Bull. Korean Math. Soc. **52** (2015), No. 3, pp. 825–836
http://dx.doi.org/10.4134/BKMS.2015.52.3.825

A NOTE ON THE COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF B-VALUED RANDOM VARIABLES

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Abstract. In this article, we discuss the complete moment convergence for arrays of B-valued random variables. We obtain some new results which improve the corresponding ones of Sung and Volodin [17].

1. Introduction

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a probability space, and let B be a separable real Banach space with norm $||\cdot||$. A random element is defined to be an $\mathcal{F}$-measurable mapping of Ω into B equipped with the Borel σ-algebra (that is, the σ-algebra generated by the open sets determined by $||\cdot||$). The expected value of a B-valued random element X is defined to be the Bochner integral and denoted by EX.

Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of random elements in a real Banach space. An array of rowwise random elements $\{X_{nk}, k \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X (write $\{X_{nk}\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n\geq 1, k\geq 1} P(||X_{nk}|| > x) \leq CP(|X| > x), \quad \forall x > 0.$$

Now we recall the following concepts of convergence which were introduced by Hsu and Robbins [6] and Chow [4], respectively.

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

Received March 1, 2014.
2010 *Mathematics Subject Classification.* 60F15, 60F05, 60B12.
Key words and phrases. complete moment convergence, weighted sums, B-valued random variables.

Definition 1.2. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \text{ for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence.

Remark 1.1. It is easily seen that the complete moment convergence is the more general version of the complete convergence (see Remark 2.1).

Hsu and Robbins [6] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. This result has been generalized and extended in several directions by many authors (see, [3, 5, 7, 8, 9, 11, 14, 15, 17, 20, 22]). Some of these generalizations are in a Banach space setting (see, [2, 7, 8, 9, 14, 15, 17]).

Chow [4] investigated the complete moment convergence for independent random variables. His result also has been generalized and extended in subsequent literatures (see, [10, 12, 13, 16, 18, 19, 21]). However, according to our knowledge, few articles discuss the complete moment convergence for weighted sums of arrays of Banach space valued random elements.

Ahmed et al. [2] established the following theorem.

Theorem A. *Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise independent random elements taking values in a separable real Banach space with $\{X_{nk}\} \prec X$. Let $\{a_{nk}, k \geq 1, n \geq 1\}$ be an array of constants such that*

$$\sup_{k\geq 1} |a_{nk}| = O(n^{-\gamma}) \quad \text{for some } \gamma > 0 \tag{1.1}$$

and

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \quad \text{for some } \alpha < \gamma. \tag{1.2}$$

Let β be such that $\alpha + \beta \neq -1$ and fix $\delta > 1$ such that $\alpha/\gamma + 1 < \delta \leq 2$. If

$$E|X|^{\nu} < \infty \text{ where } \nu = \max\{1 + (1 + \alpha + \beta)/\gamma,\ \delta\}$$

and $S_n \equiv \sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} n^{\beta} P(||S_n|| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

Sung and Volodin [17] improved and complemented Theorem A. They obtained the following result.

Theorem B. *Suppose $\beta \geq -1$. Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise independent random elements with $\{X_{nk}\} \prec X$. Let $\{a_{nk}, k \geq 1, n \geq 1\}$ be an array of constants satisfying* (1.1) *and* (1.2). *Assume that $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ in probability. Then the following statements hold*:

(i) *If* $1+\alpha+\beta<0$ *and* $E|X|<\infty$, *then* (1.3) *holds.*
(ii) *If* $1+\alpha+\beta=0$ *and* $E(|X|\log|X|)<\infty$, *then* (1.3) *holds.*
(iii) *If* $1+\alpha+\beta>0$ *and* $E|X|^{1+(1+\alpha+\beta)/\gamma}<\infty$, *then* (1.3) *holds.*

In this article, we will improve Theorem B to the complete moment convergence case and will obtain a much stronger conclusion under the same conditions of Theorem B. The symbol C denotes a positive constant which is not necessarily the same one in each appearance. $S_n \equiv \sum_{k=1}^{\infty} a_{nk}X_{nk}$.

2. Preliminaries and main result

We first present some useful lemmas which are important in the proof of our main result.

Lemma 2.1 (See Acosta [1]). *Let* $\{X_k,\ 1\le k\le n\}$ *be a sequence of independent random elements. Then there exists a positive constant* C_p *depending only on* p *such that*
(i) *for* $1\le p\le 2$,

$$E\left|\left\|\sum_{k=1}^{n}X_k\right\|-E\left\|\sum_{k=1}^{n}X_k\right\|\right|^p\le C_p\sum_{k=1}^{n}E||X_k||^p,$$

(ii) *for* $p>2$,

$$E\left|\left\|\sum_{k=1}^{n}X_k\right\|-E\left\|\sum_{k=1}^{n}X_k\right\|\right|^p\le C_p\left\{\sum_{k=1}^{n}E||X_k||^p+\left(\sum_{k=1}^{n}E||X_k||^2\right)^{p/2}\right\}.$$

Lemma 2.2 (See Sung and Volodin [17]). *Let* $\{X_{nk}, k\ge 1, n\ge 1\}$ *be an array of rowwise independent random elements. Suppose there exists* $\delta>0$ *such that* $||X_{nk}||\le\delta$ *a.s. for all* $k\ge 1$ *and* $n\ge 1$. *Put* $T_n=\sum_{k=1}^{\infty}X_{nk}$. *If* $T_n\to 0$ *in probability, then* $E||T_n||\to 0$ *as* $n\to\infty$.

Now we state our main result and its proof.

Theorem 2.1. *Suppose* $\beta\ge -1$. *Let* $\{X_{nk}, k\ge 1, n\ge 1\}$ *be an array of rowwise independent random elements with* $\{X_{nk}\}\prec X$. *Let* $\{a_{nk}, k\ge 1, n\ge 1\}$ *be an array of constants satisfying* (1.1) *and* (1.2). *Assume that* $\sum_{k=1}^{\infty}a_{nk}X_{nk}\to 0$ *in probability. Then the following statements hold*:
(i) *If* $1+\alpha+\beta<0$ *and* $E|X|<\infty$, *then*

$$\sum_{n=1}^{\infty}n^{\beta}E\big\{||S_n||-\varepsilon\big\}_{+}<\infty \quad \text{for all } \varepsilon>0. \tag{2.1}$$

(ii) *If* $1+\alpha+\beta=0$ *and* $E(|X|\log|X|)<\infty$, *then* (2.1) *holds.*
(iii) *If* $1+\alpha+\beta>0$ *and* $E|X|^{1+(1+\alpha+\beta)/\gamma}<\infty$, *then* (2.1) *holds.*

Remark 2.1. Notice that the conditions of Theorem 2.1 are same as those of Theorem B and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} E\{||S_n|| - \varepsilon\}_+ \\
&= \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P(||S_n|| - \varepsilon > t)\mathrm{d}t \\
&\geq \sum_{n=1}^{\infty} n^{\beta} \int_0^{\varepsilon} P(||S_n|| - \varepsilon > t)\mathrm{d}t \geq \varepsilon \sum_{n=1}^{\infty} n^{\beta} P(||S_n|| > 2\varepsilon),
\end{aligned}$$

hence Theorem 2.1 improves Theorem B.

Remark 2.2. It is not difficult to find that there exists some difference among the results of Kim and Ko [9], Qiu et al. [14] and Sung and Volodin [17]. Kim and Ko [9] studied the case $\beta = -1$ and $1+\alpha+\beta > 0$. Qiu et al. [14] discussed the case $\beta \geq -1$ and $1+\alpha+\beta = 0$. However, Sung and Volodin [17] investigated the case $\beta \geq -1$ and $1+\alpha+\beta \geq 0$. Since Theorem 2.1 improves Theorem B, to some extent, it also improves the results of Kim and Ko [9], Qiu et al. [14] (Theorem 2 with $\theta = 1$).

Proof of Theorem 2.1. By the conditions (1.1) and (1.2), without loss of generality, we may assume that

$$\sup_{k\geq 1} |a_{nk}| = n^{-\gamma} \tag{2.2}$$

and

$$\sum_{k=1}^{\infty} |a_{nk}| = n^{\alpha}. \tag{2.3}$$

We will prove (2.1) by considering the following four cases.

Case 1: $1+\alpha+\beta < 0$

By (2.3), $\{X_{nk}\} \prec X$ and $E|X| < \infty$, we get

$$\sum_{n=1}^{\infty} n^{\beta} E\{||S_n|| - \varepsilon\}_+ \leq \sum_{n=1}^{\infty} n^{\beta} E||S_n|| \ \leq \ C \sum_{n=1}^{\infty} n^{\alpha+\beta} E|X| < \infty.$$

Case 2: $1+\alpha+\beta = 0$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta} E\{||S_n|| - \varepsilon\}_+ \\
&= \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P(||S_n|| > \varepsilon + t)\mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{\beta} \int_0^{1} P(||S_n|| > \varepsilon + t)\mathrm{d}t + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(||S_n|| > \varepsilon + t)\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{n=1}^{\infty} n^{\beta} P(||S_n|| > \varepsilon) + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(||S_n|| > t)\mathrm{d}t \\ &=: I_1 + I_2,\end{aligned}$$

in order to prove (2.1), we need only to show that $I_1 < \infty$ and $I_2 < \infty$. From Theorem B, we get immediately $I_1 < \infty$. Then we prove $I_2 < \infty$. Let $Y_{nk} = a_{nk}X_{nk}I(||a_{nk}X_{nk}|| \leq t)$, $Z_{nk} = a_{nk}X_{nk} - Y_{nk}$, then

$$\begin{aligned}I_2 &\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \int_1^{\infty} P(||a_{nk}X_{nk}|| > t)\mathrm{d}t + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\left|\left|\sum_{k=1}^{\infty} Y_{nk}\right|\right| > t\right)\mathrm{d}t \\ &=: I_3 + I_4.\end{aligned}$$

From $\{X_{nk}\} \prec X$, (2.2), (2.3) and $E(|X| \log(|X|)) < \infty$, we have

$$\begin{aligned}I_3 &\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E||a_{nk}X_{nk}||I(||a_{nk}X_{nk}|| > 1) \\ &\leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|I(|X| > |a_{nk}|^{-1}) \quad \text{(by (2.2) and (2.3))} \\ &\leq C \sum_{n=1}^{\infty} n^{-1} E|X|I(|X| > n^{\gamma}) \\ &= C \sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E|X|I(m^{\gamma} < |X| \leq (m+1)^{\gamma}) \\ &= C \sum_{m=1}^{\infty} E|X|I(m^{\gamma} < |X| \leq (m+1)^{\gamma}) \sum_{n=1}^{m} n^{-1} \\ &\leq C \sum_{m=1}^{\infty} \log m \; E|X|I(m^{\gamma} < |X| \leq (m+1)^{\gamma}) \\ &\leq CE(|X| \log |X|) < \infty.\end{aligned}$$

For I_4, we first prove that for all $t \geq 1$, $\beta \geq -1$ and $1 + \alpha + \beta \geq 0$

$$t^{-1} E\left|\left|\sum_{k=1}^{\infty} Z_{nk}\right|\right| \to 0 \quad \text{as } n \to \infty \tag{2.4}$$

and

$$t^{-1} E\left|\left|\sum_{k=1}^{\infty} Y_{nk}\right|\right| \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

For all $t \geq 1$, by $\{X_{nk}\} \prec X$, $\beta \geq -1$ and $E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$, we have

$$t^{-1} E\left|\left|\sum_{k=1}^{\infty} Z_{nk}\right|\right| \leq t^{-1} \sum_{k=1}^{\infty} E||Z_{nk}||$$

$$
\begin{aligned}
&\leq Ct^{-1}\sum_{k=1}^{\infty} E|a_{nk}X|I(|a_{nk}X|>t)\\
&\leq C\sum_{k=1}^{\infty} E|a_{nk}X|I(|a_{nk}X|>1)\\
&\leq C\sum_{k=1}^{\infty} E|a_{nk}X|^{1+(1+\alpha+\beta)/\gamma}I(|a_{nk}X|>1)\\
&\leq Cn^{-(\beta+1)}E|X|^{1+(1+\alpha+\beta)/\gamma}I(|X|>n^{\gamma})\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Therefore, (2.4) holds. From (2.4) and the hypothesis $\sum_{k=1}^{\infty} a_{nk}X_{nk}\to 0$ in probability, we get $t^{-1}\sum_{k=1}^{\infty} Y_{nk}\to 0$ in probability for all $t\geq 1$. Noting that $||t^{-1}Y_{nk}||\leq 1$, by Lemma 2.2, we know (2.5) holds. Hence while n is sufficiently large, $E||\sum_{k=1}^{\infty} Y_{nk}||\leq t/2$ holds uniformly for $t\geq 1$. Then by the Markov inequality and Lemma 2.1, we have

$$
\begin{aligned}
I_4 &\leq \sum_{n=1}^{\infty} n^{\beta}\int_1^{\infty} P\Big(\Big|\Big|\sum_{k=1}^{\infty} Y_{nk}\Big|\Big|-E\Big|\Big|\sum_{k=1}^{\infty} Y_{nk}\Big|\Big|>t/2\Big)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\int_1^{\infty} t^{-2}E\Big(\Big|\Big|\sum_{k=1}^{\infty} Y_{nk}\Big|\Big|-E\Big|\Big|\sum_{k=1}^{\infty} Y_{nk}\Big|\Big|\Big)^2\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} t^{-2}E||Y_{nk}||^2\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} t^{-2}\Big\{E(a_{nk}X)^2I(|X|\leq |a_{nk}|^{-1}t)+t^2P(|X|>|a_{nk}|^{-1}t)\Big\}\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} t^{-2}E(a_{nk}X)^2I(|X|\leq |a_{nk}|^{-1})\mathrm{d}t\\
&\quad + C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} t^{-2}E(a_{nk}X)^2I(|a_{nk}|^{-1}<|X|\leq |a_{nk}|^{-1}t)\mathrm{d}t\\
&\quad + C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} P(|X|>|a_{nk}|^{-1}t)\mathrm{d}t\\
&=: I_5+I_6+I_7.
\end{aligned}
$$

Here, we used the fact that if a random variable X_{nk} is stochastically dominated by a random variable X, then for all $q>0$ and $x>0$

$$
E||X_{nk}||^qI(||X_{nk}||\leq x)\leq C\{E|X|^qI(|X|\leq x)+x^qP(|X|>x)\}.
$$

For I_7, by a similar argument as in the proof of $I_3<\infty$, we have

$$
I_7 \;\leq\; C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E|a_{nk}X|I(|X|>|a_{nk}|^{-1})<\infty.
$$

For I_5, we have

$$\begin{aligned}
I_5 &\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E(a_{nk}X)^2 I(|X|\le |a_{nk}|^{-1})\\
&= C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E(a_{nk}X)^2 I(|X|\le n^{\gamma})\\
&\quad + C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E(a_{nk}X)^2 I(n^{\gamma}<|X|\le |a_{nk}|^{-1})\\
&=: I_5^{*}+I_5^{**}.
\end{aligned}$$

From (2.2) and (2.3), we get

$$\begin{aligned}
I_5^{*} &\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} |a_{nk}|(\sup_{k\ge 1}|a_{nk}|)EX^2 I(|X|\le n^{\gamma})\\
&\le C\sum_{n=1}^{\infty} n^{-1-\gamma}EX^2 I(|X|\le n^{\gamma})\\
&= C\sum_{n=1}^{\infty} n^{-1-\gamma}\sum_{m=1}^{n} EX^2 I((m-1)^{\gamma}<|X|\le m^{\gamma})\\
&= C\sum_{m=1}^{\infty} EX^2 I((m-1)^{\gamma}<|X|\le m^{\gamma})\sum_{n=m}^{\infty} n^{-1-\gamma}\\
&\le C\sum_{m=1}^{\infty} m^{-\gamma}EX^2 I((m-1)^{\gamma}<|X|\le m^{\gamma}) \ \le\ CE|X|<\infty.
\end{aligned}$$

By a similar argument as in the proof of $I_3<\infty$, we have

$$\begin{aligned}
I_5^{**} &\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E|a_{nk}X|I(n^{\gamma}<|X|\le |a_{nk}|^{-1})\\
&\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty} E|a_{nk}X|I(|X|>n^{\gamma})\\
&\le C\sum_{n=1}^{\infty} n^{-1}E|X|I(|X|>n^{\gamma})<\infty.
\end{aligned}$$

Next we consider I_6. Noting that

$$\begin{aligned}
&\int_1^{\infty} t^{-2}E(a_{nk}X)^2 I(|a_{nk}|^{-1}<|X|\le |a_{nk}|^{-1}t)\mathrm{d}t\\
=&\ \sum_{m=1}^{\infty}\int_m^{m+1} t^{-2}E(a_{nk}X)^2 I(1<|a_{nk}X|\le t)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{m=1}^{\infty} m^{-2} E(a_{nk}X)^2 I(1<|a_{nk}X|\leq m+1)\\
&= \sum_{m=1}^{\infty} m^{-2} \sum_{s=1}^{m} E(a_{nk}X)^2 I(s<|a_{nk}X|\leq s+1)\\
&= \sum_{s=1}^{\infty} E(a_{nk}X)^2 I(s<|a_{nk}X|\leq s+1) \sum_{m=s}^{\infty} m^{-2}\\
&\leq \sum_{s=1}^{\infty} s^{-1} E(a_{nk}X)^2 I(s<|a_{nk}X|\leq s+1)\\
&\leq CE|a_{nk}X|I(|a_{nk}X|>1).
\end{aligned}
$$

Hence by a similar argument as in the proof of $I_3<\infty$, we have

$$
I_6 \leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|I(|X|>|a_{nk}|^{-1})<\infty.
$$

The proof of (2.1) for $1+\alpha+\beta=0$ is completed.

Case 3: $0<1+\alpha+\beta<\gamma$ (i.e., $1<1+(1+\alpha+\beta)/\gamma<2$)

As in Case 2, by Theorem B, we can prove that $I_1<\infty$. By a similar argument as in Case 2 and $E|X|^{1+(1+\alpha+\beta)/\gamma}<\infty$, we have

$$
\begin{aligned}
I_3 &\leq C\sum_{m=1}^{\infty} E|X|I(m^{\gamma}<|X|\leq (m+1)^{\gamma}) \sum_{n=1}^{m} n^{\alpha+\beta}\\
&\leq C\sum_{m=1}^{\infty} m^{\alpha+\beta+1}\ E|X|I(m^{\gamma}<|X|\leq (m+1)^{\gamma})\\
&\leq CE|X|^{1+(1+\alpha+\beta)/\gamma}<\infty.
\end{aligned}
$$

Notice that (2.4) and (2.5) also hold for $1+\alpha+\beta>0$, we need only to prove $I_4<\infty$. By similar arguments as in Case 2, we can prove that $I_5^{*}<\infty$, $I_5^{**}<\infty$, $I_6<\infty$ and $I_7<\infty$. Here we omit the details.

Case 4: $1+\alpha+\beta\geq\gamma$ (i.e., $1+(1+\alpha+\beta)/\gamma\geq 2$)

As in Case 2, we can prove that $I_1<\infty$, $I_3<\infty$, (2.4) and (2.5). Hence we need only to prove $I_4<\infty$. Take $\eta>0$ such that $\eta>\max\{1+(1+\alpha+\beta)/\gamma,\ 2(1+\beta)/(r-\alpha)\}$. By the Markov inequality and Lemma 2.1, we have

$$
\begin{aligned}
I_4 &\leq \sum_{n=1}^{\infty} n^{\beta}\int_1^{\infty} P\left(\left\|\sum_{k=1}^{\infty} Y_{nk}\right\| - E\left\|\sum_{k=1}^{\infty} Y_{nk}\right\| > t/2\right)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\int_1^{\infty} t^{-\eta}\left\{\sum_{k=1}^{\infty} E||Y_{nk}||^{\eta} + \left(\sum_{k=1}^{\infty} E||Y_{nk}||^2\right)^{\eta/2}\right\}\mathrm{d}t\\
&= C\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{\infty}\int_1^{\infty} t^{-\eta}E||Y_{nk}||^{\eta}\mathrm{d}t + C\sum_{n=1}^{\infty} n^{\beta}\int_1^{\infty} t^{-\eta}\left(\sum_{k=1}^{\infty} E||Y_{nk}||^2\right)^{\eta/2}\mathrm{d}t
\end{aligned}
$$

$=: I_8 + I_9.$

By $\{X_{nk}\} \prec X$, we have

$$
\begin{aligned}
I_8 &\le C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \int_1^{\infty} t^{-\eta}\Big\{E|a_{nk}X|^{\eta} I(|X| \le |a_{nk}|^{-1}t) + t^{\eta} P(|X| > |a_{nk}|^{-1}t)\Big\}\mathrm{d}t \\
&= C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \int_1^{\infty} t^{-\eta} E|a_{nk}X|^{\eta} I(|X| \le |a_{nk}|^{-1})\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \int_1^{\infty} t^{-\eta} E|a_{nk}X|^{\eta} I(|a_{nk}|^{-1} < |X| \le |a_{nk}|^{-1}t)\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} \int_1^{\infty} P(|X| > |a_{nk}|^{-1}t)\mathrm{d}t \\
&=: I_{10} + I_{11} + I_{12}.
\end{aligned}
$$

By a similar argument as in the proof of $I_7 < \infty$, we get $I_{12} < \infty$. By a similar argument as in the proof of $I_6 < \infty$ (by replacing exponent 2 into η), we get $I_{11} < \infty$. For I_{10}, we have

$$
\begin{aligned}
I_{10} &\le C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|^{\eta} I(|X| \le n^{\gamma}) \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|^{\eta} I(n^{\gamma} < |X| \le |a_{nk}|^{-1}) \\
&=: I_{10}^{*} + I_{10}^{**}.
\end{aligned}
$$

By a similar argument as in the proof of $I_5^{*} < \infty$, we get

$$
I_{10}^{*} \le C\sum_{n=1}^{\infty} n^{\alpha+\beta-(\eta-1)\gamma} E|X|^{\eta} I(|X| \le n^{\gamma}) \le CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty.
$$

Take $\theta > 0$ such that $1 < \theta < 1 + (1+\alpha+\beta)/\gamma$. Obviously $\theta < \eta$, then

$$
\begin{aligned}
I_{10}^{**} &\le C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} E|a_{nk}X|^{\theta} I(n^{\gamma} < |X| \le |a_{nk}|^{-1}) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha+\beta-(\theta-1)\gamma} E|X|^{\theta} I(|X| > n^{\gamma}) \\
&= C\sum_{n=1}^{\infty} n^{\alpha+\beta-(\theta-1)\gamma} \sum_{m=n}^{\infty} E|X|^{\theta} I(m^{\gamma} < |X| \le (m+1)^{\gamma}) \\
&= C\sum_{m=1}^{\infty} E|X|^{\theta} I(m^{\gamma} < |X| \le (m+1)^{\gamma}) \sum_{n=1}^{m} n^{\alpha+\beta-(\theta-1)\gamma}
\end{aligned}
$$

$$\leq C \sum_{m=1}^{\infty} m^{1+\alpha+\beta-(\theta-1)\gamma} E|X|^{\theta} I(m^{\gamma} < |X| \leq (m+1)^{\gamma})$$
$$\leq C E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty.$$

Finally, we prove $I_9 < \infty$. From (2.2), (2.3) and $\eta > 2(1+\beta)/(r-\alpha)$, we have

$$\begin{aligned} I_9 &\leq C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} t^{-\eta} \left(\sum_{k=1}^{\infty} a_{nk}^2 E||X_{nk}||^2 \right)^{\eta/2} \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} t^{-\eta} \left(\sup_{k\geq 1} |a_{nk}| \sum_{k=1}^{\infty} |a_{nk}| E|X|^2 \right)^{\eta/2} \mathrm{d}t \\ &\leq C \sum_{n=1}^{\infty} n^{\beta-(\gamma-\alpha)\eta/2} (E|X|^2)^{\eta/2} < \infty. \end{aligned}$$

The proof is complete. □

3. Complete moment convergence of moving average processes

As an application, we state one result on the complete moment convergence of moving average processes, which improves Theorem 4.1 of Sung and Volodin [17].

Theorem 3.1. *Suppose $\beta \geq -1$. Let $\{Y_k, -\infty < k < \infty\}$ be a doubly infinite sequence of independent random elements which are stochastically dominated by a random variable X. Let $\{a_k, -\infty < k < \infty\}$ be an absolutely summable sequence of real numbers and set $X_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k$, $i \geq 1$. Assume that $\sum_{i=1}^{n} X_i / n^{1/p} \to 0$ in probability, where $1 \leq p < 2$. Then the following statements hold*:

(i) *If $\beta > -1$, $1 \leq p < 2$ and $E|X|^{p(\beta+2)} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\beta-1/p} E\left\{ \left|\left| \sum_{i=1}^{n} X_i \right|\right| - \varepsilon n^{1/p} \right\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

(ii) *If $1 < p < 2$ and $E|X|^p < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-1-1/p} E\left\{ \left|\left| \sum_{i=1}^{n} X_i \right|\right| - \varepsilon n^{1/p} \right\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

(iii) *If $E(|X| \log |X|) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-2} E\left\{ \left|\left| \sum_{i=1}^{n} X_i \right|\right| - \varepsilon n \right\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof. Let $X_{nk} = Y_k$ and $a_{nk} = n^{-1/p} \sum_{i=1}^{n} a_{i+k}$ for $-\infty < k < \infty$ and $n \geq 1$. Then the result follows by Theorem 2.1 with $\alpha = 1-1/p$, $\gamma = 1/p$ and $1 \leq p < 2$ (see the proof of Theorem 4.1 in [17]). □

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