# VOLUME INEQUALITIES FOR THE $L_p$-SINE TRANSFORM OF ISOTROPIC MEASURES

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Abstract. For $p \geq 1$, sharp isoperimetric inequalities for the $L_p$-sine transform of isotropic measures are established. The corresponding reverse inequalities are obtained in an asymptotically optimal form. As applications of our main results, we present volume inequalities for convex bodies which are in $L_p$ surface isotropic position.

## 1. Introduction

The setting for this article is Euclidean $n$-space $\mathbb{R}^n$ with $n \geq 3$. We use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$ and we write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$.

A non-negative finite Borel measure $\mu$ on the unit sphere $S^{n-1}$ is said to be isotropic if it has the same moment of inertia about all lines through the origin or, equivalently, if for all $x \in \mathbb{R}^n$,

$$|x|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\mu(u).$$

Two basic examples of isotropic measures on $S^{n-1}$ are (suitably normalized) spherical Lebesgue measure and the cross measures, i.e., measures concentrated uniformly on $\{\pm b_1, \ldots, \pm b_n\}$, where $b_1, \ldots, b_n$ denote orthonormal basis vectors of $\mathbb{R}^n$. Isotropic measures have been the focus of recent studies, in particular, in relation with a variety of extremal problems for convex bodies (see, e.g., [8, 9, 16, 20, 29, 43, 45] and the references therein).

A measure on $S^{n-1}$ is said to be even if it assumes the same value on antipodal sets. Each even isotropic Borel measure $\mu$ on $S^{n-1}$ determines an $n$-dimensional subspace of $L_p$ whose unit ball we denote by $Z_p^* = Z_p^*(\mu)$. To

be specific, the $n$-dimensional subspace of $L_p = L_p(S^{n-1})$ may be taken to be $\mathbb{R}^n$ with a norm defined, for each $x \in \mathbb{R}^n$, by

$$\|x\|_{Z_p^*} = \Big( \int_{S^{n-1}} |x \cdot v|^p d\mu(v) \Big)^{\frac{1}{p}}.$$

Conversely, a theorem of Lewis [22] shows that each $n$-dimensional subspace of $L_p$ is isometric to a Banach space with such a representation for some even isotropic Borel measure $\mu$ (see, e.g. [29] for details).

Volume inequalities for the body $Z_p^* = Z_p^*(\mu)$ or its polar, $Z_p = Z_p(\mu)$, that characterize the Euclidean subspaces of $L_p$, are easily obtained by using well-known standard inequalities (such as the Urysohn and Hölder inequalities). Much more difficult to obtain are the reverse inequalities for $Z_p^*$ or $Z_p$. These have the $l_p^n$ subspaces of $L_p$ as extremals.

In 1991 Ball [2] used his normalized Brascamp-Lieb inequality to obtain the sharp reverse inequality for the volume of $Z_p^*$, for all $p \in [0, \infty]$. Ball's inequality shows that the unit ball of $l_p^n$ is extremal. The solution to the uniqueness problem for Ball's reverse inequality was obtained later by Barthe [3] for discrete measures by using his newly established equality conditions for the Brascamp-Lieb inequality. Barthe proved that indeed the unit ball of $l_p^n$ is the only extremal for Ball's inequality when $\mu$ is a discrete isotropic measure.

The reverse inequalities for the volume of $Z_p$ would prove to be more resistant. Ball [1] established the reverse inequality for the volume of $Z_p$ for the case $p = 1$ and predicted that for $p > 1$ these inequalities could be obtained from a reverse Brascamp-Lieb inequality. Again, the breakthrough was achieved by Barthe [3]. Barthe found the reverse Brascamp-Lieb inequality anticipated by Ball and used it to establish the reverse inequalities for the volume of $Z_p$ for all $p > 1$. Barthe also established the uniqueness of the extremals when $\mu$ is a discrete measure.

The problem of establishing Ball's inequalities, along with their equality conditions, for isotropic measures which are not necessarily discrete was solved in 2004 by E. Lutwak, D. Yang and G. Zhang [28]. All their inequalities were obtained along with their equality conditions and that was done for all $p \in [1, \infty]$ and all even isotropic measures $\mu$.

For $p \in [1, \infty]$, let $p^* \in [1, \infty]$ denote the Hölder conjugate of $p$; i.e., $p^*$ is defined by $\frac{1}{p} + \frac{1}{p^*} = 1$. For $n, p \in (0, \infty)$, let

$$\kappa_n(p) = 2^n \frac{\Gamma(1 + \frac{1}{p})^n}{\Gamma(1 + \frac{n}{p})}.$$

Let $\kappa_n(\infty) = 2^n$, and abbreviate $\kappa_n(2)$ by $\kappa_n$ and note that for positive integer $n$, the unit ball of $\mathbb{R}^n$ has precisely volume $\kappa_n$. Let $\Gamma$ denote the Gamma function and, for $p \in (0, \infty)$, define $c_p$ by

$$c_p = \Big( \frac{\Gamma(1 + \frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(1 + \frac{1}{2})\Gamma(\frac{n+p}{2})} \Big)^{\frac{n}{p}},$$

and define $c_\infty = \lim_{p\to\infty} c_p = 1$.

The following two theorems were obtained by E. Lutwak, D. Yang and G. Zhang in [28].

**Theorem A** ([28]). *Suppose $p \in [1, \infty]$. If $\mu$ is an even isotropic measure on $S^{n-1}$, then*

$$\frac{\kappa_n}{c_p} \leq V(Z_p^*) \leq \kappa_n(p).$$

*If $p \in [1, \infty)$ is not an even integer, then there is equality in the left inequality if and only if $\mu$ is suitably normalized Lebesgue measure. For $p \neq 2$, there is equality in the right inequality if and only if $\mu$ is a cross measure.*

**Theorem B** ([28]). *Suppose $p \in [1, \infty]$. If $\mu$ is an even isotropic measure on $S^{n-1}$, then*

$$\kappa_n(p^*) \leq V(Z_p) \leq \kappa_n c_p.$$

*If $p \in [1, \infty)$ is not an even integer, then there is equality in the right inequality if and only if $\mu$ is suitably normalized Lebesgue measure. For $p \neq 2$, there is equality in the left inequality if and only if $\mu$ is a cross measure.*

For the sine transform of even isotropic measures, sharp isoperimetric inequalities were established by G. Maresch and F. E. Schuster in [30]. The corresponding reverse inequalities were obtained in an asymptotically optimal form. The authors also showed that these new inequalities have direct applications to strong volume estimates for convex bodies from data about their sections or projections.

The sine transform $\mathscr{S}\mu$ of a finite Borel measure $\mu$ on $S^{n-1}$ is the continuous function defined by

$$(\mathscr{S}\mu)(x) = \int_{S^{n-1}} |x|u^\perp| d\mu(u), \qquad x \in \mathbb{R}^n.$$

Here, $|x|u^\perp|$ is the length of the orthogonal projection of $x$ onto the hyperplane orthogonal to $u$. If $\mu$ is even and not concentrated on two antipodal points, its sine transform uniquely determines a norm $\|\cdot\|_{S_\mu^*}$ on $\mathbb{R}^n$ whose unit ball we denote by $S_\mu^* = S_\mu^*(\mu)$ and its polar by $S_\mu = S_\mu(\mu)$.

In this article we obtain sharp isoperimetric inequalities for the spherical $L_p$-sine transform of isotropic measures.

**Definition.** For $p \geq 1$, the $L_p$-sine transform $\mathscr{S}_p\mu$ of a finite Borel measure $\mu$ on $S^{n-1}$ is the continuous function defined by

$$(\mathscr{S}_p\mu)(x) = \left( \int_{S^{n-1}} |x|u^\perp|^p d\mu(u) \right)^{\frac{1}{p}}, \qquad x \in \mathbb{R}^n.$$

If $\mu$ is even and not concentrated on two antipodal points, its $L_p$-sine transform uniquely determines a norm $\|\cdot\|_{S_p^*}$ on $\mathbb{R}^n$ whose unit ball we denote by $S_p^* = S_p^*(\mu)$ and its polar by $S_p = S_p(\mu)$.

Let $p \geq 1$, define

$$\alpha_{n,p} := \frac{\Gamma(1 + \frac{n}{p})p^{\frac{n}{n-1}}n^{\frac{n}{p}-n}(n-1)^{\frac{n}{p}-\frac{n}{n-1}+n}}{\Gamma(\frac{n-1}{p})^{\frac{n}{n-1}}}$$

and

$$\gamma_{n,p} := \frac{(n-1)\kappa_{n-1}\Gamma(\frac{n+p-1}{2})\Gamma(\frac{1}{2})}{\kappa_n\Gamma(\frac{n+p}{2})}.$$

The main results of this article are the following two theorems.

**Theorem 1.** *For $p \geq 1$, if $\mu$ is an isotropic measure on $S^{n-1}$, then*

$$\frac{\kappa_n}{\gamma_{n,p}^n} \leq V(S_p^*) \leq \frac{\kappa_n\gamma_{n,p}^n}{\alpha_{n,p}}.$$

*If $\mu$ is even and $p$ is not an even integer, then there is equality in the left inequality if and only if $\mu$ is normalized Lebesgue measure.*

**Theorem 2.** *For $p \geq 1$, if $\mu$ is an isotropic measure on $S^{n-1}$, then*

$$\frac{\kappa_n\alpha_{n,p}}{\gamma_{n,p}^n} \leq V(S_p) \leq \kappa_n\gamma_{n,p}^n.$$

*If $\mu$ is even and $p$ is not an even integer, then there is equality in the right inequality if and only if $\mu$ is normalized Lebesgue measure.*

The reverse volume bounds in Theorem 1 and Theorem 2 are asymptotically optimal as $n \to \infty$. The ideas and techniques of K. Ball [1, 2], F. Barthe [3, 4], E. Lutwak, D. Yang, G. Zhang [28], G. Maresch and F. E. Schuster [30] play a critical role throughout this paper. It would be impossible to overstate our reliance on their work.

## 2. Background material

For quick later reference, we collect in this section background material regarding convex bodies (see e.g. the books of Gardner [6] and Schneider [36]).

Let $\mathbb{R}^n$ denote the Euclidean $n$-dimensional space with corresponding Euclidean norm $|\cdot|$. Let $B_p^n = \{x \in \mathbb{R}^n : |x \cdot e_1| + \cdots + |x \cdot e_n|^p \leq 1\}$ denote the unit ball of the $n$-dimensional $l_p^n$-space, where $e_1, \ldots, e_n$ denotes the canonical basis for $\mathbb{R}^n$. The set $S^{n-1}$ is the unit sphere of $B_2^n$. Write $\kappa_n$ for $V(B_2^n)$, the volume of $B_2^n$, and let $\omega_n$ denote the surface area of $B_2^n$.

A convex body $K$ is a compact, convex set with non-empty interior. Let $V(K)$ denote the volume of $K$ and $\mathscr{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric. A convex body $K \in \mathbb{R}^n$ is uniquely determined by its support function $h_K$ defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

The polar body $K^*$ of a convex body $K$ containing the origin in its interior is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}.$$

Let $\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$, $x \in \mathbb{R}^n \setminus \{0\}$, denote the radial function of the convex body $K$ containing the origin. The Minkowski functional $\|\cdot\|_K$ is defined by $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$. Clearly, $\rho_K(u) = \|u\|_K^{-1}$ for $u \in S^{n-1}$. It follows from the definitions of support functions and radial functions, and the definition of the polar body of $K$, that

$$(2.1) \qquad \rho_{K^*}(\cdot) = h_K(\cdot)^{-1} \quad \text{and} \quad h_{K^*}(\cdot) = \rho_K(\cdot)^{-1}.$$

Using (2.1) and the polar coordinate formula for volume, it is easy to see that the volume of a convex body $K \in \mathbb{R}^n$ containing the origin in its interior is given by

$$(2.2) \qquad V(K) = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-h_{K^*}(x)^p} dx.$$

The classical Urysohn inequality (see, e.g. [36, p. 318]) provides an upper bound for the volume of a convex body in terms of the average value of its support function: If $K \in \mathscr{K}^n$ has non-empty interior, then

$$(2.3) \qquad \left(\frac{V(K)}{\kappa_n}\right)^{\frac{1}{n}} \leq \frac{1}{n\kappa_n} \int_{S^{n-1}} h_K(u) du,$$

with equality if and only if $K$ is a ball. Here the integral is with respect to spherical Lebesgue measure.

In order to prove our theorems, we shall require the following rank $n-1$ case of the multidimensional Brascamp-Lieb inequality and its reverse form.

In the following we write $\pi_u$, $u \in S^{n-1}$, for the orthogonal projection onto the hyperplane $u^\perp$.

**Lemma 2.1** ([23], The Brascamp-Lieb Inequality). *Let* $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, *and* $c_1, \ldots, c_m > 0$ *such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = Id.$$

*If* $f_i : u_i^\perp \to [0, \infty)$, $1 \leq i \leq m$, *are integrable functions, then*

$$(2.4) \qquad \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x|u_i^\perp)^{c_i} dx \leq \prod_{i=1}^{m} \left( \int_{u_i^\perp} f_i \right)^{c_i}.$$

*There is equality if the* $f_i$, $1 \leq i \leq m$, *are identical Gaussian densities.*

**Lemma 2.2** ([3], The Reverse Brascamp-Lieb Inequality). *Let* $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, *and* $c_1, \ldots, c_m > 0$ *such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = Id.$$

*If* $f_i : u_i^\perp \to [0, \infty)$, $1 \leq i \leq m$, *are integrable functions, then*

$$(2.5) \qquad \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^\perp \right\} dx \geq \prod_{i=1}^{m} \left( \int_{u_i^\perp} f_i \right)^{c_i}.$$

*There is equality if the $f_i$, $1 \leq i \leq m$, are identical Gaussian densities.*

The proof of the reverse Brascamp-Lieb inequality by Barthe relies on the existence and uniqueness of a certain measure preserving map, the so called Brenier map, between two sufficiently regular probability measures (see e.g. [5, 31]). Barthe's proof also exploited a classical principle dating back to Kantorovich which states that the problem of optimal mass transportation admits two dual formulations. In particular, this duality principle made it possible to derive both the Brascamp-Lieb inequality and its inverse form from a single inequality which is stated in the following theorem.

**Lemma 2.3.** *Let $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = Id.$$

*If $f_i, g_i : u_i^{\perp} \to [0, \infty)$, $1 \leq i \leq m$, are integrable functions, such that*

$$\int_{u_i^{\perp}} f_i = \int_{u_i^{\perp}} g_i = 1.$$

*Then*

$$(2.6) \quad \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x|u_i^{\perp})^{c_i} dx \leq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} g_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^{\perp} \right\} dx.$$

Note that equality in (2.6) can only hold if the $f_i$ are extremizers for the Brascamp-Lieb inequality and the $g_i$ are extremizers for the reverse Brascamp-Lieb inequality.

## 3. Proof of the main results

**Theorem 3.1.** *For $p \geq 1$, if $\mu$ is an isotropic measure on $S^{n-1}$, then*

$$V(S_p^*) \leq \frac{V(S_p)}{\alpha_{n,p}}.$$

*Proof.* First assume that $\mu$ is discrete and let $\mathrm{supp}\mu = \{u_1, \ldots, u_m\}$ and $\mu(\{u_i\}) =: \bar{c}_i > 0$. Since $\mu$ is isotropic, if follows that $\mu(S^{n-1}) = \sum_{i=1}^{m} \bar{c}_i = n$. Since $\pi_u = Id - u \otimes u$, it follows that

$$(3.1) \quad \frac{1}{n-1} \sum_{i=1}^{m} \bar{c}_i \pi_{u_i} = \sum_{i=1}^{m} \bar{c}_i u_i \otimes u_i = Id.$$

From (2.2) and the definition of the $L_p$-sine transform, it follows that

$$V(S_p^*) = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_{S_p^*}^p} dx$$

$$= \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{m} \bar{c}_i |x|u_i^{\perp}|^p} dx$$

$$(3.2) \qquad = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \prod_{i=1}^{m} \left( e^{-(n-1)|x|u_i^{\perp}|^p} \right)^{c_i} dx,$$

where $c_i := \frac{\bar{c}_i}{(n-1)}$, $i = 1, \ldots, m$.

From the fact that $|x|u^{\perp}| = h(B^n|u^{\perp}, x)$, $\sum_{i=1}^{m} \bar{c}_i = n$ and Jensen's inequality, it follows that, for all $u \in S^{n-1}$,

$$h_{S_p}^p(u) = \|u\|_{S_p^*}^p = \sum_{i=1}^{m} \bar{c}_i |u|u_i^{\perp}|^p$$

$$= \sum_{i=1}^{m} \bar{c}_i h_{B^n|u_i^{\perp}}^p(u)$$

$$= n \frac{\sum_{i=1}^{m} \bar{c}_i h_{B^n|u_i^{\perp}}^p(u)}{\sum_{i=1}^{m} \bar{c}_i}$$

$$\geq n \left( \frac{\sum_{i=1}^{m} \bar{c}_i h_{B^n|u_i^{\perp}}(u)}{\sum_{i=1}^{m} \bar{c}_i} \right)^p$$

$$(3.3) \qquad = n^{1-\frac{1}{p}} \left( h_{\sum_{i=1}^{m} \bar{c}_i B^n|u_i^{\perp}}(u) \right)^p,$$

with equality if and only if $|u|u_1^{\perp}| = \cdots = |u|u_m^{\perp}|$ or $p = 1$. Hence, we have

$$S_p \supseteq \left\{ x \in \mathbb{R}^n : x = n^{\frac{1}{p}-1} \sum_{i=1}^{m} \bar{c}_i y_i, y_i \in B^n|u_i^{\perp} \right\}$$

$$= \left\{ x \in \mathbb{R}^n : x = n^{\frac{1}{p}-1}(n-1) \sum_{i=1}^{m} c_i y_i, y_i \in B^n|u_i^{\perp} \right\}.$$

Consequently, we obtain

$$(3.4) \quad V(S_p) \geq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} 1_{[0,(n-1)n^{\frac{1}{p}-1}]}(|y_i|)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^{\perp} \right\} dx.$$

Define functions $f_i : u_i^{\perp} \to [0, \infty)$, $1 \leq i \leq m$, by

$$(3.5) \qquad f_i(y) = \frac{p(n-1)^{\frac{n-1}{p}-1}}{\Gamma(\frac{n-1}{p})\kappa_{n-1}} e^{-(n-1)|y|^p},$$

and $g_i : u_i^{\perp} \to [0, \infty)$, $1 \leq i \leq m$, by

$$(3.6) \qquad g_i(y) = \frac{1}{\left( (n-1)n^{\frac{1}{p}-1} \right)^{n-1} \kappa_{n-1}} 1_{[0,(n-1)n^{\frac{1}{p}-1}]}(|y|).$$

Note that the normalizations are chosen such that

$$\int_{u_i^{\perp}} f_i = \int_{u_i^{\perp}} g_i = 1.$$

From the fact that $\sum_{i=1}^{n} c_i = \frac{n}{n-1}$, (3.2) and (3.5), we have

$$V(S_p^*) = \frac{1}{\Gamma(1+\frac{n}{p})}\Big(\frac{\Gamma(\frac{n-1}{p})\kappa_{n-1}}{p(n-1)^{\frac{n-1}{p}-1}}\Big)^{\frac{n}{n-1}}$$

$$\cdot \int_{\mathbb{R}^n} \prod_{i=1}^{m} \Big(\frac{p(n-1)^{\frac{n-1}{p}-1}}{\Gamma(\frac{n-1}{p})\kappa_{n-1}} e^{-(n-1)|x|u_i^\perp|^p}\Big)^{c_i} dx$$

$$(3.7) \qquad = \frac{1}{\Gamma(1+\frac{n}{p})}\Big(\frac{\Gamma(\frac{n-1}{p})\kappa_{n-1}}{p(n-1)^{\frac{n-1}{p}-1}}\Big)^{\frac{n}{n-1}} \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i^{c_i}(x|u_i^\perp)dx.$$

From the fact that $\sum_{i=1}^{n} c_i = \frac{n}{n-1}$, (3.4) and (3.6), we have

$$V(S_p)$$

$$\geq \Big(\big((n-1)n^{\frac{1}{p}-1}\big)^{n-1}\kappa_{n-1}\Big)^{\frac{n}{n-1}}$$

$$\cdot \int_{\mathbb{R}^n} \sup\Big\{\prod_{i=1}^{m}\Big(\frac{1}{\big((n-1)n^{\frac{1}{p}-1}\big)^{n-1}\kappa_{n-1}}1_{[0,(n-1)n^{\frac{1}{p}-1}]}(|y_i|)\Big)^{c_i}:$$

$$x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^\perp\Big\}dx$$

$$(3.8) \quad = \big((n-1)n^{\frac{1}{p}-1}\big)^{n}\kappa_{n-1}^{\frac{n}{n-1}} \int_{\mathbb{R}^n} \sup\Big\{\prod_{i=1}^{m} g_i^{c_i}(y_i) : x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^\perp\Big\}dx.$$

Since $\sum_{i=1}^{m} c_i \pi_{u_i} = \frac{1}{n-1}\sum_{i=1}^{m} \bar{c}_i \pi_{u_i} = Id$ and $\int_{u_i^\perp} f_i = \int_{u_i^\perp} g_i = 1$, by (3.7), (3.8) and Lemma 2.3, we obtain,

$$V(S_p^*) \leq \frac{1}{\Gamma(1+\frac{n}{p})}\Big(\frac{\Gamma(\frac{n-1}{p})\kappa_{n-1}}{p(n-1)^{\frac{n-1}{p}-1}}\Big)^{\frac{n}{n-1}}\frac{1}{\big((n-1)n^{\frac{1}{p}-1}\big)^{n}\kappa_{n-1}^{\frac{n}{n-1}}}V(S_p)$$

$$= \frac{\Gamma(\frac{n-1}{p})^{\frac{n}{n-1}}}{\Gamma(1+\frac{n}{p})p^{\frac{n}{n-1}}n^{\frac{n}{p}-n}(n-1)^{\frac{n}{p}-\frac{n}{n-1}+n}}V(S_p)$$

$$(3.9) \qquad = \frac{V(S_p)}{\alpha_{n,p}}.$$

Now let $\mu$ be an arbitrary isotropic measure on $S^{n-1}$. As in [4], we can construct a sequence $\mu_k, k \in \mathbb{N}$, of discrete isotropic measures such that $\mu_k$ converges weakly to $\mu$ as $k \to \infty$. It follows that $\lim_{k\to\infty} h_{S_{p_k}(\mu)}(v) = h_{s_p(\mu)}(v)$ for every $v \in S^{n-1}$. Since the pointwise convergence of support functions implies the convergence of the respective convex bodies in the Hausdorff metric (see e.g. [38, Chapter 1]), the continuity of volume and polarity on convex bodies containing the origin in their interiors finishes the proof. $\square$

In order to prove Theorems 1 and 2, we now only need the following result.

**Theorem 3.2.** *For $p \geq 1$, if $\mu$ is an isotropic measure on $S^{n-1}$, then*

$$\frac{\kappa_n}{\gamma_{n,p}^n} \leq V(S_p^*) \qquad and \qquad V(S_p) \leq \kappa_n \gamma_{n,p}^n.$$

*If $\mu$ is even and $p$ is not an even integer, then there is equality in either inequality if and only if $\mu$ is normalized Lebesgue measure.*

*Proof.* It follows from the polar coordinate formula for volume, (2.1), and the Hölder inequality that

$$\left(\frac{V(S_p^*)}{\kappa_n}\right)^{-\frac{1}{n}} = \left(\frac{1}{n\kappa_n}\int_{S^{n-1}} h_{S_p}(u)^{-n}du\right)^{-\frac{1}{n}} \leq \frac{1}{n\kappa_n}\int_{S^{n-1}} h_{S_p}(u)du$$

with equality if and only if $h_{S_p}(\cdot)$ is constant, i.e., $S_p$ is a ball. From the definition of the sine transform and Fubini's theorem, we obtain

$$
\begin{aligned}
\frac{1}{n\kappa_n}\int_{S^{n-1}} h_{S_p}(u)du &= \frac{1}{n\kappa_n}\int_{S^{n-1}}\int_{S^{n-1}}\left(1-(u\cdot v)^2\right)^{\frac{p}{2}}dud\mu(v)\\
&= \frac{1}{n\kappa_n}\int_{S^{n-1}}d\mu(v)(n-1)\kappa_{n-1}\int_{-1}^{1}(1-t^2)^{\frac{p}{2}+\frac{n-3}{2}}dt\\
&= \frac{(n-1)\kappa_{n-1}\Gamma(\frac{n+p-1}{2})\Gamma(\frac{1}{2})}{\kappa_n\Gamma(\frac{n+p}{2})}\\
&= \gamma_{n,p}.
\end{aligned}
$$

Consequently,

$$\left(\frac{V(S_p^*)}{\kappa_n}\right)^{-\frac{1}{n}} \leq \gamma_{n,p}$$

with equality if and only if $S_p$ is a ball. Using standard techniques from the theory of spherical harmonics (see the book [15] or the recent articles [11, 14, 17, 39, 41]) it is not difficult to show that the $L_p$ sine transform is injective on even measures if $p$ is not an even integer. This yields the equality conditions for even isotropic measures.

In order to establish the second inequality, we apply the classical Urysohn inequality (2.3) to obtain

$$\left(\frac{V(S_p)}{\kappa_n}\right)^{\frac{1}{n}} \leq \frac{1}{n\kappa_n}\int_{S^{n-1}} h_{S_p}(u)du = \gamma_{n,p}$$

with equality if and only if $S_p$ is a ball. Again, this yields the equality conditions for even isotropic measures. $\qquad\square$

The proof that the reverse inequalities in Theorems 3.1 and 3.2 are asymptotically optimal is now almost verbatim the same as the corresponding proof for $p = 1$ in [30].

## 4. Volume estimates from $L_p$-sine transform

In this section, we obtain results which are analogues of volume estimates due to E. Lutwak, D. Yang and G. Zhang [27] for $p$-projection bodies (see also [19, 24, 32] for more information on $L_p$ projection bodies). However, in contrast to $L_p$ projection bodies, the operators we consider will not be compatible with general linear transformations but merely with rotations (see [10, 12, 13, 21, 33, 37, 40, 42] for recent results concerning such operators). Therefore we will put the convex bodies in $L_p$ surface isotropic position, a notion that we will recall in the following.

An important part of geometric tomography deals with the estimation of the volume and other geometric quantities of a convex or star body from data about the projections or the sections of the body (see e.g. [1, 7, 18, 35, 44, 46] and, in particular, [6, Chapter 9] and the references therein).

For $p \geq 1$, convex bodies $K, L \in \mathscr{K}^n$, and $\varepsilon > 0$, the Firey $L_p$-combination $K + \varepsilon \cdot L$ is defined as the convex body whose support function is given by

$$h_{K+\varepsilon \cdot L}(\cdot)^p = h_K(\cdot)^p + \varepsilon h_L(\cdot)^p.$$

For $p \geq 1$, the $L_p$-mixed volume, $V_p(K, L)$, of the convex bodies $K, L$ was defined in [25] by:

$$\frac{n}{p}V_p(K, L) = \lim_{\varepsilon \to 0+} \frac{V(K + \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

That this limit exists was demonstrated in [25]. It was shown in [25], that corresponding to each convex body $K$ containing the origin, there is a positive Borel measure, $S_p(K, \cdot)$, on $S^{n-1}$ such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(u)^p dS_p(K, u)$$

for each convex body $Q$. The measure $S_1(K, \cdot)$ is just the classical surface area measure of $K$.

Let $\mathscr{K}_o^n$ denote the space of convex bodies that contain the origin in their interior. We define an operator $\Phi_p : \mathscr{K}_o^n \to \mathscr{K}_o^n$ by

$$h_{\Phi_p K}(v) = \left( \frac{1}{\kappa_n \gamma_{n,p}} \int_{S^{n-1}} |v|u^\perp|^p dS_p(K, u) \right)^{\frac{1}{p}},$$

where $\gamma_{n,p}$ is defined as in the introduction. Here, the normalization is chosen such that $\Phi_p B_2^n = B_2^n$. We will denote the polar of the body $\Phi_p K$ by $\Phi_p^* K$. It is important to note that while $\Phi_p$ still commutes with orthogonal transformations, it does not intertwine affine transformations like the $L_p$-projection body map. Consequently, the quantities $V(\Phi_p K)$ and $V(\Phi_p^* K)$ are rigid motion invariant but not invariant under volume preserving linear transformations. In fact, for a convex body $K$ of given volume, $V(\Phi_p K)$ may be arbitrarily large and $V(\Phi_p^* K)$ arbitrarily small, respectively. We will therefore fix a position of

the body, to be more precise, the $L_p$ surface isotropic position, to bound the quantities $V(\Phi_p K)$ and $V(\Phi_p^* K)$.

**Definition.** A convex body $K \in \mathscr{K}_o^n$ is said to be in $L_p$ surface isotropic position if and only if its $L_p$ surface area measure $S_p(K, \cdot)$ is, up to normalization, isotropic. In this case, we denote the total measure $S_p(K, S^{n-1})$ by $\partial_p(K)$.

**Theorem 4.1.** *If $K \in \mathscr{K}^n$ is in $L_p$ surface isotropic position, for $1 < p < \infty$, then*

$$n^{\frac{n}{p}} \kappa_n^{1+\frac{n}{p}} \gamma_{n,p}^{\frac{n}{p}-n} \leq V(\Phi_p^* K) \partial_p(K)^{\frac{n}{p}} \leq \frac{n^{\frac{n}{p}} \kappa_n^{1+\frac{n}{p}} \gamma_{n,p}^{\frac{n}{p}+n}}{\alpha_{n,p}}.$$

*There is equality in the left inequality among centrally symmetric convex bodies if and only if $K$ is a ball. Moreover,*

$$\frac{n^{-\frac{n}{p}} \kappa_n^{1-\frac{n}{p}} \alpha_{n,p}}{\gamma_{n,p}^{n+\frac{n}{p}}} \leq \frac{V(\Phi_p K)}{\partial_p(K)^{\frac{n}{p}}} \leq n^{-\frac{n}{p}} \kappa_n^{1-\frac{n}{p}} \gamma_{n,p}^{n-\frac{n}{p}},$$

*with equality in the right inequality among centrally symmetric convex bodies if and only if $K$ is a ball.*

*Proof.* Define the non-negative Borel measure $\mu$ on $S^{n-1}$ by

$$\mu = \frac{n}{\partial_p(K)} S_p(K, \cdot).$$

Since $K$ is in $L_p$ surface isotropic position, it follows that $\mu$ is isotropic. From the definitions of $S_p$ and the map $\Phi_p$, we have

$$
\begin{aligned}
h_{S_p}(x) &= \left( \int_{S^{n-1}} |x|u^\perp|^p d\mu(u) \right)^{\frac{1}{p}} \\
&= \left( \frac{n \kappa_n \gamma_{n,p}}{\partial_p(K)} \right)^{\frac{1}{p}} \left( \frac{1}{\kappa_n \gamma_{n,p}} \int_{S^{n-1}} |x|u^\perp|^p dS_p(K, u) \right)^{\frac{1}{p}} \\
&= \left( \frac{n \kappa_n \gamma_{n,p}}{\partial_p(K)} \right)^{\frac{1}{p}} h_{\Phi_p K}(x).
\end{aligned}
$$

Hence $S_p = \left( \frac{n \kappa_n \gamma_{n,p}}{\partial_p(K)} \right)^{\frac{1}{p}} \Phi_p K$ and $S_p^* = \left( \frac{\partial_p(K)}{n \kappa_n \gamma_{n,p}} \right)^{\frac{1}{p}} \Phi_p^* K$. An application of Theorem 3.1 and Theorem 3.2 completes the proof. $\square$

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