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AN IDENTITY BETWEEN THE m-SPOTTY ROSENBLOOM-TSFASMAN WEIGHT ENUMERATORS OVER FINITE COMMUTATIVE FROBENIUS RINGS

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Abstract. This paper is devoted to presenting a MacWilliams type identity for m-spotty RT weight enumerators of byte error control codes over finite commutative Frobenius rings, which can be used to determine the error-detecting and error-correcting capabilities of a code. This provides the relation between the m-spotty RT weight enumerator of the code and that of the dual code. We conclude the paper by giving three illustrations of the results.

1. Introduction

The error control codes play an important role in improving reliability in communications and computer memory system [5]. Recently, high-density RAM chips with wide I/O data, called a byte, have been increasedly used in computer memory systems. These chips are very likely to have multiple random bit errors when exposed to strong electromagnetic waves, radio-active particles or high-energy cosmic rays. To make these memory systems more reliable, spotty [21] and m-spotty [20] byte errors are introduced, which can be effectively detected or corrected using byte error-control codes. To make clear the error-detecting and error-correcting capabilities of a code, the research has been done on some special types of polynomials, called weight enumerators.

In general, the weight enumerator of a code is a polynomial describing certain properties of the code, and an identity which relates the weight enumerator of a code with that of its dual code is called the MacWilliams type identity. For the past few years, various weight enumerators with respect to m-spotty Hamming Weight (Lee weight and RT weight) have been studied for various types of codes. Suzuki et al. [19] proved a MacWilliams type identity for binary byte error-control codes. M. Özen and V. Siap [8] and I. Siap [17] extended this result

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to arbitrary finite fields and to the ring $\mathbb{F}_2+u\mathbb{F}_2$ with $u^2=0$, respectively, which was generalized to ring $\mathbb{F}_2+u\mathbb{F}_2+\cdots+u^{m-1}\mathbb{F}_2$ with $u^m=0$ in [15]. I. Siap [16] derived a MacWilliams type identity for m-spotty Lee weight enumerator of byte error-control codes over $\mathbb{Z}_4$. A. Sharma and A. K. Sharma introduced joint m-spotty weight enumerators of two byte error-control codes over the ring of integers modulo ℓ and over arbitrary finite fields with respect to m-spotty Lee weight [14], r-fold joint m-spotty weight [12] and m-spotty Hamming weight [13]. They also discussed some of their applications and derived MacWilliams type identities for these enumerators.

In this paper, we will consider a MacWilliams type identity for m-spotty RT weight enumerators of linear codes over finite commutative Frobenius ring, which generalizes the results (case of binary field) of [9] to arbitrary finite commutative Frobenius ring. The organization of this paper is as follows: Section 2 provides definitions of m-spotty RT weight and m-spotty RT distance. Section 3 presents MacWilliams type identities for m-spotty RT weight, and Section 4 illustrates the weight distribution of the m-spotty byte error control code by three examples. Finally, the paper concludes in Section 5.

2. Preliminaries

In this section, we begin by giving some basic definitions that we need to derive our results. Let R be a finite commutative Frobenius ring with unity and N be a positive integer. Let us recall some basic knowledge about R as describe in [3]. Writing the identity element 1 of the ring as the sum of the primitive idempotents of R, we obtain an isomorphism

$$R \xrightarrow[\phi]{\cong} R_1 \oplus \cdots \oplus R_s,$$

where $R_1,\ldots,R_s$ are local commutative rings. The finite commutative ring R is called a *Frobenius ring* if R is self-injective (i.e., the regular module is injective), or equivalently, $(C^{\perp})^{\perp}=C$ for any submodule C of any free R-module R^n, where $C^{\perp}$ denotes the orthogonal submodule of C with respect to the usual Euclidean inner product on R^n. Moreover, in this case, $|C^{\perp}||C|=|R|^n$ for any submodule C of R^n, where $|C|$ denotes the cardinality of C. This is one of the reasons why only finite Frobenius rings are suitable for coding alphabets. With the isomorphism ϕ, R is Frobenius if and only if every local component R_i is Frobenius, and the finite local Frobenius ring R_i is Frobenius if and only if R_i has a unique minimal ideal.

A character of R is a homomorphism π,

$$\pi : (R,+) \to (\mathbb{C}^{\times},\cdot).$$

R is Frobenius if and only if there exists a character χ of R such that ker π contains no nonzero left (right) ideal of R. This π is a generating character. Let $\widehat{R}$ be all characters of R, for any character $\pi\in\widehat{R}$, there are two homomorphisms

$R \to \widehat{R}$:

$$r \mapsto {}^r\pi,$$
$$r \mapsto \pi^r.$$

The first is left linear; The second is right linear. A character is left (right) generating character if the first (second) map is surjective. The reader may refer to [22] for more details on Frobenius rings.

Let R^N be the R-module of all N-tuples over R. For a positive divisor b of N, a byte error-control code of length N and byte length b over R is defined as an R-submodule of R^N.

The RT weight and the RT distance over R are defined as follows:

Definition 2.1 (see [1, 7, 11])**.** Let $c = (c_1, c_2, \ldots, c_{bn}) \in R^{bn}$, and

$$w_{RT}(c) \quad = \quad \begin{cases} \max\{i : c_i \neq 0\} & c \neq 0, \\ 0 & c = 0. \end{cases}$$

$w_{RT}(c)$ is called the RT weight of c.

Definition 2.2 (see [21])**.** A spotty byte error is defined as t or fewer bits errors within a b-bit byte, where $1 \leq t \leq b$. When none of the bits in a byte is in error, we say that no spotty byte error has occurred.

An s-spotty byte error is defined as a random t-bit error within a byte. If there are more than t-bit errors in a byte, the errors are defined as m-spotty byte errors. We can define the m-spotty RT weight and the m-spotty RT distance over R as follows.

Definition 2.3. Let $e \in R^N$ be an error vector and $e_i \in R^b$ be the i-th byte of e, where $N = nb$ and $1 \leq i \leq n$. The number of t/b-errors in e, denoted by $w_{MRT}(e)$, and m-spotty RT weight is defined as

$$w_{MRT}(e) = \sum_{i=1}^{n} \left\lceil \frac{w_{RT}(e_i)}{t} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer not less than x.

Definition 2.4. Let c and v be codewords of m-spotty byte error control code C over R. Here c_i and v_i are the i-th bytes of c and v, respectively. Then, m-spotty RT distance function between c and v, denoted by d_{MRT}, is defined as follows:

$$d_{MRT}(c, v) = \sum_{i=1}^{n} \left\lceil \frac{d_{RT}(c_i, v_i)}{t} \right\rceil.$$

In Definition 2.4, if we take $t = b = 1$, then the m-spotty RT metric coincides with both RT and Hamming metrics. Also, in the case of $t = n = 1$, the m-spotty RT metric coincides with RT metric.

Remark 2.5. Similar to the proof of Theorem 2.5 in [9], m-spotty RT distance over R is a metric, that is, this function satisfies the metric axioms.

3. The MacWilliams identity over finite commutative Frobenius rings

Hereinafter, codes will be taken to be of length N where N is a multiple of byte length b, i.e., $N = bn$.

Let $c = (c_1, c_2, \ldots, c_N)$ and $v = (v_1, v_2, \ldots, v_N)$ be two elements of R^N. The inner product of c and v, denoted by $\langle c, v\rangle$, is defined as follows: $\langle c, v\rangle = \sum_{i=1}^{n} \langle c_i, v_i\rangle = \sum_{i=1}^{n} \left(\sum_{j=1}^{b} c_{(i,j)} v_{(i,b-j+1)} \right)$. Here, $\langle c_i, v_i\rangle = \sum_{j=1}^{b} c_{(i,j)} v_{(i,b-j+1)}$ denotes the inner product of c_i and v_i, respectively. Also $c_{(i,j)}$ and $v_{(i,b-j+1)}$ are the j-th bits of c_i and v_i, respectively. The inner product for each byte is taken in reverse order similar to the RT case where $n = 1$.

Now we recall some examples of finite commutative Frobenius rings and their generating characters, most of them can be found in [2] and [22].

Remark 3.1. Here are some examples of finite commutative Frobenius rings.

(i) Let $R = \mathbb{F}$ be a finite field. A generating character χ on $R = \mathbb{F}$ is given by $\chi(x) = \xi^{Tr(x)}$, where $\xi = e^{\frac{2\pi i}{p}}$ and $\mathrm{Tr} : \mathbb{F} \to \mathbb{F}_p$ is the trace function from $\mathbb{F}$ to $\mathbb{F}_p$.

(ii) Let $R = \mathbb{Z}_\ell$. Set $\xi = e^{\frac{2\pi i}{\ell}}$. Then $\chi(x) = \xi^x$, $x \in \mathbb{Z}_\ell$, is a generating character.

(iii) The finite direct sum of Frobenius rings is Frobenius. If $R_1, \ldots, R_n$ each has generating characters $\chi_1, \ldots, \chi_n$, then $R = \oplus R_i$ has generating character $\chi = \prod \chi_i$.

(iv) Any Galois ring is Frobenius. A Galois ring $R = GR(p^n, r) \cong \mathbb{Z}_{p^n}[x]/\langle f\rangle$ is a Galois extension of $\mathbb{Z}_{p^n}$ of degree r, where f is a monic irreducible polynomial in $\mathbb{Z}_{p^n}[x]$ of degree r. Because f is monic, any element a of R is represented by a unique polynomial $r = \sum_{i=1}^{r-1} a_i x^i$, with $a_i \in \mathbb{Z}_{p^n}$. Set $\xi = e^{\frac{2\pi i}{p^n}}$. Then $\chi(a) = \xi^{a_{r-1}}$.

(v) $R = \mathbb{F}_2[u_1, \ldots, u_k]/\left\langle u_i^2 = 0, u_i u_j = u_j u_i\right\rangle$ is a Frobenius ring. Let $r_k = \sum_{A \subseteq \{1,2,\ldots,k\}} c_A u_A \in R$. Then (c_A) can be thought of as a binary vector of length 2^k. Let $wt(c_A)$ be the Hamming weight of this vector. Then $\chi(r_k) = (-1)^{wt(c_A)}$.

In order to prove our main theorem, we should first prove the following two lemmas. From now onwards, we assume χ be a generating character over finite commutative Frobenius rings in Remark 3.1. ℓ denotes the cardinality of R, i.e., $|R| = \ell$.

Lemma 3.2. *Let $c = (c_1, \ldots, c_b) \in R^b$ with $w_{RT}(c) = j$. For any $0 \le k \le b$, we have*

$$S^{(\ell)}(k, j) := \sum_{w_{RT}(v)=k} \chi_c(v)$$

$$= \begin{cases} 1, & \text{if } k = 0; \\ \ell^{k-1}(\ell - 1), & \text{if } 1 \le k \le b - j; \\ -\ell^{k-1}, & \text{if } k = b + 1 - j; \\ 0, & \text{if } k \ge b + 2 - j. \end{cases}$$

Proof. It is easy to verify the result when $k = 0$. Let us assume $1 \le k \le b$ from here.

$$\begin{aligned} S^{(\ell)}(k, j) &= \sum_{w_{RT}(v)=k} \chi(\langle c, v\rangle) \\ &= \sum_{w_{RT}(v)=k} \chi(v_1 c_b + \cdots + v_k c_{b+1-k}) \\ &= \left(\prod_{i=1}^{k-1} \Big(\sum_{v_i \in R} \chi(c_{b+1-i} v_i)\Big)\right) \times \Big(\sum_{v_k \in \mathbb{R}^*} \chi(c_{b+1-k} v_k)\Big). \end{aligned}$$

Denote $T_i = \sum_{v_i \in R} \chi(c_{b+1-i} v_i)$ $(1 \le i \le k-1)$, and $T_k = \sum_{v_k \in R^*} \chi(c_{b+1-k} v_k)$. If $k \le b - j$, then we have $c_b = \cdots = c_{b+1-k} = 0$ and

$$T_i = \sum_{v_i \in R} \chi(0 \cdot v_i) = \sum_{v_i \in R} 1 = \ell, \; T_k = \sum_{v_k \in R^*} 1 = \ell - 1.$$

Hence $S^{(\ell)}(k, j) = \ell^{k-1}(\ell - 1)$. If $k = b + 1 - j$, we get $c_b = \cdots = c_{b+2-k} = 0$ and $c_{b+1-k} = c_j \neq 0$, and then

$$T_i = \ell, \; T_k = \sum_{v_k \in R} \chi(c_j v_k) - \chi(c_j \cdot 0) = -1.$$

Hence, $S^{(\ell)}(k, j) = -\ell^{k-1}$. If $k \ge b + 2 - j$, then the last $k - 1$ positions of codeword c contain at least one nonzero element, suppose for some $j \ge b-k+2$, $c_j \neq 0$, we have

$$T_{b+1-j} = \sum_{v_{b+1-j} \in R} \chi(c_j v_{b+1-j}) = 0.$$

Hence $S^{(\ell)}(k, j) = 0$. This proves the lemma. $\square$

The following theorem gives a partial information for $V^{(t,\ell)}(z)$.

Theorem 3.3. *Let $c = (c_1, c_2, \ldots, c_b)$ and $v = (v_1, v_2, \ldots, v_b)$ be two elements of R^b, with $w_{RT}(c) = j$. Then we have the following*

(i)

$$\sum_{v \in R^b} \chi_c(v) z^{\lceil w_{RT}(v)/t \rceil} = V_j^{(t,\ell)}(z),$$

where $V_j^{(t,\ell)}(z) = \sum_{k=0}^{b} S^{(\ell)}(k, j) z^{\lceil k/t \rceil}$.

(ii) *Let* $z = 1$ *in* $V_j^{(t,\ell)}(z)$. *Then*

$$V_j^{(t,\ell)}(1) = \begin{cases} \ell^b, & \text{if } j = 0; \\ 0, & \text{if } j \neq 0. \end{cases}$$

(iii) *For* $0 \leq j \leq b$, *let* $s = b - j$ *in Lemma 3.2, then*

$$V_j^{(t,\ell)}(z) = \begin{cases} 1 + (\ell^b - 1)z, & \text{if } j = 0,\ k \leq t \leq b; \\ 1 - z, & \text{if } j = b - s \geq 1,\ s + 1 \leq t \leq b, \end{cases}$$

where k *is the parameter of* $S^{(\ell)}(k, j)$ *in the expression of* $V_j^{(t,\ell)}(z)$.

Proof. (i) Using Lemma 3.2, we can obtain

$$\begin{aligned} \sum_{v \in R^b} \chi_c(v) z^{\lceil w_{RT}(v)/t \rceil} &= \sum_{k=0}^{b} \sum_{w_{RT}(v)=k} \chi_c(v) z^{\lceil k/t \rceil} \\ &= \sum_{k=0}^{b} z^{\lceil k/t \rceil} \left(\sum_{w_{RT}(v)=k} \chi_c(v) \right) \\ &= \sum_{k=0}^{b} S^{(\ell)}(k, j) z^{\lceil k/t \rceil} = V_j^{(t,\ell)}(z). \end{aligned}$$

(ii) Let $z = 1$. Suppose $j = 0$, then according to Lemma 3.2, we have

$$V_0^{(t,\ell)}(1) = \sum_{k=0}^{b} S^{(\ell)}(k, 0) = 1 + \sum_{k=1}^{b} S^{(\ell)}(k, 0) = 1 + \sum_{k=1}^{b} \ell^{k-1}(\ell - 1) = \ell^b.$$

Suppose $j \neq 0$, according to Lemma 3.2, we can get

$$\begin{aligned} V_j^{(t,\ell)}(1) = \sum_{k=0}^{b} S^{(\ell)}(k, j) &= 1 + \sum_{k=1}^{b-j} S^{(\ell)}(k, j) + \sum_{k=b-j+1}^{b} S^{(\ell)}(k, j) \\ &= 1 + \sum_{k=1}^{b-j} \ell^{k-1}(\ell - 1) - \ell^{b-j} = 0. \end{aligned}$$

(iii) Suppose $j = 0$ and $k \leq t \leq b$. By applying Lemma 3.2, then

$$V_0^{(t,\ell)}(z) = \sum_{k=0}^{b} S^{(\ell)}(k, 0) z^{\lceil k/t \rceil} = 1 + \sum_{k=1}^{b} S^{(\ell)}(k, 0) z = 1 + (\ell^b - 1)z.$$

Suppose $j = b - s \geq 1$ and $s + 1 \leq t \leq b$. According to Lemma 3.2, then

$$\begin{aligned} V_{b-s}^{(t,\ell)}(z) &= \sum_{k=0}^{b} S^{(\ell)}(k, b - s) z^{\lceil k/t \rceil} = 1 + \left(\sum_{k=1}^{s} S^{(\ell)}(k, b - s) - \ell^s \right) z \\ &= 1 + \left(\sum_{k=1}^{s} \ell^{k-1}(\ell - 1) - \ell^s \right) z = 1 - z. \end{aligned}$$

This proves the results. $\square$

Let $(G, +)$ be a finite abelian group and V be a vector space over the complex numbers. The set $\widehat{G}$ of all characters of G forms an abelian group under pointwise multiplication. For any function $f : G \longrightarrow V$, define its *Fourier transform* $\hat{f} : \widehat{G} \longrightarrow V$ by

$$\widehat{f}(\pi) = \sum_{x \in G} \pi(x) f(x), \pi \in \widehat{G}.$$

Given a subgroup $H \subseteq G$, define an *annihilator* $(\widehat{G} : H) = \{\pi \in \widehat{G} : \pi(H) = 1\}$. Moreover, we have $|(\widehat{G} : H)| = |G|/|H|$.

The Poission summation formula relates the sums of a function over a subgroup to the sum of its Fourier transform over the annihilator of the subgroup. The following lemma can be found in [22], which plays an important role in deriving the MacWilliams identity for m-spotty RT weight.

Lemma 3.4 (Poisson Summation Formula)**.** *Let $H \subset G$ be a subgroup, and let $f : G \longrightarrow V$ be any function from G to a complex vector space V. Then*

$$\sum_{x \in H} f(x) = \frac{1}{|(\widehat{G} : H)|} \sum_{\pi \in (\widehat{G}:H)} \widehat{f}(\pi).$$

Let $\alpha_j = \#\{i : w_{RT}(c_i) = j,\ 1 \leq i \leq n\}$. That is, α_j is the number of bytes having RT weight j, $0 \leq j \leq b$, in a codeword. The summation of $\alpha_0, \alpha_1, \ldots, \alpha_b$ is equal to the code length in bytes, that is $\sum_{j=0}^{b} \alpha_j = n$. The RT weight distribution vector $(\alpha_0, \alpha_1, \ldots, \alpha_b)$ is determined uniquely for the codeword c. Then, the m-spotty RT weight of the codeword c is expressed as $w_{MRT}(c) = \sum_{j=0}^{b} \lceil j/t \rceil \cdot \alpha_j$. Let $A_{(\alpha_0, \alpha_1, \ldots, \alpha_b)}(C)$ be the number of codewords with RT weight distribution vector $(\alpha_0, \alpha_1, \ldots, \alpha_b)$. For example, let $c = (010\ 012\ 020\ 202\ 000\ 200)$ be a codeword over $\mathbb{F}_3$ with byte 3. Then, the RT weight distribution vector of the codeword is $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 2, 2)$. Therefore, $A_{(1,1,2,2)}(C)$ is the number of codewords with RT weight distribution vector $(1, 1, 2, 2)$.

We are now ready to define the m-spotty RT weight enumerator of a byte error control code over R.

Definition 3.5. The weight enumerator for m-spotty byte error control code C is defined as

$$W_C(z) = \sum_{c \in C} z^{w_{MRT}(c)}.$$

By using the parameter $A_{(\alpha_0,\alpha_1,\ldots,\alpha_b)}(C)$, which denotes the number of codewords with RT weight distribution vector $(\alpha_0,\alpha_1,\ldots,\alpha_b)$, $W_C(z)$ can be expressed as follows:

$$W_C(z) \quad = \quad \sum_{\substack{(\alpha_0,\ldots,\alpha_b)\\ \alpha_0,\ldots,\alpha_b\geq 0\\ \alpha_0+\cdots+\alpha_b=n}} A_{(\alpha_0,\ldots,\alpha_b)}(C)\prod_{j=0}^{b}(z^{\lceil j/t\rceil})^{\alpha_j}.$$

The next theorem holds for the weight enumerator $W_C(z)$ of the code and that of the dual code $C^\perp$, expressed as $W_{C^\perp}(z)$.

Theorem 3.6. *Let C be a linear code and $C^\perp$ be its dual code. The relation between the m-spotty RT weight enumerators of C and $C^\perp$ is given by*

$$\begin{aligned} W_{C^\perp}(z) &= \sum_{\substack{(\alpha_0,\ldots,\alpha_b)\\ \alpha_0,\ldots,\alpha_b\geq 0\\ \alpha_0+\cdots+\alpha_b=n}} A^{\perp}_{(\alpha_0,\ldots,\alpha_b)}(C)\prod_{j=0}^{b}(z^{\lceil j/t\rceil})^{\alpha_j} \\ &= \frac{1}{|C|}\sum_{\substack{(\alpha_0,\ldots,\alpha_b)\\ \alpha_0,\ldots,\alpha_b\geq 0\\ \alpha_0+\cdots+\alpha_b=n}} A_{(\alpha_0,\ldots,\alpha_b)}(C)\prod_{j=0}^{b}(V_j^{(t,\ell)}(z))^{\alpha_j}. \end{aligned}$$

Moreover, $W_{C_1}(z)=W_{C_2}(z)$ if and only if $W_{C_1^\perp}(z)=W_{C_2^\perp}(z)$.

Proof. Given a linear code $C\subset R^n$, we apply the Poisson Summation Formula with $G=R^n$, $H=C$, and $V=\mathbb{C}[z]$, the polynomial ring over $\mathbb{C}$ in one indeterminate. The first task is to identify the character-theoretic annihilator $(\widehat{G}:H)=(\widehat{R}^n:C)$ with $C^\perp$. Let ρ be a generating character of R_ℓ. We use ρ to define a homomorphism $\beta: R\longrightarrow \widehat{R}$. For $r\in R$, the character $\beta(r)\in\widehat{R}$ has the form $\beta(r)(s)=(r\rho)(s)=\rho(sr)$ for $s\in R$. One can verify that β is an isomorphism of R-modules. In particular, $\mathrm{wt}(r)=\mathrm{wt}(\beta r)$, where $\mathrm{wt}(r)=0$ for $r=0$, and $\mathrm{wt}(r)\neq 0$ for $r\neq 0$.

Extend β to an isomorphism $\beta: R^n\longrightarrow\widehat{R}^n$ of R-modules, via $\beta(x)(y)=\rho(yx)$, for $x,y\in R^n$. Again, $\mathrm{w_{RT}}(x)=\mathrm{w_{RT}}(\beta x)$. For $x\in R^n$, $\beta(x)\in(\widehat{R}:C)$ means $\beta(x)(C)=\beta(C\cdot x)=1$. This means that the ideal $C\cdot x$ of R is contained in $\ker(\rho)$. Because ρ is a generating character, which implies that $C\cdot x=0$. Thus $x\in C^\perp$. The converse is obvious. Thus $C^\perp$ corresponds to $(\widehat{R}:C)$ under the isomorphism β.

Remember that $\beta: R^n\longrightarrow\widehat{R}^n$ is an isomorphism of R-modules and $(C^\perp)^\perp=C$. Thus the Poisson Summation Formula becomes

$$\sum_{v\in C^\perp} f(v)=\frac{1}{|C|}\sum_{c\in C}\widehat{f}(c),$$

where the Fourier transform is

$$\widehat{f}(c) = \sum_{v \in R^N} \chi_c(v) f(v).$$

Define $f(v) = \prod_{i=1}^n z^{\lceil w_{RT}(v_i)/t \rceil}$, where v_i denotes the i-th byte of v. Then we can get

$$\widehat{f}(c) = \sum_{v \in R^N} \chi_c(v) \prod_{i=1}^n z^{\lceil w_{RT}(v_i)/t \rceil} = \sum_{v \in R^{nb}} \prod_{i=1}^n \chi_{c_i}(v_i) \prod_{i=1}^n z^{\lceil w_{RT}(v_i)/t \rceil} \tag{1}$$

$$= \prod_{i=1}^n \left(\sum_{v_i \in R^b} \chi_{c_i}(v_i) z^{\lceil w_{RT}(v_i)/t \rceil} \right) = \prod_{i=1}^n V_{w_{RT}(c_i)}^{(t,\ell)}(z). \tag{2}$$

Assume that RT weight of the fixed vector c_i is $w_{RT}\,(c_i) = j$, and c has the RT weight distribution vector $(\alpha_0, \ldots, \alpha_b)$, then we have

$$\widehat{f}(c) = \prod_{j=0}^b (V_j^{(t,\ell)}(z))^{\alpha_j}. \tag{3}$$

Thus we have

$$\sum_{c \in C^\perp} \prod_{j=0}^b (z^{\lceil j/t \rceil})^{\alpha_j} = \frac{1}{|C|} \sum_{c \in C} \prod_{j=0}^b (V_j^{(t,\ell)}(z))^{\alpha_j}.$$

After rearranging the summations on both sides according to the RT weight distribution vectors of codewords in $C^\perp$ and C respectively, we have the result

$$\begin{aligned} &\sum_{\substack{(\alpha_0,\ldots,\alpha_b) \\ \alpha_0,\ldots,\alpha_b \geq 0 \\ \alpha_0+\cdots+\alpha_b = n}} A_{(\alpha_0,\ldots,\alpha_b)}(C^\perp) \prod_{j=0}^b (z^{\lceil j/t \rceil})^{\alpha_j} \\ &= \frac{1}{|C|} \sum_{\substack{(\alpha_0,\ldots,\alpha_b) \\ \alpha_0,\ldots,\alpha_b \geq 0 \\ \alpha_0+\cdots+\alpha_b = n}} A_{(\alpha_0,\ldots,\alpha_b)}(C) \prod_{j=0}^b (V_j^{(t,\ell)}(z))^{\alpha_j}. \end{aligned} \tag{4}$$

According to Equations (1)-(3) and Definition 3.5, it is easily checked that $W_C(z)$ is uniquely determined by $V_j^{(t,\ell)}(z)$, on the other hand, Equation (4) implies that $W_C^\perp(z)$ is also uniquely determined by $V_j^{(t,\ell)}(z)$ and Equation (4) is none other than the MacWilliams identity of the code C and its dual code $C^\perp$. Thus $W_{C_1}(z) = W_{C_2}(z)$ if and only if $W_{C_1^\perp}(z) = W_{C_2^\perp}(z)$. This proves the main results. $\square$

Note that if $t = 1$ and $n = 1$, then according to Definition 2.4, the m-spotty RT metric coincides with RT metric, then the MacWilliams identity with respect to the m-spotty RT weight enumerators in Theorem 3.6 becomes explicitly the MacWilliams identity with respect to the RT enumerators. If

TABLE 1. RT weight distribution vectors of the codewords in C and the number of codewords.

RT weight vector	number
$(3,0,0,0)$	1
$(0,1,1,1)$	4
$(1,0,1,1)$	2
$(0,2,1,0)$	2

TABLE 2. Polynomials $V_j^{(2,3)}(z)$ for $t=2$ and $b=3$.

$V_0^{(2,3)}(z) = 1 + 8z + 18z^2$
$V_1^{(2,3)}(z) = 1 + 8z - 9z^2$
$V_2^{(2,3)}(z) = V_3^{(2,3)}(z) = 1 - z$

$t=1$ and $n \neq 1$, then it does not always become explicitly the MacWilliams identity with respect to the RT enumerators.

4. Application examples

In Section 3, we present a proof of a MacWilliams identity that is valid over any finite commutative Frobenius ring. In this section, we take three examples to illustrate Theorem 3.6, where Tables 2, 4 and Table 6 also demonstrate the results of Theorem 3.3 with respect to the proposition of polynomial $V_j^{(t,\ell)}(z)$.

Example 4.1. Let

$$G = \begin{pmatrix} 1 & 0 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the generator matrix of a linear code C of length 9 over $\mathbb{F}_3$ (which is a finite field). C has 9 codewords. The dual code of C is a ternary linear code of length 9 and it has 2187 codewords.

Before computing the m-spotty weight enumerator of C, we illustrate how to apply the formulae. It is easy to show that the codeword $c = (011\ 010\ 000)$ belongs to C. Let $b=3$ and $t=2$. Then, the RT weight distribution vector of the codeword is $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1,0,1,1)$. The RT weight distribution vectors of the codewords of C, the number of codewords, and polynomials $V_j^{(t,\ell)}(z)$ for $b=3$ and $t=2$ are shown in Tables 1 and 2 for the necessary computations to apply Theorem 3.6.

According to the expression of $W_C(z)$ and Table 1, we obtain the m-spotty weight enumerator of C as $W_C(z) = 1 + 4z^3 + 4z^4$. By applying Theorem 3.6

TABLE 3. RT weight distribution vectors of the codewords in C and the number of codewords.

RT weight vector	number
$(2,0,0,0)$	1
$(0,0,0,2)$	18
$(0,1,0,1)$	1
$(0,0,1,1)$	3
$(0,1,1,0)$	1

TABLE 4. Polynomials $V_j^{(2,6)}(z)$ for $t=2$ and $b=3$.

$V_0^{(2,6)}(z) = 1 + 35z + 180z^2$
$V_1^{(2,6)}(z) = 1 + 35z - 36z^2$
$V_2^{(2,6)}(z) = V_3^{(2,6)}(z) = 1 - z$

and Table 2, we obtain

$$\begin{aligned} W_{C^\perp}(z) &= \frac{1}{|C|} \sum_{\alpha_0+\alpha_1+\alpha_2+\alpha_3=3} A_{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}(C) \prod_{j=0}^{b} (V_j^{(2,3)}(z))^{\alpha_j} \\ &= \frac{1}{9}(V_0^{(2,3))}(z))^3 + \frac{4}{9}(V_1^{(2,3))}(z))(V_2^{(2,3))}(z))(V_3^{(2,3))}(z)) \\ &\quad + \frac{2}{9}(V_0^{(2,3))}(z))(V_2^{(2,3))}(z))(V_3^{(2,3))}(z)) + \frac{2}{9}(V_1^{(2,3))}(z))^2(V_3^{(2,3))}(z)) \\ &= 1 + 10z + 24z^2 + 116z^3 + 542z^4 + 846z^5 + 648z^6. \end{aligned}$$

Example 4.2. Let

$$G = \begin{pmatrix} 1 & 1 & 1 & 5 & 4 & 2 \\ 0 & 3 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 & 0 & 3 \end{pmatrix}$$

be the generator matrix of a linear code C over $\mathbb{Z}_6$ (which is a residue class ring) of length 6. C has 24 codewords. The dual code of C is also a linear code of length 6 and it has 1944 codewords.

The number of codewords, and polynomials $V_j^{(t,\ell)}(z)$ for $b=3$ and $t=2$ are shown in Tables 3 and 4 for the necessary computations to apply Theorem 3.6.

According to the expression of $W_C(z)$ and Table 3, we obtain the m-spotty weight enumerator of C as

$$W_C(z) = 1 + z^2 + 4z^3 + 18z^4.$$

TABLE 5. RT weight distribution vectors of the codewords in C and the number of codewords.

RT weight vector	number
$(3,0,0,0)$	1
$(0,1,1,1)$	12
$(1,1,0,1)$	3
$(0,0,0,3)$	11
$(0,0,1,2)$	5

TABLE 6. Polynomials $V_j^{(2,16)}(z)$ for $t=2$ and $b=3$.

$V_0^{(2,6)}(z) = 1 + 255z + 3840z^2$
$V_1^{(2,6)}(z) = 1 + 255z - 256z^2$
$V_2^{(2,6)}(z) = V_3^{(2,6)}(z) = 1 - z$

According to Theorem 3.6 and Table 4, we obtain

$$\begin{aligned} W_{C^\perp}(z) &= \frac{1}{|C|} \sum_{\alpha_0+\alpha_1+\alpha_2+\alpha_3=3} A_{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}(C) \prod_{j=0}^{b} (V_j^{(2,6)}(z))^{\alpha_j} \\ &= \frac{1}{24}\big[(V_0^{(2,6)}(z))^2 + 18(V_3^{(2,6))}(z))^2 + (V_1^{(2,6)}(z))(V_2^{(2,6)}(z)) \\ &\quad + (V_1^{(2,6)}(z))^2(V_3^{(2,6)}(z)) + 3(V_2^{(2,6)}(z))^2(V_3^{(2,6)}(z))\big] \\ &= \frac{1}{24}\big[(1+35z+180z^2)^2 + 18(1-z)^2 + 2(1+35z-36z^2)(1-z) \\ &\quad + 3(1-z)^2\big] \\ &= 1 + 4z + 61z^2 + 528z^3 + 1350z^4. \end{aligned}$$

Example 4.3. Let C be a byte error-control code over $R_{16} = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ (which is not a chain ring) generated by the follow set

$$\{(1,0,0,u,v,1,0,0,u), (0,0,uv,uv,0,0,0,uv,uv)\},$$

where $u^2 = v^2 = 0$ and $uv = vu$. Its length is 9 and byte length is 3. It is easy to check that the generators are independent, hence the code has type $(16)^1(2)^1$ and $|C| = 32$. Its dual code $C^\perp$, which is also a byte error-control code over R_{16} of length 9, contains 2147483648 ($|C^\perp|$ is very large) codewords.

The number of codewords, and polynomials $V_j^{(t,\ell)}(z)$ for $b=3$ and $t=2$ are shown in Tables 5 and 6 for the necessary computations to apply Theorem 3.6.

According to the definition of $W_C(z)$ and Table 5, we obtain the m-spotty weight enumerator of C as

$$W_C(z) = 1 + 3z^3 + 12z^4 + 5z^5 + 11z^6.$$

Combining Theorem 3.6 with Table 4, we obtain

$$\begin{aligned}
W_{C^\perp}(z) &= \frac{1}{|C|}\sum_{\alpha_0+\alpha_1+\alpha_2+\alpha_3=3} A_{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}(C)\prod_{j=0}^{b}(V_j^{(2,16)}(z))^{\alpha_j}\\
&= \frac{1}{32}\big[(V_0^{(2,16)}(z))^3 + 12(V_1^{(2,16))}(z))(V_2^{(2,16))}(z))(V_3^{(2,16))}(z))\\
&\quad + 3(V_0^{(2,16))}(z))(V_1^{(2,16))}(z))(V_3^{(2,16))}(z)) + 11(V_3^{(2,16)}(z))^3\\
&\quad + 5(V_2^{(2,16)}(z))(V_3^{(2,16)}(z))^2\big]\\
&= 1 + 165z + 12555z^2 + 781303z^3 + 24613464z^4 + 352604160z^5\\
&\quad + 1769472000z^6.
\end{aligned}$$

5. Conclusion

In this paper, we derive a MacWilliams identity for m-spotty RT weight enumerators over arbitrary finite commutative Frobenius rings from the Poisson summation formula, which includes [9] as a special case and extends the results of [18]. This provides the relation between the m-spotty RT weight enumerator of the code and that of the dual code, which can be used to determine the error-detecting and error-correcting capabilities of a code. Especially when the size of $C^\perp$ of a code C is very large (Example 4.3), it is easy to determine the RT weight distribution of m-spotty byte error-control codes by Theorem 3.6.

During the revision of the paper anonymous reviewers pointed out references [4] and [10] that also had studied a generalization version of RT metric.

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