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POINTWISE ESTIMATES AND BOUNDEDNESS OF GENERALIZED LITTLEWOOD-PALEY OPERATORS IN $BMO(\mathbb{R}^n)$

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Abstract. In this paper, we study the generalized Littlewood-Paley operators. It is shown that the generalized g-function, Lusin area function and g_λ^*-function on any BMO function are either infinite everywhere, or finite almost everywhere, respectively; and in the latter case, such operators are bounded from $BMO(\mathbb{R}^n)$ to $BLO(\mathbb{R}^n)$, which improve and generalize some previous results.

1. Introduction

The classic Littlewood-Paley operators (i.e., g-function, Lusin area function and g_λ^* function) on Euclid space $\mathbb{R}^n$ were first introduced by Stein in [12]. Let $\varphi(x)$ be a real-valued integrable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \varphi(x)dx = 0, \tag{1.1}$$

$$|\varphi(x)| \le \frac{C}{(1+|x|)^{n+1}}, \tag{1.2}$$

$$|\nabla\varphi(x)| \le \frac{C}{(1+|x|)^{n+2}}, \tag{1.3}$$

where

$$\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n),$$

and C is a constant independent of x.

For a function φ, which satisfies the above conditions, we set

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad t > 0,\ x \in \mathbb{R}^n.$$

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Then the classic Littlewood-Paley operators are defined as

$$g(f)(x) = \Big\{ \int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t} \Big\}^{1/2},$$

$$S(f)(x) = \Big\{ \int \int_{\Gamma(x)} |f * \varphi_t(y)|^2 \frac{dydt}{t^{n+1}} \Big\}^{1/2},$$

$$g_\lambda^*(f)(x) = \Big\{ \int \int_{\mathbb{R}_+^{n+1}} (\frac{t}{t+|x-y|})^{n\lambda} |f * \varphi_t(y)|^2 \frac{dydt}{t^{n+1}} \Big\}^{1/2}, \quad \lambda \in (1, \infty),$$

where $\mathbb{R}_+^{n+1} = \{(y,t) \in \mathbb{R}^{n+1} : y \in \mathbb{R}^n, \ t > 0\}$, $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| < t\}$ for $x \in \mathbb{R}^n$.

As is well known, the Littlewood-Paley operators play important roles in harmonic analysis. Many authors have studied boundedness and various properties of these operators. The literature on the study of the Littlewood-Paley operators is by now quite vast. We will recount here some closely related results. In 1984, Wang proved in [15] that if $f \in BMO(\mathbb{R}^n)$, then $g(f)$ is either infinite everywhere or is finite almost everywhere and in the latter case $g(f)$ is bounded on $BMO(\mathbb{R}^n)$. Subsequently, Kurtz obtained similar results for the Lusin area function S and the Littlewood-Paley g_λ^* function in [7]. Later on, Leckband in [8] proved the boundedness of the above three operators from $L^\infty(\mathbb{R}^n)$ to $BLO(\mathbb{R}^n)$. Recently, Meng and Yang in [10] improved this result with replacing $L^\infty(\mathbb{R}^n)$ by the larger space $BMO(\mathbb{R}^n)$. In addition, the mapping properties for such operators in the Campanato spaces and the Lipschitz spaces $Lip_\alpha(\mathbb{R}^n)$ and other various spaces are also established (see [6, 11, 16] for more details).

In this paper, we will focus on the generalized Littlewood-Paley operators defined as follows:

$$g_r(f)(x) = \left\{ \int_{\mathbb{R}^n} |f * \varphi_t(x)|^r \frac{dt}{t} \right\}^{1/r}, \tag{1.4}$$

$$S_r(f)(x) = \left\{ \int \int_{\Gamma(x)} |f * \varphi_t(y)|^r \frac{dydt}{t^{n+1}} \right\}^{1/r}, \tag{1.5}$$

$$g_{\lambda,r}^*(f)(x) = \left\{ \int \int_{\mathbb{R}_+^{n+1}} (\frac{t}{t+|x-y|})^{n\lambda} |f * \varphi_t(y)|^r \frac{dydt}{t^{n+1}} \right\}^{1/r}, \tag{1.6}$$

where $r > 1$. Clearly, g_2, S_2 and $g_{\lambda,2}^*$ are the classic Littlewood-Paley function $g(f)$, Lusin area function $S(f)$ and g_λ^* function, respectively.

For the generalized Littlewood-Paley operators, several attentions have been attracted. For example, Chen [3] showed that if $r \geq 2$, $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then g_r is bounded on $L^p(\mathbb{R}^n)$; Sun [14] proved that if $2 \leq r < \infty$, $f \in BMO(\mathbb{R}^n)$, then $g_r(f)(x)$ is finite almost everywhere and g_r is bounded on $BMO(\mathbb{R}^n)$ if $g_r(f)(x)$ is finite in a set of positive measure; Bao and Tao [1] obtained the following results: if $r \geq 2$, and f belongs to the Campanato space

$\mathcal{E}^{\alpha,p}$ for $0 \leq \alpha < 1$, $1 < p < \infty$, then $g_r(f)(x)$ is either infinite everywhere or is finite almost everywhere and in the latter case g_r is bounded on the Campanato space $\mathcal{E}^{\alpha,p}$.

Comparing with the results of Meng and Yang in [10], it is natural to ask whether these generalized operators are also bounded from $BMO(\mathbb{R}^n)$ to $BLO(\mathbb{R}^n)$? The main purpose of this paper is to address this question.

To state our results, we first recall the definitions of BMO and BLO spaces. For a complex-valued integrable function f on $\mathbb{R}^n$, set

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f(x) - m_B(f)|dx,$$

where the supremum is taken over all ball B in $\mathbb{R}^n$ and $m_B(f)$ denotes the integral average of f on the ball B. The function f is called to be bounded mean oscillation if $\|f\|_{BMO} < \infty$ and $BMO(\mathbb{R}^n)$ is the set of all locally integrable functions f on $\mathbb{R}^n$ with $\|f\|_{BMO(\mathbb{R}^n)} < \infty$.

As is well known, for all $p \in (1,\infty)$ we have

$$\left(\frac{1}{|B|}\int_B |f(x) - m_B(f)|^p dx\right)^{\frac{1}{p}} \leq C_p\|f\|_{BMO(\mathbb{R}^n)}. \tag{1.7}$$

Next, we will recall the definition of the space $BLO(\mathbb{R}^n)$ introduced by R. Coifman and R. Rochberg in [5]. A complex-valued integrable function f is said to belong to $BLO(\mathbb{R}^n)$ if there exists a constant C_2 such that for any ball B we have

$$\frac{1}{|B|}\int_B [f(x) - \inf_{y\in B} f(y)]dx \leq C_2. \tag{1.8}$$

The infimum of the constant C_2 on the right side is defined as the BLO norm of f, denoted by $\|f\|_{BLO(\mathbb{R}^n)}$.

Remark 1.1. As in [9], it is not difficult to show that $BLO(\mathbb{R}^n)$ is a proper subspace of $BMO(\mathbb{R}^n)$. For example, take $f(x) = (\log|x|)\chi_{\{|x|\leq 1\}}(x)$ for all $x \in \mathbb{R}$, then it is easy to show that $f \in BMO(\mathbb{R})$, but $f \notin BLO(\mathbb{R})$. Notice that the above function is nonpositive. However, it is not so easy to show that there exists a non-negative function which is in $BMO(\mathbb{R}^n)$, but not in $BLO(\mathbb{R}^n)$. The authors in [9] (see Proposition 2.1 of [9]) constructed an interesting counterexample of this kind, which further indicates the meaning of the following theorems, since all Littlewood-Paley operators are nonnegative. Furthermore, it can be proved that $BLO(\mathbb{R}^n)$ is not a linear space and $\|\cdot\|_{BLO(\mathbb{R}^n)}$ is not a norm.

Now, we formulate our main results as follows.

Theorem 1.1. *Suppose that $f \in BMO(\mathbb{R}^n)$, and $r \geq 2$. Then $g_r(f)$ is either infinite everywhere, or is finite almost everywhere, and in the latter case there exists a constant C independent of f such that*

$$\|g_r(f)\|_{BLO(\mathbb{R}^n)} \leq C\|f\|_{BMO(\mathbb{R}^n)}. \tag{1.9}$$

Theorem 1.2. *Suppose that $f \in BMO(\mathbb{R}^n)$, $r \geq 2$. Then $S_r(f)$ is either infinite everywhere, or is finite almost everywhere, and in the latter case there exists a constant independent of f such that*

$$\|S_r(f)\|_{BLO(\mathbb{R}^n)} \leq C\|f\|_{BMO(\mathbb{R}^n)}. \tag{1.10}$$

Theorem 1.3. *Given $\lambda \in (1,\infty)$, then for any $f \in BMO(\mathbb{R}^n)$ and $r \geq 2$, $g^*_{\lambda,r}(f)$ is either infinite everywhere, or is finite almost everywhere, and in the latter case there exists a constant independent of f such that*

$$\|g^*_{\lambda,r}(f)\|_{BLO(\mathbb{R}^n)} \leq C\|f\|_{BMO(\mathbb{R}^n)}. \tag{1.11}$$

Remark 1.2. Obviously, when $r = 2$, our results recover the results of Meng and Yang in [10], which improved the results of Leckband in [8]. It is not clear whether the conclusions of Theorems 1.1-1.3 are also true for $1 < r < 2$, which is interesting. But, please notice that the conclusion of Lemma 2.1 below holding true also needs $r \in [2,\infty)$.

Remark 1.3. We remark that to obtain (1.9), it suffices to prove

$$\|[g_r(f)]^r\|_{BLO(\mathbb{R}^n)} \leq C\|f\|^r_{BMO(\mathbb{R}^n)}. \tag{1.12}$$

Indeed, observe that for any given ball B and $x \in B$, if $\inf_{x' \in B} g_r(f)(x') < \infty$, then

$$g_r(f)(x) - \inf_{x' \in B} g_r(f)(x') \leq \Big\{[g_r(f)(x)]^r - \inf_{x' \in B}[g_r(f)(x')]^r\Big\}^{1/r}. \tag{1.13}$$

Therefore

$$\begin{aligned}
&\frac{1}{|B|}\int_B [g_r f(x) - \inf g_r(f)]dx \\
&\leq \Big(\frac{1}{|B|}\int_B (g_r f(x) - \inf g_r(f))^r dx\Big)^{1/r} \\
&\leq \Big(\frac{1}{|B|}\int_B ([g_r f(x)]^r - \inf[g_r(f)]^r)dx\Big)^{1/r}.
\end{aligned}$$

Combining this with (1.12) we get (1.9). Similarly, we can deal with (1.10) and (1.11).

We should point out that our proofs of main theorems are motivated much by the methods used by Meng and Yang in [10].

The rest of this paper is organized as follows. After recalling and establishing some auxiliary lemmas in Section 2, we give the proofs of our main results in Section 3.

Throughout the paper, the letter C, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but independent of the essential variables.

2. Auxiliary lemmas

In this section, we will recall and establish some auxiliary lemmas, which will be useful in the proofs of main theorems.

Lemma 2.1 (cf. [3]). *For $r \geq 2$, $1 < p < \infty$, $g_r(f)$ is bounded on $L^p(\mathbb{R}^n)$.*

Lemma 2.2. *Suppose that $2 \leq r \leq p < \infty$. Then*

$$\|g^*_{\lambda,r}(f)\|_p \leq C_{p,\lambda,r}\|f\|_p.$$

Proof. Let $h(x)$ be a non-negative measurable function on $\mathbb{R}^n$. We claim that

$$\int_{\mathbb{R}^n} \Big(g^*_{\lambda,r}(f)(x)\Big)^r h(x)dx \leq C_\lambda \int_{\mathbb{R}^n} \Big(g_r(f)(x)\Big)^r M(h)(x)dx, \tag{2.1}$$

where M denotes the Hardy-Littlewood maximal function.

Indeed, since the left side of (2.1) equals to

$$\int_0^\infty \int_{\mathbb{R}^n} |f * \varphi_t(y)|^r \frac{1}{t}\left[\int_{\mathbb{R}^n} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} t^{-n}h(x)dx\right] dydt,$$

we need only to prove that

$$\sup_{t>0} \int_{\mathbb{R}^n} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} t^{-n}h(x)dx \leq C_\lambda M(h)(y),$$

which follows from the following inequality

$$\sup_{\epsilon>0}(h * \varphi_\epsilon)(y) \leq C\|\varphi\|_{L^1} M(h)(y),$$

where φ is a nonnegative radial decreasing function with $\|\varphi\|_{L^1} < \infty$ and $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{x}{\epsilon})$. Here we set $\varphi(x) = (1+|x|)^{-\lambda n}$, $\epsilon = t$.

To prove this lemma for the case $p = r$, it is sufficient to set $h(x) \equiv 1$ in (2.1).

For $p > r$, there exists $q > 1$ such that $1/q + r/p = 1$. We take the supremum on the left side of the inequality (2.1) for all non-negative function h such that $\|h\|_q \leq 1$, getting

$$\|g^*_{\lambda,r}(f)\|_p^r \leq \|g_r(f)\|_p^r \|M(h)\|_q.$$

By Lemma 2.1 and the L^q $(q > 1)$ boundedness of the maximal function, we obtain the desired conclusion. $\square$

Lemma 2.3. *For any $x \in \mathbb{R}^n$ and $r > 1$, $S_r(f)(x) \leq C_\lambda\ g^*_{\lambda,r}(f)(x)$.*

Proof. The proof of this lemma follows from the following simple fact that

$$\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} \geq 2^{-n\lambda}$$

for any $(y,t) \in \Gamma(x)$. $\square$

By Lemmas 2.2 and 2.3, we immediately get the following lemma.

Lemma 2.4. *Suppose that $2 \le r \le p < \infty$. Then*

$$\|S_r(f)\|_p \le C_{p,r}\|f\|_p.$$

Lemma 2.5 (cf. [13])**.** *Suppose that $p \in (1,\infty)$, $\beta \in (0,n)$ and $1/q = 1/p - \beta/n$. Then the following fractal integral operator*

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}dy$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

3. Proofs of main results

This section is devoted to the proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1. By Remark 1.2, it suffices to prove that for any $f \in BMO(\mathbb{R}^n)$, if there exists a point $x_0 \in \mathbb{R}^n$ such that $g_r(f)(x_0) < \infty$, then $g_r(f)$ is finite almost everywhere in $\mathbb{R}^n$ and for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \Big([g_r f(x)]^r - \inf_{x'\in B}[g_r f(x')]^r\Big)dx \le C\|f\|_{BMO}^r.$$

By homogeneity, we may assume $\|f\|_{BMO} = 1$. Therefore it is enough to prove

$$\frac{1}{|B|}\int_B \left([g_r f(x)]^r - \inf_{x'\in B}[g_r f(x')]^r\right)dx \le C.$$

We temporarily assume that $g_r(f)$ is finite almost everywhere in $\mathbb{R}^n$, which will be proved later.

Denote by ρ the radius of the ball B. For any $x \in B$, let

$$g_{r,\rho}(f)(x) := \left(\int_0^{4\rho} |f*\varphi_t(x)|^r\frac{dt}{t}\right)^{1/r},$$
$$g_{r,\infty}(f)(x) := \left(\int_{4\rho}^{\infty} |f*\varphi_t(x)|^r\frac{dt}{t}\right)^{1/r}.$$

Since φ has mean value zero, we get

$$\begin{aligned}(3.1)\qquad &\frac{1}{|B|}\int_B \left([g_r f(x)]^r - \inf_{x'\in B}[g_r(f)]^r\right)dx\\
&\le C\left\{\frac{1}{|B|}\int_B \Big(g_{r,\rho}([f-m_B(f)]\chi_{8B})(x)\Big)^r dx\right.\\
&\quad + \frac{1}{|B|}\int_B \Big(g_{r,\rho}([f-m_B(f)]\chi_{\mathbb{R}^n\setminus 8B})(x)\Big)^r dx\\
&\quad \left.+\frac{1}{|B|}\int_B \sup_{x'\in B}\Big|[g_{r,\infty}(f)(x)]^r - [g_{r,\infty}(f)(x')]^r\Big|dx\right\}\\
&:= A_1 + A_2 + A_3.\end{aligned}$$

In what follows, we estimate the above three terms, respectively. At first, in view of (1.7) and Lemma 2.1, we have $A_1 \le C$. Next, it is obvious that for $x \in B$, $z \in \mathbb{R}^n \backslash 8B$, $|x-z| \sim |x_B - z|$. Thus by (1.2) we get

$$\begin{aligned}A_2 &\le C\frac{1}{|B|}\int_B \int_0^{4\rho}\left[\int_{\mathbb{R}^n\backslash 8B}\frac{|f(z)-m_B(f)|}{|x_B-z|^{n+1}}dz\right]^r t^{r-1}dtdx\\ &\le C\left\{\rho^r\left[\sum_{k=3}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f(z)-m_{2^{k+1}B}(f)|}{(2^k\rho)^{n+1}}dz\right.\right.\\ &\qquad\left.\left.+\sum_{k=3}^{\infty}\frac{|m_{2^{k+1}B}(f)-m_B(f)|}{2^k\rho}\right]^r\right\}\\ &\le C.\end{aligned}$$

Finally, to estimate A_3, we need only to show that for any x, $x' \in B$,

$$\left|[g_{r,\infty}(f)(x)]^r-[g_{r,\infty}(f)(x^{'})]^r\right| \le \int_{4\rho}^{\infty}\left||f*\varphi_t(x)|^r-|f*\varphi_t(x^{'})|^r\right|\frac{dt}{t} \le C. \tag{3.2}$$

For any $t \ge 4\rho$, choose $k_0 \in \mathbb{N}$ such that $2^{k_0}\rho \le t < 2^{k_0+1}\rho$, then by the cancelation property of φ and condition (1.2), we know that for any $x \in B$,

$$\begin{aligned}&\||f*\varphi_t(x)|\\ &\le C\Big(\int_{2^{k_0}B}\frac{|f(z)-m_{2^{k_0}B}(f)|t}{(t+|x-z|)^{n+1}}dz+\sum_{k=k_0}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f(z)-m_{2^{k_0}B}(f)|t}{(t+|x-z|)^{n+1}}dz\Big)\\ &\le C\Big(\frac{1}{t^n}\int_{2^{k_0}B}|f(z)-m_{2^{k_0}B}(f)|dz\\ &\qquad+\sum_{k=k_0}^{\infty}\frac{t}{(2^k\rho)^{n+1}}\int_{2^{k+1}B}|f(z)-m_{2^{k_0}B}(f)|dz\Big)\\ &\le C.\end{aligned}$$

Consequently, by the mean value theorem, we get that for any x, $x' \in B$,

$$\begin{aligned}\left||f*\varphi_t(x)|^r-|f*\varphi_t(x')|^r\right| &\le C|f*\varphi_t(x)-f*\varphi_t(x')|\Big(\max_{x\in B}|f*\varphi_t(x)|\Big)^{r-1}\\ &\le C|f*\varphi_t(x)-f*\varphi_t(x')|.\end{aligned}$$

This implies

$$\begin{aligned}\left|[g_{r,\infty}(f)(x)]^r-[g_{r,\infty}(f)(x^{'})]^r\right| &= \int_{4\rho}^{\infty}\left||f*\varphi_t(x)|^r-|f*\varphi_t(x^{'})|^r\right|\frac{dt}{t}\\ &\le C\int_{4\rho}^{\infty}|f*\varphi_t(x)-f*\varphi_t(x^{'})|\frac{dt}{t}.\end{aligned}$$

On the other hand, by the fact that φ has mean value zero, $|\varphi_t(z)| \le t^{-n}$ and (1.3), we have

$$
\begin{aligned}
&|f*\varphi_t(x) - f*\varphi_t(x')| \\
&\le C\Big(\big|[f-m_B(f)]\chi_{\mathbb{R}^n\setminus 8B} * \varphi_t(x) - [f-m_B(f)]\chi_{\mathbb{R}^n\setminus 8B} * \varphi_t(x')\big| \\
&\qquad + \big|[f-m_B(f)]\chi_{8B} * \varphi_t(x)\big| + \big|[f-m_B(f)]\chi_{8B} * \varphi_t(x')\big|\Big) \\
&\le C\Big(\int_{\mathbb{R}^n\setminus 8B} |f(x)-m_B(f)||\varphi(\frac{x-z}{t}) - \varphi(\frac{x'-z}{t})|t^{-n}dz \\
&\qquad + \int_{8B} \frac{|f-m_B(f)|}{t^n}dz\Big) \\
&\le C\Big(\int_{\mathbb{R}^n\setminus 8B} |f(x)-m_B(f)|\frac{|x-x'|t}{(t+|x-z|)^{n+2}}dz + \int_{8B}\frac{|f-m_B(f)|}{t^n}dz\Big).
\end{aligned}
$$

Therefore, for x, $x' \in B$, we obtain

$$
\begin{aligned}
&\Big|[g_{r,\infty}(f)(x)]^r - [g_{r,\infty}(f)(x^{'})]^r\Big| \\
&\le C\Big(\int_{\mathbb{R}^n\setminus 8B} |f(x)-m_B(f)|\Big[\int_{4\rho}^{|x-z|} \frac{\rho}{(t+|x-z|)^{n+2}}dt \\
&\qquad + \int_{|x-z|}^{\infty} \frac{\rho}{(t+|x-z|)^{n+2}}dt\Big]dz + \int_{8B}\int_{4\rho}^{\infty} \frac{|f(z)-m_B(f)|}{t^{n+1}}dtdz\Big) \\
&\le C\Big(\int_{\mathbb{R}^n\setminus 8B} \frac{|f(z)-m_B(f)|r}{|x-z|^{n+1}} + \frac{1}{\rho^n}\int_{8B}|f-m_B(f)|dz\Big) \\
&\le C.
\end{aligned}
$$

This proves that $A_3 \le C$, which together with the estimates of A_1 and A_2 implies that

$$
\frac{1}{|B|}\int_B ([g_r f(x)]^r - \inf_{x'\in B}[g_r f(x')]^r)dx < \infty. \tag{3.3}
$$

To end the proof of this theorem, it remains to prove that $g_r(f)$ is finite almost everywhere. First observe that with x' replaced by x_0 in (3.2) and repeat the same process in the proof of (3.2), we get

$$
|[g_{r,\infty}(f)(x)]^r - [g_{r,\infty}(f)(x_0)]^r| \le C, \quad \forall\, x \in B.
$$

Consequently,

$$
|g_{r,\infty}(f)(x)| < \infty, \quad \forall\, x \in B
$$

since $|g_{r,\infty}(f)(x_0)| < \infty$. This avoid the case of $\infty - \infty$ in the expression of A_3 in (3.1).

On the other hand, we assume that $B \ni x_0$ in (3.3). Then

$$
g_r(f)(x) < \infty, \quad \text{a.e. } x \in B.
$$

Note that the ball B is chosen arbitrarily. Hence $g_r(f)(x)$ is finite almost everywhere in $\mathbb{R}^n$. Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. As in proving Theorem 1.1, it suffices to prove that for any $f \in BMO(\mathbb{R}^n)$ with $\|f\|_{BMO} = 1$, if there exists a point $x_0 \in \mathbb{R}^n$ such that $S_r(f)(x_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \left([S_r f(x)]^r - \inf_{x' \in B}[S_r f(x')]^r\right) dx \le C.$$

Let

$$S_{r,\rho}(f)(x) := \left(\int_0^{4\rho}\int_{|y-x|<t} |f * \varphi_t(x)|^r \frac{dydt}{t^{n+1}}\right)^{1/r},$$

$$S_{r,\infty}(f)(x) := \left(\int_{4\rho}^{\infty}\int_{|y-x|<t} |f * \varphi_t(x)|^r \frac{dydt}{t^{n+1}}\right)^{1/r}.$$

Then the cancelation property of φ yields

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left([S_r f(x)]^r - \inf_{x' \in B}[S_r f(x')]^r\right) dx \\
&\le C\Big\{\frac{1}{|B|}\int_B [S_{r,\rho}([f - m_B(f)]\chi_{8B})(x)]^r dx \\
&\qquad + \frac{1}{|B|}\int_B \Big[S_{r,\rho}([f - m_B(f)]\chi_{\mathbb{R}^n\setminus 8B})(x)\Big]^r dx \\
&\qquad + \frac{1}{|B|}\int_B \sup_{x' \in B} |[S_{r,\infty}(f)(x)]^r - [S_{r,\infty}(f)(x')]^r| dx\Big\} \\
&:= G_1 + G_2 + G_3.
\end{aligned}$$

Now, we estimate G_1, G_2 and G_3, respectively. Firstly, (1.7) and Lemma 2.4 implies $G_1 \le C$. Secondly, for any $x \in B$, $z \in \mathbb{R}^n \setminus 8B$ and $y \in \mathbb{R}^n$ satisfying $|y - x| < t$ ($t \in (0, 4\rho)$), we have $|y - z| \sim |z - x_B|$. Therefore by (1.2)

$$\begin{aligned}
G_2 &\le C\frac{1}{|B|}\int_B\int_0^{4\rho}\int_{|y-x|<t}\left[\int_{\mathbb{R}^n\setminus 8B}\frac{|f(z) - m_B(f)|}{|x_B - z|^{n+1}}dz\right]^r \frac{dydt}{t^{n+1-r}}dx \\
&\le C\left\{\rho^r\left[\sum_{k=3}^{\infty}\int_{2^{k+1}B\setminus 2^k B}\frac{|f(z) - m_{2^{k+1}B}(f)|}{(2^k\rho)^{n+1}}dz\right.\right. \\
&\qquad \left.\left. + \sum_{k=3}^{\infty}\frac{|m_{2^{k+1}B}(f) - m_B(f)|}{2^k\rho}\right]^r\right\} \\
&\le C.
\end{aligned}$$

Finally, to estimate G_3, it is enough to prove that for any $x, x' \in B$, we have

$$\left|[S_{r,\infty}(f)(x)]^r - [S_{r,\infty}(f)(x')]^r\right|$$

$$\begin{aligned}&\le C\int_{4\rho}^{\infty}\int_{|y|<t}\Big||f*\varphi_t(x+y)|^r-|f*\varphi_t(x^{'}+y)|^r\Big|\frac{dydt}{t^{n+1}}\\&\le C.\end{aligned}$$

Note that for any $x\in B$, $z\in\mathbb{R}^n$ and $|y|\le t$ $(t\ge 4\rho)$, we have $t+|x+y-z|\sim t+|x-z|$. Then, for any $x,x'\in B$, and $y\in\mathbb{R}^n$ satisfying $|y|\le t$ $(t\ge 4\rho)$, we have

$$\begin{aligned}&|f*\varphi_t(x+y)|\\&\le C\Big(\int_{2^{k_0}B}\frac{|f(z)-m_{2^{k_0}B}(f)|t}{(t+|x-z|)^{n+1}}dz+\sum_{k=k_0}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f(z)-m_{2^{k_0}B}(f)|t}{(t+|x-z|)^{n+1}}dz\Big)\\&\le C\Big(\frac{1}{t^n}\int_{2^{k_0}B}|f(z)-m_{2^{k_0}B}(f)|dz\\&\qquad+\sum_{k=k_0}^{\infty}\frac{t}{(2^k\rho)^{n+1}}\int_{2^{k+1}B}|f(z)-m_{2^{k_0}B}(f)|dz\Big)\\&\le C,\end{aligned}$$

and

$$\begin{aligned}&|f*\varphi_t(x+y)-|f*\varphi_t(x'+y)|\\&\le C\left(\int_{\mathbb{R}^n\backslash 8B}|f(x)-m_B(f)|\frac{|x-x'|t}{(t+|x-z|)^{n+2}}dz+\int_{8B}\frac{|f-m_B(f)|}{t^n}dz\right),\end{aligned}$$

where k_0 is a positive integer satisfying $2^{k_0}\rho\le t<2^{k_0+1}\rho$. Therefore by the mean value theorem we can prove that for any $x,x'\in B$, we have

$$\begin{aligned}&\Big|[S_{r,\infty}(f)(x)]^r-[S_{r,\infty}(f)(x^{'})]^r\Big|\\&\le C\int_{4\rho}^{\infty}\int_{|y|<t}|f*\varphi_t(x+y)-f*\varphi_t(x^{'}+y)|\frac{dydt}{t^{n+1}}\\&\le C\left(\int_{4\rho}^{\infty}\int_{|y|<t}\frac{dydt}{t^{2n+1}}\int_{8B}|f(z)-m_B(f)|dz\right.\\&\qquad\left.+\int_{\mathbb{R}^n\backslash 8B}\int_{4\rho}^{\infty}\int_{|y|<t}\frac{|f(z)-m_B(f)|r}{(t+|x-z|)^{n+2}t^n}dydtdz\right)\\&\le C.\end{aligned}$$

This implies that $G_3\le C$ and we complete the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. As before, it suffices to prove that for any $f\in BMO(\mathbb{R}^n)$, $\|f\|_{BMO}=1$, if there exists a point $x_0\in\mathbb{R}^n$ such that $g^*_{\lambda,r}(f)(x_0)<\infty$, then for any $B\subset\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B\Big([g^*_{\lambda,r}f(x)]^r-\inf_{x'\in B}[g^*_{\lambda,r}f(x^{'})]^r\Big)dx\le C.$$

We denote the radius and center of the ball B by ρ and x_B, respectively. For any non-negative integer k, let

$$J(k) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x_B| < 2^{k-2}\rho,\ 0 < t < 2^{k-2}\rho\}.$$

Since φ has mean value zero, we obtain

$$\begin{aligned}
&\frac{1}{|B|}\int_B \Big([g^*_{\lambda,r}f(x)]^r - \inf_{x'\in B}[g^*_{\lambda,r}f(x^{'})]^r\Big)dx \\
&\le C\Big(\frac{1}{|B|}\int_B [g^*_{\lambda,r,0}([f - m_B(f)]\chi_{8B})(x)]^r dx \\
&\qquad + \frac{1}{|B|}\int_B [g^*_{\lambda,r,0}([f - m_B(f)]\chi_{\mathbb{R}^n\setminus 8B})(x)]^r dx \\
&\qquad + \frac{1}{|B|}\int_B \sup_{x'\in B} |[g^*_{\lambda,r,\infty}(f)(x)]^r - [g^*_{\lambda,r,\infty}(f)(x^{'})]^r| dx\Big) \\
&:= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$g^*_{\lambda,r,0}(f)(x) = \left\{\int\int_{J(0)} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |f * \varphi_t(y)|^r \frac{dydt}{t^{n+1}}\right\}^{1/r},$$

$$g^*_{\lambda,r,\infty}(f)(x) = \left\{\int\int_{\mathbb{R}^{n+1}_+\setminus J(0)} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |f * \varphi_t(y)|^r \frac{dydt}{t^{n+1}}\right\}^{1/r}.$$

By (1.7) and Lemma 2.2, it is easy to check that $I_1 \le C$.

In what follows, we estimate I_2 and I_3, respectively. Since for any $(y,t) \in J(0)$, $x \in B$, $z \in \mathbb{R}^n\setminus 8B$, we have $|z - y| \sim |z - x_B|$, $|y - x| < 2\rho$, so by (1.2) we get

$$\begin{aligned}
I_2 &\le C\frac{1}{|B|}\int_B \int\int_{J(0)} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\int_{\mathbb{R}^n\setminus 8B} \frac{|f(z) - m_B(f)|}{(t+|y-z|)^{n+1}}dz\right)^r \frac{dydt}{t^{n+1-r}}dx \\
&\le C\frac{1}{|B|}\int_B \int\int_{J(0)} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\sum_{k=3}^{\infty}\int_{2^{k+1}B\setminus 2^k B} \frac{|f(z) - m_B(f)|}{(2^k\rho)^{n+1}}dz\right)^r \frac{dydt}{t^{n-1}}dx \\
&\le C\frac{1}{|B|}\int_B \int_0^{\rho/4}\int_{|y-x_B|<2\rho} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{1}{\rho^r}\frac{dydt}{t^{n+1-r}}dx \\
&\le C\frac{1}{|B|}\int_B \int_0^{\rho/4}\int_{|s-\frac{x_B-x}{t}|<\frac{2\rho}{t}} \left(\frac{t}{t+t|s|}\right)^{\lambda n}\frac{1}{\rho^r}t^{r-1}dsdtdx \\
&\le C\frac{1}{|B|}\int_B \int_0^{\rho/4}\int_{|s|<\frac{4\rho}{t}} \frac{1}{(1+|s|)^{n-1}}\frac{1}{\rho^r}t^{r-1}dsdtdx \\
&\le C.
\end{aligned}$$

To estimate I_3, first observe that by the cancelation property of φ, for any $x, x' \in B$ we have

$$\begin{aligned}
&\big|[g^*_{\lambda,r,\infty}(f)(x)]^r - [g^*_{\lambda,r,\infty}(f)(x')]^r\big| \\
&\leq C\int\int_{\mathbb{R}^{n+1}_+\backslash J(0)} \left|\left(\frac{t}{t+|x-y|}\right)^{\lambda n} - \left(\frac{t}{t+|x'-y|}\right)^{\lambda n}\right| |f*\varphi_t(y)|^r \frac{dydt}{t^{n+1}} \\
&\leq C\int\int_{\mathbb{R}^{n+1}_+\backslash J(0)} \frac{\rho}{(t+|x-y|)^{n\lambda+1}} |f*\varphi_t(y)|^r \frac{dydt}{t^{n+1-n\lambda}} \\
&\leq C\Bigg\{\sum_{k=1}^{\infty}\rho\int\int_{J(k)\backslash J(k-1)}\left[\int_{\mathbb{R}^n\backslash 2^{k+1}B}\frac{|f(z)-m_B(f)|}{(t+|y-z|)^{n+1}}dz\right]^r\left(\frac{t}{2^k\rho}\right)^{n\lambda+1}\frac{dydt}{t^{n+2-r}} \\
&\qquad + \sum_{k=1}^{\infty}\rho\int\int_{J(k)\backslash J(k-1)}\left[\int_{2^{k+1}B}\frac{|f(z)-m_B(f)|}{(t+|y-z|)^{n+1}}dz\right]^r\left(\frac{t}{2^k\rho}\right)^{n\lambda+1}\frac{dydt}{t^{n+2-r}}\Bigg\} \\
&:= E_1 + E_2.
\end{aligned}$$

Since $\frac{t}{2^k\rho} \leq 1$ for $(y,t) \in J_k$, we have

$$\left(\frac{t}{2^k\rho}\right)^{n\lambda_1+1} \leq \left(\frac{t}{2^k\rho}\right)^{n\lambda_2+1} \quad (\lambda_1 > \lambda_2 > 1).$$

So, we always assume that $\lambda \in (1,2)$. Note that

$$t + |z-y| \sim 2^k\rho + |z - x_B|, \quad \forall x \in B,\, z \notin 2^{k+1}B,\, (y,t) \in J(k)\backslash J(k-1),$$

which implies

$$\begin{aligned}
E_1 &\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int\int_{J(k)\backslash J(k-1)}\left[\sum_{l=k+1}^{\infty}\int_{2^{l+1}B\backslash 2^lB}\frac{|f(z)-m_B(f)|}{(2^k\rho+|z-x_B|)^{n+1}}dz\right]^r\frac{dydt}{t^{n+1-r-n\lambda}} \\
&\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int\int_{J(k)\backslash J(k-1)}\left[\sum_{l=k+1}^{\infty}\int_{2^{l+1}B}\frac{|f(z)-m_B(f)|}{(2^l\rho)^{n+1}}dz\right]^r\frac{dydt}{t^{n+1-r-n\lambda}} \\
&\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int_0^{2^{k-2}\rho}\int_{|y-x_B|<2^{k-2}\rho}\left(\frac{k}{2^k\rho}\right)^r\frac{dydt}{t^{n+1-r-n\lambda}} \\
&\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}(2^{k-2}\rho)^{n\lambda+r-n}(2^{k-2}\rho)^n\left(\frac{k}{2^k\rho}\right)^r \\
&= \sum_{k=1}^{\infty}\frac{k^r}{2^k} \\
&\leq C.
\end{aligned}$$

On the other hand, by the Minkowski inequality and the boundedness of the fractional integral operator $I_{\frac{n}{r}(\lambda-1)}$ from $L^{r/\lambda}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, we have

$$
\begin{aligned}
E_2 &\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int_{|y-x_B|<2^{k-2}\rho} \\
&\quad \left[\int_{2^{k+1}B}|f(z)-m_B(f)|\left(\int_0^{2^{k-2}\rho}\frac{t^{r+n\lambda-n-1}}{(t+|y-z|)^{nr+r}}dt\right)^{1/r}dz\right]^r dy \\
&\leq C\left\{\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int_{|y-x_B|<2^{k-2}\rho}\left[\int_{2^{k+1}B}|f(z)-m_B(f)|\right.\right. \\
&\quad \left.\left.\left(\int_0^{|y-z|}\frac{t^{r+n\lambda-n-1}}{(t+|y-z|)^{nr+r}}dt+\int_{|y-z|}^{\infty}\frac{t^{r+n\lambda-n-1}}{(t+|y-z|)^{nr+r}}dt\right)^{\frac{1}{r}}dz\right]^r dy\right\} \\
&\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\int_{\mathbb{R}^n}\left[\int_{2^{k+1}B}\frac{|f(z)-m_B(f)|}{|z-y|^{n-\frac{n\lambda-n}{r}}}dz\right]^r dy \\
&\leq C\sum_{k=1}^{\infty}\frac{\rho}{(2^k\rho)^{n\lambda+1}}\left[\int_{2^{k+1}B}|f(z)-m_B(f)|^{\frac{r}{\lambda}}dz\right]^{\lambda} \\
&\leq C.
\end{aligned}
$$

Then, $I_3 \leq C$ and Theorem 1.3 is proved. □

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