n+1} - s_n)}{1 - \ell_0(s_{n+1} - s_0 + s_n)} \qquad \text{for } n = 1, 2, \ldots.$$

is also a majorizing sequence for $\{x_n\}$ which is tighter than $\{t_n\}$.

(b) In practice constant c depends on initial guesses x_{-1} and x_0 which can be chosen to be as close to each other as we wish. Therefore, in particular, we can always choose

$$\ell c < \frac{3 - \sqrt{1 + 4\frac{\ell_0}{\ell}}}{1 + \sqrt{1 + 4\frac{\ell_0}{\ell}}},$$

which according to (iv) in Lemma 2.1 implies that the new sufficient convergence criterion (2.14) is weaker than the old one given by (1.3).

3. Numerical examples

Example 3.1. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, equipped with the max-norm. Consider the following nonlinear boundary value problem

$$\begin{cases} u'' = -u^3 - \gamma u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)(u^3(t) + \gamma u^2(t))dt, \tag{3.1}$$

where $\mathcal{Q}$ is the Green function:

$$\mathcal{Q}(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |\mathcal{Q}(s,t)| dt = \frac{1}{8}.$$

Then problem (3.1) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)(x^3(t) + \gamma x^2(t))dt.$$

The Fréchet derivative of the operator F is given by

$$[F'(x)y](s) = y(s) - 3\int_0^1 \mathcal{Q}(s,t)x^2(t)y(t)dt - 2\gamma \int_0^1 \mathcal{Q}(s,t)x(t)y(t)dt.$$

Then, we have that

$$[(I - F'(x_0))(y)](s) = 3\int_0^1 \mathcal{Q}(s,t)x_0^2(t)y(t)dt + 2\gamma \int_0^1 \mathcal{Q}(s,t)x_0(t)y(t)dt.$$

Hence, if $2\gamma < 5$, then

$$\|I - F'(x_0)\| \le 2(\gamma - 2) < 1.$$

It follows that $F'(x_0)^{-1}$ exists and

$$\|F'(x_0)^{-1}\| \le \frac{1}{5 - 2\gamma}.$$

We also have that $\|F(x_0)\| \leq 1+\gamma$. Define the divided difference defined by

$$\delta F(x,y) = \int_0^1 F'(y+t(x-y))dt.$$

Choosing $x_{-1}(s)$ such that $\|x_{-1}-x_0\| \leq c$ and $k_0 c < 1$. Then, we have

$$\|\delta F(x_{-1},x_0)^{-1}F(x_0)\| \leq \|\delta F(x_{-1},x_0)^{-1}F'(x_0)\|\|F'(x_0)F(x_0)\|$$

and

$$\|\delta F(x_{-1},x_0)^{-1}F'(x_0)\| \leq \frac{1}{(1-k_0 c)},$$

where k_0 is such that

$$\|F'(x_0)^{-1}(F'(x_0)-A_0)\| \leq k_0 c.$$

Set $u_0(s)=s$ and $\mathcal{D}=U(u_0,R)$. It is easy to verify that $U(u_0,R)\subset U(0,R+1)$ since $\| u_0 \|=1$. If $2\gamma<5$, and $k_0c<1$ the operator F' satisfies conditions of Theorem 2.6, with

$$\eta = \frac{1+\gamma}{(1-k_0c)(5-2\gamma)}, \quad l = \frac{\gamma+6R+3}{8(5-2\gamma)(1-k_0c)}, \quad l_0 = \frac{2\gamma+3R+6}{16(5-2\gamma)(1-k_0c)}.$$

Choosing $R_0=0.9$, $\gamma=0.5$ and $c=1$ we obtain that

$$k_0 = 0.1938137822\cdots,$$
$$\eta = 0.465153\cdots,$$
$$l = 0.344989\cdots$$

and

$$l_0 = 0.187999\cdots.$$

Then, criterion (1.3) is not satisfied since $lc+2\sqrt{l\eta} = 1.14617\cdots > 1$, but criterion (2.14) is satisfied since

$$\eta + c = 1.46515\cdots \leq \frac{4l}{(l^2+\sqrt{l^2+4l_0l})^2} = 1.49682\cdots.$$

As a consequence the convergence of the secant-method is guaranteed by Theorem 2.4.

Example 3.2. Let $\mathcal{X}=\mathcal{Y}=\mathbb{R}$ and let consider the real functions

$$F(x) = x^3 - k,$$

where $k\in\mathbb{R}$ and we are going to apply secant-method to find the solution of $F(x)=0$. We take the starting point $x_0=1$ we consider the domain $\Omega = B(x_0,1)$ and we let x_{-1} free in order to find a relation between k and x_{-1} for which criterion (1.3) is not satisfied but new criterion (2.14) is satisfied. In this case, we obtain

$$\eta = |(1-k)(1+x_{-1}+x_{-1}^2)|,$$
$$l = \frac{6}{|1+x_{-1}+x_{-1}^2|},$$

$$l_0 = \frac{9}{2|1 + x_{-1} + x_{-1}^2|}.$$

Taking all this data into account we obtain the following criteria:

(i) If $55/54 < k \leq 25/24$ and

$$\alpha < x_{-1} \leq \frac{2 - 27k}{2(-29 + 27k)} - \frac{1}{2}\sqrt{3}\sqrt{-\frac{2164 - 3024k + 729k^2}{(-29 + 27k)^2}},$$

where α is the smallest positive root of

$$p(t) - 73 + 24k + (22 + 48k)t + (-111 + 72k)t^2 + (-38 + 48k)t^3 + (-25 + 24k)t^4.$$

(ii) If $25/24 < k < 29/27$ and

$$1 < x_1 \leq \frac{2 - 27k}{2(-29 + 27k)} - \frac{1}{2}\sqrt{3}\sqrt{-\frac{2164 - 3024k + 729k^2}{(-29 + 27k)^2}}.$$

(iii) If $55/54 < k < 25/24$ and

$$\frac{56 - 27k}{2(-29 + 27k)} + \frac{1}{2}\sqrt{3}\sqrt{-\frac{-968 - 108k + 729k^2}{(-29 + 27k)^2}} \leq x_{-1} < \alpha,$$

where α is the greatest positive root of

$$p(t) = -49 + 24k + (22 + 48k)t + (-111 + 72k)t^2 + (-62 + 48k)t^3 + (-25 + 24k)t^4.$$

(iv) If $25/24 \leq k < 29/27$ and

$$\frac{56 - 27k}{2(-29 + 27k)} + \frac{1}{2}\sqrt{3}\sqrt{-\frac{-968 - 108k + 729k^2}{(-29 + 27k)^2}} \leq x_{-1} < 1.$$

(v) If $25/27 < k < 23/24$ and

$$1 \leq x_{-1} < \frac{52 - 27k}{2(-25 + 27k)} - \frac{1}{2}\sqrt{3}\sqrt{-\frac{-968 + 108k + 729k^2}{(-25 + 27k)^2}}.$$

(vi) If $23/24 \leq k < 53/54$ and

$$\alpha \leq x_{-1} < \frac{52 - 27k}{2(-25 + 27k)} - \frac{1}{2}\sqrt{3}\sqrt{-\frac{-968 + 108k + 729k^2}{(-25 + 27k)^2}},$$

where α is the smallest positive root of

$$p(t) = 25 + 24k + (-118 + 48k)t + (-33 + 72k)t^2 + (-58 + 48k)t^3 + (-23 + 24k)t^4.$$

(vii) If $25/27 < k \leq 23/24$ and

$$\frac{-2 - 27k}{2(-25 + 27k)} + \frac{1}{2}\sqrt{3}\sqrt{-\frac{1732 - 2808k + 729k^2}{(-25 + 27k)^2}} \leq x_{-1} < 1.$$

(viii) If $23/24 < k < 53/54$ and

$$\frac{-2-27k}{2(-25+27k)} + \frac{1}{2}\sqrt{3}\sqrt{-\frac{1732-2808k+729k^2}{(-25+27k)^2}} \leq x_{-1} < \alpha,$$

where α is the greatest positive root of

$p(t) = 1+24k+(-118+48k)t+(-33+72k)t^2+(-34+48k)t^3+(-23+24k)t^4.$

Now we consider a case in which both criteria (1.3) and (2.14) are satisfied to compare the majorizing sequences. We choose $k = 0.99$ and $x_{-1} = 1.2$ and we obtain

$$c = 0.2, \qquad \eta = 0.0364\cdots, \qquad l = 1.64835, \qquad l_0 = 1.23626.$$

Moreover, criterion (1.3)

$$lc + 2\sqrt{l\eta} = 0.819568 < 1,$$

is satisfied and criterion (2.14)

$$c + \eta = 0.2364\cdots \leq 0.26963\cdots = \frac{4l}{(l^2+\sqrt{l^2+4l_0 l})^2},$$

is also satisfied. In Table 1 it is shown that $\{s_n\}$, $\{t_n\}$ and $\{w_n\}$ are majorizing sequences and it is shown also that the tighter sequence is $\{s_n\}$.

Table 1. Comparison between the sequences $\{s_n\}$, $\{t_n\}$ and $\{w_n\}$

n	$\|s_{n+1}-s_n\|$	$\|t_{n+1}-t_n\|$	$\|w_{n+1}-w_n\|$
1	$0.0150308\cdots$	$0.0200411\cdots$	$0.0232399\cdots$
2	$0.00197814\cdots$	$0.00292257\cdots$	$0.00446203\cdots$
3	$0.0000890021\cdots$	$0.000181477\cdots$	$0.000339709\cdots$
4	4.88677×10^{-7}	1.53289×10^{-6}	4.52784×10^{-6}
5	1.16179×10^{-10}	7.63675×10^{-10}	4.32958×10^{-9}
6	1.66533×10^{-16}	3.16414×10^{-15}	5.45120×10^{-14}

Conclusion

We present a new semilocal convergence analysis for the secant method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. We showed that the new convergence criteria can be always weaker than the corresponding ones in earlier studies such as [1, 14, 17, 19, 21–24, 27, 28]. Numerical examples where the old results cannot guarantee the convergence but our new convergence criteria can are also provided in this study.

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