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PARABOLIC MARCINKIEWICZ INTEGRALS ASSOCIATED TO POLYNOMIALS COMPOUND CURVES AND EXTRAPOLATION

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Abstract. In this note we consider the parametric Marcinkiewicz integrals with mixed homogeneity along polynomials compound curves. Under the rather weakened size conditions on the integral kernels both on the unit sphere and in the radial direction, the L^p bounds of such operators are given by an extrapolation argument. Some previous results are greatly extended and improved.

1. Introduction

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $\alpha_j \geq 1\,(j = 1, \ldots, n)$ be fixed real numbers. Define the function $F : \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{R}$ by $F(x, \rho) = \sum_{j=1}^{n} x_j^2 \rho^{-2\alpha_j}$, $x = (x_1, x_2, \ldots, x_n)$. It is clear that for each fixed $x \in \mathbb{R}^n$, the function $F(x, \rho)$ is a decreasing function in $\rho > 0$. We let $\rho(x)$ denote the unique solution of the equation $F(x, \rho) = 1$. It was showed in [15] that $(\mathbb{R}^n, \rho)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^{n}$. For $\lambda > 0$, let A_λ be the diagonal $n \times n$ matrix $A_\lambda = \mathrm{diag}\{\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_n}\}$. For a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $y \in \mathbb{R}^n$, we denote $A_{\phi(\rho(y))}y'$ by $A_\phi(y)$, where $\mathbb{R}^+ := (0, \infty)$ and $y' = A_{\rho(y)^{-1}}y \in S^{n-1}$.

The change of variables related to the spaces $(\mathbb{R}^n, \rho)$ is given by the transformation

$$\begin{aligned}
x_1 &= \rho^{\alpha_1} \cos\theta_1 \cdots \cos\theta_{n-2} \cos\theta_{n-1},\\
x_2 &= \rho^{\alpha_2} \cos\theta_1 \cdots \cos\theta_{n-2} \sin\theta_{n-1},\\
&\;\;\vdots\\
x_{n-1} &= \rho^{\alpha_{n-1}} \cos\theta_1 \sin\theta_2,
\end{aligned}$$

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$$x_n = \rho^{\alpha_n} \sin\theta_1.$$

Thus $dx = \rho^{\alpha-1}J(x')d\rho d\sigma(x')$, where $\rho^{\alpha-1}J(x')$ is the Jacobian of the above transform and $\alpha = \sum_{j=1}^n \alpha_j$, $J(x') = \sum_{j=1}^n \alpha_j(x'_j)^2$. Obviously, $J(x') \in \mathcal{C}^\infty(S^{n-1})$ and there exists $M > 0$ such that

$$1 \le J(x') \le M, \ \ \forall\, x' \in S^{n-1}.$$

It is easy to see that

$$\rho(x) = |x|, \quad \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

Let Ω be integrable on S^{n-1} and satisfy

$$\int_{S^{n-1}} \Omega(u)J(u)d\sigma(u) = 0, \tag{1.1}$$

$$\Omega(A_s x) = \Omega(x), \ \forall\, s > 0 \text{ and } x \in \mathbb{R}^n. \tag{1.2}$$

For $d \ge 2$ and a suitable mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^d$, we define the parabolic parametric Marcinkiewicz integral operators $\mathscr{M}^{\varrho}_{\Omega,h,\Phi}$ on $\mathbb{R}^d$ by

$$\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)(x) = \Big(\int_0^\infty \Big| \frac{1}{t^\varrho} \int_{\rho(y)\le t} \frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\varrho}} f(x-\Phi(y))dy \Big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{1.3}$$

where $f \in \mathscr{S}(\mathbb{R}^d)$ (the Schwartz class), $\varrho = \sigma + i\tau$ ($\sigma, \tau \in \mathbb{R}$ with $\sigma > 0$) and $h \in \Delta_1(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+)$ for $\gamma \ge 1$ denotes the set of all measurable functions h on $\mathbb{R}^+$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{R>0} \Big(R^{-1} \int_0^R |h(t)|^\gamma dt \Big)^{1/\gamma} < \infty.$$

It is easy to check that $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+)$ for $0 < \gamma_1 < \gamma_2 < \infty$. Also, let $\mathcal{N}_\alpha(\mathbb{R}^+)$, $\alpha > 0$, be the set of all measurable functions h on $\mathbb{R}^+$ satisfying $N_\alpha(h) = \sum_{m=1} m^\alpha 2^m d_m(h) < \infty$ with $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k,m)|$, where $E(k,1) = \{t \in (2^k, 2^{k+1}] : \ |h(t)| \le 2\}$, and

$$E(k,m) = \{t \in (2^k, 2^{k+1}] : \ 2^{m-1} < |h(t)| \le 2^m\} \ \text{ for } m \ge 2.$$

It follows from [22] that $\Delta_\gamma(\mathbb{R}^+) \subsetneq \mathcal{N}_\alpha(\mathbb{R}^+)$ for any $\alpha > 0$ and $1 < \gamma < \infty$.

As is well known, the parabolic operators have a long history. It may go back to Fabes and Rivière [15], Madych [20] and Calderón and Torchinsky [6]. If $n = d$, $h(t) = \varrho = 1$ and $\Phi(y) = y$, the operator $\mathscr{M}^{\varrho}_{\Omega,h,\Phi}$ recovers the classical parabolic Marcinkiewicz integral operator denoted by $\mathscr{M}_\Omega$, which was discussed extensively by many authors. Xue, Ding and Yabuta [28] first proved that $\mathscr{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in L^q(S^{n-1})$ for fixed $q > 1$. Afterwards, Chen and Ding [7] (resp., [8]) extended the above result to the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ (resp., $\Omega \in H^1(S^{n-1})$). Moreover, it follows from Wang, Chen and Yu's work [25] (also see [3, 19]) that $\mathscr{M}_\Omega$ is of type (p, p) for $2\beta/(2\beta-1) < p < 2\beta$ if $\Omega \in \mathcal{F}_\beta(S^{n-1})$ for some $\beta > 1$, where

$$\mathcal{F}_\beta(S^{n-1}) := \Big\{\Omega \in L^1(S^{n-1}) : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \Big(\log \frac{1}{|\xi \cdot y'|}\Big)^\beta d\sigma(y') < \infty\Big\}$$

for all $\beta > 0$.

Note that

$$\bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \text{ and } \bigcap_{\beta>1} \mathcal{F}_\beta(S^{n-1}) \nsubseteq L\log^+ L(S^{n-1}); \tag{1.4}$$

$$L^q(S^{n-1}) \subsetneq L(\log^+ L)(S^{n-1}) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}); \tag{1.5}$$

$$L(\log^+ L)^{\beta_1}(S^{n-1}) \subsetneq L(\log^+ L)^{\beta_2}(S^{n-1}), \ \forall\ 0 < \beta_2 < \beta_1; \tag{1.6}$$

$$L(\log^+ L)^\beta(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq L(\log^+ L)^\beta(S^{n-1}), \quad \forall\ 0 < \beta < 1; \tag{1.7}$$

$$L(\log^+ L)^\beta(S^{n-1}) \subsetneq H^1(S^{n-1}), \quad \forall\ \beta \geq 1. \tag{1.8}$$

For the general operator $\mathscr{M}^\varrho_{\Omega,h,\Phi}$ in the Euclidean setting, i.e., the case of $\alpha_1 = \cdots = \alpha_n = 1$, we denote $\mathscr{M}^\varrho_{\Omega,h,\Phi}$ by $\mu^\varrho_{\Omega,h,\Phi}$. If $n = d$, $\Phi(y) = y$ and $h(t) = 1$, the operator $\mu^\varrho_{\Omega,h,\Phi}$ reduces to the classical parametric Marcinkiewicz integral operator denoted by μ^ϱ_Ω. The L^p bounds of μ^ϱ_Ω have been discussed extensively by many authors. For example, see [5, 23, 24, 27] for the case $\varrho \equiv 1$, [4, 17] for the case $\varrho > 0$, [12, 21] for the case $\varrho \in \mathbb{C}$ with $\mathrm{Re}\varrho > 0$. On the other hand, the investigation of the parametric Marcinkiewicz integral operators $\mu^\varrho_{\Omega,h,\Phi}$ with rough kernels on the unit sphere as well as in the radial direction have also received a large amount of attention of many authors (see [2, 9, 10, 11, 13] et al.). In particular, Al-Qassem and Pan [2] obtained the following result.

Theorem A. *Let $\Phi(y) = \mathcal{P}(y) = (P_1(y), \ldots, P_d(y))$ with P_j being polynomial on $\mathbb{R}^n$. Suppose that Ω satisfies* (1.1)-(1.2) *and $\mathcal{P}(y) = -\mathcal{P}(-y)$.*

(i) *If $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$, then*

$$\|\mu^\varrho_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^d)}, \ 2 \leq p < \infty;$$

(ii) *If $\Omega \in L(\log^+ L)(S^{n-1})$ and $h \in \mathcal{N}_1(\mathbb{R}^+)$, then*

$$\|\mu^\varrho_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(1+\|\Omega\|_{L(\log^+ L)(S^{n-1})})(1+N_1(h))\|f\|_{L^p(\mathbb{R}^d)}, \ 1 < p < 2.$$

The constant $C_p = C_{\varrho,n,d,p,\max_{1\leq j\leq d} \deg (P_j)}$ is independent of the coefficients of P_j for all $1 \leq j \leq d$.

In light of the aforementioned facts concerning the above Marcinkiewicz integrals, we find it is natural to ask whether Theorem A holds if replacing $\Phi(y) = \mathcal{P}(y)$ by $\Phi(y) = P(|y|)y'$ with P being a polynomial on $\mathbb{R}$. Moreover, another question, which arises from the above result, is the following:

Question 1. *For the general case $\alpha_j \geq 1\,(j = 1, \ldots, n)$, is $\mathscr{M}^{\varrho}_{\Omega,h,\Phi}$ bounded on $L^p(\mathbb{R}^n)$ under the same assumptions on Ω and h in Theorem A, even in the special case $n = d$ and $\Phi(y) = y$?*

In this paper, we will give an affirmative answer to these questions by the following:

Theorem 1. *Let $n = d$ and $\Phi(y) = (P_1(\varphi(\rho(y)))y'_1, \ldots, P_n(\varphi(\rho(y)))y'_n)$ with P_j being real valued polynomials on $\mathbb{R}$ satisfying $P_j(0) = 0$ and $\varphi \in \mathfrak{F}$. Here, the function class $\mathfrak{F}$ is the set of all function φ satisfying the condition* (a) *or* (b).

(a) *$\varphi : \mathbb{R}^+ \to (0, \infty)$ is a positive increasing $\mathcal{C}^1$ function such that $t\varphi'(t) \geq C_\varphi \varphi(t)$ and $\varphi(2t) \leq c_\varphi \varphi(t)$ for all $t > 0$, where C_φ and c_φ are independent of t.*

(b) *$\varphi : \mathbb{R}^+ \to (0, \infty)$ is a positive decreasing $\mathcal{C}^1$ function such that $t\varphi'(t) \leq -C_\varphi \varphi(t)$ and $\varphi(t) \leq c_\varphi \varphi(2t)$ for all $t > 0$, where C_φ and c_φ are independent of t.*

Suppose that Ω satisfies (1.1)-(1.2).

(i) *If $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$, then*

$$\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \leq C(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)},\ 2 \leq p < \infty;$$

(ii) *If $\Omega \in L(\log^+ L)(S^{n-1})$ and $h \in \mathcal{N}_1(\mathbb{R}^+)$, then*

$$\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \leq C(1+\|\Omega\|_{L(\log^+ L)(S^{n-1})})(1+N_1(h))\|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < 2.$$

The constant $C = C_{n,\varrho,p,\max_{1\leq j\leq n}\deg(P_j),\varphi}$ is independent of the coefficients of P_j for all $1 \leq j \leq n$.

Remark 1. There are some model examples in the class $\mathfrak{F}$, such as $t^\alpha\,(\alpha > 0)$, $t^\alpha(\ln(1+t))^\beta$ $(\alpha, \beta > 0)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. We note that for any $\varphi \in \mathfrak{F}$, there exists a constant $B_\varphi > 1$ such that $\varphi(2t) \geq B_\varphi \varphi(t)$ for all $t > 0$ if φ satisfies the condition (a), and $\varphi(t) \geq B_\varphi \varphi(2t)$ if φ satisfies the condition (b) (see [3, 13]). It should be pointed out that Theorem 1 is also new even for the case $\alpha_1 = \cdots = \alpha_n = 1$, in the Euclidean setting.

Remark 2. Theorem 1 essentially improve the result of [1] (see [1, Theorem 1.8]), even in the case $\alpha_1 = \cdots = \alpha_n = 1$, $n = d$ and $P_1(t) = \cdots = P_n(t) = \varphi(t) = t$. One the other hand, by (1.4) and (1.7), Theorem 1 is distinct from the results of [13, 26], even in the case $\alpha_1 = \cdots = \alpha_n = 1$, $n = d$ and $P_1(t) = \cdots = P_n(t) = \varphi(t) = t$. Moreover, Theorem 1 greatly generalize and improve the main result in [5], even in the case $\alpha_1 = \cdots = \alpha_n = 1$, $n = d$ and $P_1(t) = \cdots = P_n(t) = \varphi(t) = t$.

As several applications of Theorem 1, we have the following corollaries.

Corollary 1. *Let $n = d$ and $\Phi(y) = A_{P_N(\varphi)}(y)$ with $\varphi \in \mathfrak{F}$, and $P_N(t) = \sum_{i=1}^N a_i t^i$ with satisfying $P_N(t) > 0$ if $t \neq 0$. Suppose that Ω satisfies* (1.1)-(1.2).

(i) *If $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$, then*

$$\begin{aligned}&\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)}\\ &\le C(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)},\ 2 \le p < \infty;\end{aligned}$$

(ii) *If $\Omega \in L(\log^+ L)(S^{n-1})$ and $h \in \mathcal{N}_1(\mathbb{R}^+)$, then*

$$\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \le C(1+\|\Omega\|_{L(\log^+ L)(S^{n-1})})(1+N_1(h))\|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < 2.$$

The constant $C = C_{n,p,\varrho,N,\varphi}$ is independent of the coefficients of P_N.

Corollary 2. *Let $n = d$ and $\alpha_1 = \cdots = \alpha_n = 1$. Let $\Phi(y) = P_N(\varphi(|y|))y'$ with $\varphi \in \mathfrak{F}$, and $P_N(t) = \sum_{i=1}^N a_i t^i$. Suppose that Ω satisfies* (1.1)-(1.2).

(i) *If $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$, then*

$$\begin{aligned}&\|\mu^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)}\\ &\le C(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)},\ 2 \le p < \infty;\end{aligned}$$

(ii) *If $\Omega \in L(\log^+ L)(S^{n-1})$ and $h \in \mathcal{N}_1(\mathbb{R}^+)$, then*

$$\|\mu^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \le C(1+\|\Omega\|_{L(\log^+ L)(S^{n-1})})(1+N_1(h))\|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < 2.$$

The constant $C = C_{n,p,\varrho,N,\varphi}$ is independent of the coefficients of P_N.

The paper is organized as follows. In Section 2 we will present some notations and lemmas. The proof of Theorem 1 will be given in Section 3. We remark that our main methods in the proof of Theorem 1 are taken from [2, 22], but we add some new techniques. Especially, the proofs of (2.6) and (2.29) in this paper are different from [2]. Throughout the paper, we let p' denote the conjugate index of p which satisfies $1/p + 1/p' = 1$. The letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables.

2. Preliminary lemmas

Let $N = \max_{1\le j\le n} \deg(P_j)$. For $1 \le l \le n$, let $P_l(t) = \sum_{i=1}^N a_{i,l}t^i$. For $1 \le s \le N$ and $1 \le l \le n$, let $P_l^{(s)}(t) = \sum_{i=1}^s a_{i,l}t^i$ and $P^{(s)}(t) = (P_1^{(s)}(t), \dots, P_n^{(s)}(t))$. Set $P^{(0)}(t) = 0$ and

$$\Phi_s(y) = (P_1^{(s)}(\varphi(\rho(y)))y_1', \dots, P_n^{(s)}(\varphi(\rho(y)))y_n').$$

We can write

$$\begin{aligned}\Phi_s(y)\cdot\xi &= \sum_{l=1}^{n}\xi_l y_l' P_l^{(s)}(\varphi(\rho(y)))\\ &= \sum_{l=1}^{n}\sum_{i=1}^{s}\xi_l y_l' a_{i,l}\varphi(\rho(y))^i\\ &= \sum_{i=1}^{s}(L_i(\xi)\cdot y')\varphi(\rho(y))^i,\end{aligned}$$

where $L_i:\mathbb{R}^n\to\mathbb{R}^n$ is the linear transformation given by

$$L_i(\xi)=(a_{i,1}\xi_1,\dots,a_{i,n}\xi_n).$$

For $1\le s\le N$, let $\lambda_s=\mathrm{rank}(L_s)$. By [16, Lemma 6.1], there exist two nonsingular linear transformations $R_s:\mathbb{R}^{\lambda_s}\to\mathbb{R}^{\lambda_s}$ and $Q_s:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$|R_s\pi_{\lambda_s}^n Q_s(\xi)|\le|L_s(\xi)|\le C|R_s\pi_{\lambda_s}^n Q_s(\xi)|, \tag{2.1}$$

where C depends only on n and $\pi_{\lambda_s}^n$ is a projection operator from $\mathbb{R}^n$ to $\mathbb{R}^{\lambda_s}$.

For $1\le s\le N$ and $t>0$, we define the family of measures $\{\sigma_{h,t}^s\}$ and the related maximal operators $\sigma_{h,s}^*$ and $M_{h,q,\gamma,s}$ on $\mathbb{R}^n$ by

$$\begin{aligned}\widehat{\sigma_{h,t}^s}(\xi) &= \frac{1}{t^\varrho}\int_{t/2<\rho(y)\le t}\exp(-2\pi i\xi\cdot\Phi_s(y))\tfrac{h(\rho(y))\Omega(y)}{\rho(y)^{\alpha-\varrho}}dy;\\ \sigma_{h,s}^*(f)(x) &= \sup_{t\in\mathbb{R}^+}||\sigma_{h,t}^s|*f(x)|;\\ M_{h,q,\gamma,s}(f)(x) &= \sup_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}||\sigma_{h,t}^s|*f(x)|\tfrac{1}{t}dt,\end{aligned}$$

where $|\sigma_{h,t}^s|$ is defined in the same way as $\sigma_{h,t}^s$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$.

The following result follows from Lemmas 2.2 and 2.3 in [18].

Lemma 2.1. *Let $\Omega\in L^q(S^{n-1})$ for some $q>1$ and $P_\lambda(t)=\sum_{i=1}^{\lambda}a_it^i$ for some $\lambda\in\mathbb{N}\backslash\{0\}$. If $\varphi\in\mathfrak{F}$, then for any $0<\epsilon<\min\{1/q',1/\lambda\}$ and $\xi\in\mathbb{R}^n$, we have*

$$\begin{aligned}&\int_{r/2}^{r}\Big|\int_{S^{n-1}}\Omega(u')\exp(-iP_\lambda(\varphi(t))\xi\cdot u')d\sigma(u')\Big|^2\tfrac{1}{t}dt\\ &\le C(\varphi)\|\Omega\|_{L^q(S^{n-1})}^2|\varphi(r)^\lambda a_\lambda\xi|^{-\epsilon}\end{aligned}$$

for $\lambda\in\{1,2,\dots,N\}$ and any $r>0$. The constant $C(\varphi)$ is independent of Ω, q and the coefficients of P_λ, but depends on φ.

Lemma 2.2. *Let $\Omega \in L^q(S^{n-1})$ satisfying (1.1)-(1.2) and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some q, $\gamma \in (1,2]$. Suppose that $\varphi \in \mathfrak{F}$. Then for $1 \le s \le N$ and $t > 0$, $\xi \in \mathbb{R}^n$, there exists a constant $C > 0$ such that*

$$\max\left\{\|\sigma^s_{h,t}\|, |\widehat{\sigma^s_{h,t}}(\xi)|, |\widehat{|\sigma^s_{h,t}|}(\xi)|\right\} \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}; \tag{2.2}$$

$$\begin{aligned}&\max\left\{|\widehat{\sigma^s_{h,t}}(\xi) - \widehat{\sigma^{s-1}_{h,t}}(\xi)|, \left|\widehat{|\sigma^s_{h,t}|}(\xi) - \widehat{|\sigma^{s-1}_{h,t}|}(\xi)\right|\right\}\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}(\varphi(t)^s|L_s(\xi)|)^{1/(2s\gamma'q')};\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\max\left\{|\widehat{\sigma^s_{h,t}}(\xi)|, |\widehat{|\sigma^s_{h,t}|}(\xi)|\right\}\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}(\varphi(t)^s|L_s(\xi)|)^{-1/(2s\gamma'q')}.\end{aligned} \tag{2.4}$$

The constant C is independent of Ω, h, q, γ, but depends on φ.

Proof. Obviously, (2.2) holds. By a change of variable and Hölder's inequality, we have

$$\begin{aligned}&\left|\widehat{\sigma^s_{h,t}}(\xi) - \widehat{\sigma^{s-1}_{h,t}}(\xi)\right|\\ &= \left|\frac{1}{t^\varrho}\int_{t/2<\rho(y)\le t}(\exp(-2\pi i\xi\cdot\Phi_s(y)) - \exp(-2\pi i\xi\cdot\Phi_{s-1}(y)))\tfrac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\varrho}}dy\right|\\ &\le \int_{t/2<\rho(y)\le t}|L_s(\xi)\varphi(\rho(y))^s|\tfrac{|\Omega(y)h(\rho(y))|}{\rho(y)^\alpha}dy\\ &\le \int_{t/2}^t |h(r)|\tfrac{1}{r}dr\|\Omega\|_{L^1(S^{n-1})}\varphi(t)^s|L_s(\xi)|\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\varphi(t)^s|L_s(\xi)|,\end{aligned}$$

which combining with (2.2) implies that

$$\left|\widehat{\sigma^s_{h,t}}(\xi) - \widehat{\sigma^{s-1}_{h,t}}(\xi)\right| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}(\varphi(t)^s|L_s(\xi)|)^{1/(2s\gamma'q')}.$$

Similarly,

$$\left|\widehat{|\sigma^s_{h,t}|}(\xi) - \widehat{|\sigma^{s-1}_{h,t}|}(\xi)\right| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}(\varphi(t)^s|L_s(\xi)|)^{1/(2s\gamma'q')}.$$

Thus (2.3) holds. On the other hand, by a change of variable and Hölder's inequality, invoking Lemma 2.1 we have

$$\begin{aligned}&|\widehat{\sigma^s_{h,t}}(\xi)|\\ &\le C\int_{t/2}^t\left|\int_{S^{n-1}}\exp(-2\pi i\xi\cdot\Phi_s(y))\Omega(y')d\sigma(y')\right||h(r)|\tfrac{1}{r}dr\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|^{1-2/\gamma'}_{L^1(S^{n-1})}\Big(\int_{t/2}^t\left|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_s(y))d\sigma(y')\right|^2\tfrac{1}{r}dr\Big)^{1/r'}\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}(\varphi(t)^s|L_s(\xi)|)^{-\epsilon/\gamma'}\end{aligned}$$

for any $0 < \epsilon < \min\{1/q', 1/s\}$. Taking $\epsilon = 1/(2q's)$, we have

$$|\widehat{\sigma^s_{h,t}}(\xi)| \leq C\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}(\varphi(t)^s|L_s(\xi)|)^{-1/(2sq'\gamma')}.$$

Similarly,

$$|\widehat{|\sigma^s_{h,t}|}(\xi)| \leq C\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}(\varphi(t)^s|L_s(\xi)|)^{-1/(2sq'\gamma')}.$$

This completes the proof of Lemma 2.2. □

Motivated by the idea in [14], we have the following result, which will play a key role in the estimates on some vector-valued norm inequalities.

Lemma 2.3. *Let Ω, h, φ be as in Lemma 2.2. Then for $0 \leq s \leq N$ and any $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\|\sigma^*_{h,s}(f)\|_{L^p(\mathbb{R}^n)} \leq C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}; \tag{2.5}$$

$$\|M_{h,q,\gamma,s}(f)\|_{L^p(\mathbb{R}^n)} \leq C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.6}$$

The constant $C = C_{n,p,\varphi}$ is independent of Ω, h, q, γ and the coefficients of P_j for $1 \leq j \leq n$.

Proof. For convenience, we set $A = (q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}$. It is easy to verify that

$$\sigma^*_{h,s}(f)(x) \leq \sup_{k\in\mathbb{Z}} \frac{1}{2^{q'\gamma'(k+1)}} \int_{2^{q'\gamma'k}<\rho(y)\leq 2^{q'\gamma'(k+1)}} |f(x-\Phi_s(y))| \frac{|\Omega(y)h(\rho(y)|}{\rho(y)^{\alpha-1}} dy. \tag{2.7}$$

For $0 \leq s \leq N$, we define the family of measures $\{\mu_{k,s}\}$ and maximal operators μ^*_s on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f(x)d\mu_{k,s}(x) = \frac{1}{2^{q'\gamma'(k+1)}} \int_{2^{q'\gamma'k}<\rho(y)\leq 2^{q'\gamma'(k+1)}} \frac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-1}} f(\Phi_s(y))du,$$

$$\mu^*_s(f)(x) = \sup_{k\in\mathbb{Z}} ||\mu_{k,s}| * f(x)|,$$

where $|\mu_{k,s}|$ is defined in the same way as $\mu_{k,s}$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$. Thus, we get form (2.7) that

$$\sigma^*_{h,s}(f)(x) \leq \mu^*_s(|f|)(x). \tag{2.8}$$

Therefore, to prove (2.5), it suffices to prove that

$$\|\mu^*_s(f)\|_{L^p(\mathbb{R}^n)} \leq C_pA\|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty. \tag{2.9}$$

Then by the proof of Lemma 2.2 and a straightforward calculation we get for $1 \leq s \leq N$,

$$|\widehat{|\mu_{k,s}|}(\xi)| \leq CA(\min\{1, (\varphi(2^{q'\gamma'k})^s|L_s(\xi)|)^{-1}\})^{1/(2sq'\gamma')}, \tag{2.10}$$

$$\big|\widehat{|\mu_{k,s}|}(\xi) - \widehat{|\mu_{k,s-1}|}(\xi)\big| \leq CA(\varphi(2^{q'\gamma'k})^s|L_s(\xi)|)^{1/(2sq'\gamma')}. \tag{2.11}$$

In what follows, we prove (2.9) by induction on s.

Case 1. Without loss of generality, we may assume that f is nonnegative and $\|f\|_{L^p(\mathbb{R}^n)} < \infty$, It is easy to check that $\mu_0^*(f)(x) \le CAf(x)$, which implies (2.9) for $s = 0$.

Case 2. Suppose that (2.9) holds for $s = m-1$, $m \in \{1, \ldots, N\}$. We shall prove (2.9) for $s = m$. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ be supported in $\{|t| \le 1\}$ and $\psi(t) \equiv 1$ for $|t| \le 1/2$. Define the Borel measures $\{\omega_{k,s}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\widehat{\omega_{k,m}}(\xi) = \widehat{|\mu_{k,m}|}(\xi) - \psi(\varphi(2^{q'\gamma' k})^m |R_m \pi^n_{\lambda_m} Q_m(\xi)|)\widehat{|\mu_{k,m-1}|}(\xi) \tag{2.12}$$

for $\xi \in \mathbb{R}^n$, where R_m, $\pi^n_{\lambda_m}$, Q_m are as in (2.1). It follows from (2.1) and (2.10)-(2.11) that

$$|\widehat{\omega_{k,m}}(\xi)| \le CA(\min\{1, \varphi(2^{q'\gamma' k})^m |L_m(\xi)|, (\varphi(2^{q'\gamma' k})^m |L_m(\xi)|)^{-1}\})^{1/(2mq'\gamma')}. \tag{2.13}$$

In addition, by (2.13) and a well-known result on maximal functions (see [16]), we have

$$\|\mu_m^*(f)\|_{L^p(\mathbb{R}^n)} \le \|G_m(f)\|_{L^p(\mathbb{R}^n)} + C\|\mu^*_{m-1}(f)\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty, \tag{2.14}$$

$$\|\omega_m^*(f)\|_{L^p(\mathbb{R}^n)} \le \|G_m(f)\|_{L^p(\mathbb{R}^n)} + C\|\mu^\star_{m-1}(f)\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty, \tag{2.15}$$

where

$$\omega_m^*(f) = \sup_{k\in\mathbb{Z}} ||\omega_{k,m}| * f| \quad\text{and}\quad G_m(f) = \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,m} * f|^2\Big)^{1/2}.$$

By our assumption we have

$$\|\mu^*_{m-1}(f)\|_{L^p(\mathbb{R}^n)} \le C_p A \|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty, \tag{2.16}$$

where C_p is independent of q, γ and the coefficients of P_j for all $1 \le j \le n$. It remains to prove that

$$\|G_m(f)\|_{L^p(\mathbb{R}^n)} \le C_p A \|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty, \tag{2.17}$$

where C_p is as above. By a well-known property of Rademacher's functions, (2.17) follows from

$$\|V_\epsilon^m(f)\|_{L^p(\mathbb{R}^n)} \le C_{p,\varphi} A \|f\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty, \tag{2.18}$$

where $V_\epsilon^m(f) = \sum_{k\in\mathbb{Z}} \epsilon_k \omega_{k,m} * f$ with $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1 and $C_{p,\varphi}$ is independent of q, γ and the coefficients of $\{P_j\}$ for all $1 \le j \le n$. Below we prove (2.18). Choose a sequence of nonnegative functions $\{\Psi_k\}_{k\in\mathbb{Z}}$ in $\mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\Psi_k) \subset [\varphi(2^{q'\gamma'(k+1)})^{-m}, \varphi(2^{q'\gamma'(k-1)})^{-m}], \quad \sum_{k\in\mathbb{Z}} \Psi_k^2(t) = 1,$$

$$\left|(d/dt)^j \Psi_k(t)\right| \le C_j |t|^{-j}\ (j = 1, 2, \ldots), \quad \forall\ t > 0,\ j \in \mathbb{N},$$

where C_j are independent of q, γ and k. Define the Fourier multiplier operator S_j by

$$\widehat{S_j f}(\xi) = \Psi_j(|R_m \pi^n_{\lambda_m} Q_m(\xi)|)\hat{f}(\xi) \quad\text{for } j \in \mathbb{Z}.$$

Then

$$(2.19)\quad \begin{aligned} V_\epsilon^m(f) &= \sum_{k\in\mathbb{Z}} \epsilon_k \omega_{k,m} * \sum_{j\in\mathbb{Z}} S_{j+k}S_{j+k}f \\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \epsilon_k S_{j+k}(\omega_{k,m} * S_{j+k}f) := \sum_{j\in\mathbb{Z}} V_j^m(f). \end{aligned}$$

By the Littlewood-Paley theory, we have

$$(2.20)\quad \|V_j^m(f)\|_{L^p(\mathbb{R}^n)} \leq C_p\Big\|\Big(\sum_{k\in\mathbb{Z}} |\omega_{k,m} * S_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\ 1<p<\infty.$$

This combining with Plancherel's theorem yields

$$\begin{aligned} \|V_j^m(f)\|^2_{(\mathbb{R}^n)} &\leq C\Big\|\Big(\sum_{k\in\mathbb{Z}} |\omega_{k,m} * S_{j+k}f|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)} \\ &\leq C\sum_{k\in\mathbb{Z}}\int_{D_{j+k}} |\widehat{\omega_{k,m}}(\xi)|^2|\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

where

$$D_k = \{\xi\in\mathbb{R}^n:\ \varphi(2^{q'\gamma'(k+1)})^{-m} \leq |R_m\pi^n_{\lambda_m}Q_m(\xi)| \leq \varphi(2^{q'\gamma'(k-1)})^{-m}\}.$$

We get from (2.1), (2.13) and Remark 1 that

$$(2.21)\quad \|V_j^m(f)\|_{L^2(\mathbb{R}^n)} \leq CA(B_\varphi^{(2-j)/2}\chi_{\{j\geq 2\}} + B_\varphi^{(j+1)/2}\chi_{\{j<2\}})\|f\|_{L^2(\mathbb{R}^n)}.$$

This together with (2.19) implies

$$\|V_\epsilon^m(f)\|_{L^2(\mathbb{R}^n)} \leq CA\|f\|_{L^2(\mathbb{R}^n)}.$$

Thus,

$$\|G_m(f)\|_{L^2(\mathbb{R}^n)} \leq CA\|f\|_{L^2(\mathbb{R}^n)},$$

which combining the Littlewood-Paley theory, (2.13), (2.15)-(2.16) with the proof of Lemma in [14, p. 544] implies that

$$(2.22)\quad \|V_j^m(f)\|_{L^p(\mathbb{R}^n)} \leq C_pA\|f\|_{L^p(\mathbb{R}^n)},\ p=4 \text{ or } p=4/3.$$

Interpolating between (2.21) and (2.22) and combining with (2.19), we get

$$\|V_\epsilon^m(f)\|_{L^p(\mathbb{R}^n)} \leq CA\|f\|_{L^p(\mathbb{R}^n)},\ 4/3<p<4,$$

which leads to

$$(2.23)\quad \|G_m(f)\|_{L^p(\mathbb{R}^n)} \leq CA\|f\|_{L^p(\mathbb{R}^n)},\ 4/3<p<4.$$

Reasoning as above, (2.13), (2.15)-(2.16), (2.23), the proof of Lemma in [14, p.544], the Littlewood-Paley theory and interpolation imply

$$\|V_\epsilon^m(f)\|_{L^p(\mathbb{R}^n)} \leq CA\|f\|_{L^p(\mathbb{R}^n)},\ 8/7<p<8.$$

By using this argument repeatedly, we can obtain ultimately that

$$\|V_\epsilon^m(f)\|_{L^p(\mathbb{R}^n)} \leq CA\|f\|_{L^p(\mathbb{R}^n)},\ 1<p<\infty.$$

This proves (2.18) and completes the proof of (2.5).

Below we prove (2.6). It suffices to prove that

$$\|M_{h,q,\gamma,s}(|f|)\|_{L^p(\mathbb{R}^n)} \le C_p A \|f\|_{L^p(\mathbb{R}^n)} \tag{2.24}$$

for all $0 \le s \le N$ and $1 < p < \infty$, C_p is independent of Ω, h, q, γ and the coefficients of P_j for all $1 \le j \le n$. We shall prove (2.24) by induction on s. When $s = 0$, it is easy to see that

$$M_{h,q,\gamma,s}(|f|)(x) \le CA|f(x)|,$$

which implies (2.24) for $s = 0$. Assume that (2.24) holds for $s = m-1, m \in \{1, \dots, N\}$, we will prove that (2.24) holds for $s = m$. Let $\psi \in C_0^\infty(\mathbb{R})$ be as in (2.12). Define the family of measures $\{\lambda_{k,m}\}_{k\in\mathbb{Z}}$ by

$$\begin{aligned}\widehat{\lambda_{k,m}}(\xi) = &\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \widehat{|\sigma_{h,t}^m|}(\xi) \tfrac{1}{t} dt - \psi(\varphi(2^{q'\gamma' k})^m |R_m \pi_{\lambda_m}^n Q_m(\xi)|)\\ &\times \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \widehat{|\sigma_{h,t}^{m-1}|}(\xi) \tfrac{1}{t} dt.\end{aligned} \tag{2.25}$$

By Lemma 2.2 and (2.1), one can easily check that

$$|\widehat{\lambda_{k,m}}(\xi)| \le CA(\min\{1, \varphi(2^{q'\gamma' k})^m |L_m(\xi)|, (\varphi(2^{q'\gamma' k})^m |L_m(\xi)|)^{-1}\})^{1/(2mq'\gamma')}. \tag{2.26}$$

By the definition of $\lambda_{k,m}$ and a well known result on maximal function (see [16]), we have

$$\begin{aligned}&\|M_{h,q,\gamma,m}(|f|)\|_{L^p(\mathbb{R}^n)}\\ &\le \|g_m(|f|)\|_{L^p(\mathbb{R}^n)} + C\|M_{h,q,\gamma,m-1}(|f|)\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty,\end{aligned} \tag{2.27}$$

$$\begin{aligned}&\|\lambda_m^*(|f|)\|_{L^p(\mathbb{R}^n)}\\ &\le \|g_m(|f|)\|_{L^p(\mathbb{R}^n)} + C\|M_{h,q,\gamma,m-1}(|f|)\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty,\end{aligned} \tag{2.28}$$

where

$$g_m(f) = \Big(\sum_{k\in\mathbb{Z}} |\lambda_{k,m} * f|^2\Big)^{1/2} \quad \text{and} \quad \lambda_m^*(f) = \sup_{k\in\mathbb{Z}} ||\lambda_{k,m}| * f|.$$

From our assumption, (2.26)-(2.28) and the similar arguments as in getting (2.5), we get (2.24) for $s = m$. Thus (2.24) holds. Lemma 2.3 is proved. □

Lemma 2.4. *Let h, Ω, φ be as in Lemma 2.2. Then for $1 \le s \le N$ and any $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t}^s * g_k|^2 \tfrac{1}{t} dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\\ &\le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\ 2 \le p < \infty;\end{aligned} \tag{2.29}$$

(2.30)

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma_{h,t}^{s}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leq C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\ 1<p<2.$$

The constant $C=C_{n,p,\varphi}$ is independent of Ω, h, q, γ and the coefficients of $\{P_j\}$ for all $1\leq j\leq n$.

Proof. We shall use the method in [2]. First we prove (2.29). For fixed $2\leq p<\infty$, by duality, there exists a nonnegative function f in $L^{(p/2)'}(\mathbb{R}^n)$ with $\|f\|_{L^{(p/2)'}(\mathbb{R}^n)}\leq 1$ such that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma_{h,t}^{s}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^2\\ &=\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma_{h,t}^{s}*g_k|^2\tfrac{1}{t}dtf(x)dx.\end{aligned}\tag{2.31}$$

By a change of variable and Hölder's inequality, we obtain

$$\begin{aligned}&|\sigma_{h,t}^{s}*g_k(x)|^2\\ &\leq\Big(\frac{1}{t}\int_{t/2<\rho(y)\leq t}|g_k(x-\Phi_s(y))|\tfrac{|h(\rho(y))\Omega(y)|}{\rho(y)^{\alpha-1}}dy\Big)^2\\ &\leq\Big(\int_{t/2}^{t}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))||\Omega(y')|J(y')d\sigma(y')|h(r)|\tfrac{1}{r}dr\Big)^2\\ &\leq C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}^{\gamma}\|\Omega\|_{L^q(S^{n-1})}\\ &\quad\times\Big(\int_{t/2}^{t}\int_{S^{n-1}}|g_k(x-\Phi_s(A_ry'))|^2|\Omega(y')|J(y')d\sigma(y')|h(r)|^{2-\gamma}\tfrac{1}{r}dr\Big).\end{aligned}\tag{2.32}$$

Thus by (2.31)-(2.32) and Hölder's inequality, one can check that

(2.33)

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma_{h,t}^{s}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^2$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}^{\gamma}\|\Omega\|_{L^q(S^{n-1})}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2\tilde{M}_{|h|^{2-\gamma},q,\gamma,s}(\tilde{f})(-x)dx$$
$$\leq C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}^{\gamma}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}^2\|\tilde{M}_{|h|^{2-\gamma},q,\gamma,s}(\tilde{f})\|_{L^{(p/2)'}(\mathbb{R}^n)},$$

where $\tilde{f}(x)=f(-x)$ and $\tilde{M}_{|h|^{2-\gamma},q,\gamma,s}(f)$ denotes $M_{|h|^{2-\gamma},q,\gamma,s}$ with $\varrho=1$. Note that $|h(\cdot)|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)$, then by (2.6) we have

$$\|\tilde{M}_{|h|^{2-\gamma},q,\gamma,s}(|\tilde{f}|)\|_{L^{(p/2)'}(\mathbb{R}^n)}$$

$$\le C(q-1)^{-1}(\gamma-1)^{-1}\||h|^{2-\gamma}\|_{\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^{(p/2)'}(\mathbb{R}^n)}$$
$$\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})},$$

which combining with (2.33) implies (2.29). Next, we prove (2.30). Let $1 < p < 2$. By duality, there exist functions $\{f_k(x,t)\}$ define on $\mathbb{R}^n \times \mathbb{R}^+$ with $\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^n,\ell^2(L^2([2^{q'\gamma'k},2^{q'\gamma'(k+1)}],dt/t)))} \le 1$ such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma^s_{h,t}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \tag{2.34}$$
$$\le \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}\sigma^s_{h,t}*g_k(x)f_k(x,t)\tfrac{1}{t}dtdx$$
$$\le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\|(H(f))^{1/2}\|_{L^{p'}(\mathbb{R}^n)},$$

where

$$H(f)(x)=\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma^s_{h,t}*\tilde{f}_k(x,t)|^2\tfrac{1}{t}dt \quad\text{and}\quad \tilde{f}_k(x,t)=f(-x,t).$$

Since $p'>2$, there exists a nonnegative function $u\in L^{(p'/2)'}(\mathbb{R}^n)$ such that

$$\|H(f)\|_{L^{p'/2}(\mathbb{R}^n)}=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\sigma^s_{h,t}*\tilde{f}_k(x,t)|^2\tfrac{1}{t}dt\,u(x)dx.$$

By a similar argument as in (2.32) and (2.5), we have

(2.35)

$$\|H(f)\|_{L^{p'/2}(\mathbb{R}^n)}$$
$$\le C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\int_{\mathbb{R}^n}\tilde{\sigma}^*_{|h|^{2-\gamma},s}(\tilde{u})(-x)\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\tilde{f}_k(x,t)|^2\tfrac{1}{t}dt\Big)dx$$
$$\le C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|f_k(x,t)|^2\tfrac{1}{t}dt\Big)\Big\|_{L^{p'/2}(\mathbb{R}^n)}\|\tilde{\sigma}^*_{|h|^{2-\gamma},s}(\tilde{u})\|_{L^{(p'/2)'}(\mathbb{R}^n)}$$
$$\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|^2_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|^2_{L^q(S^{n-1})},$$

where $\tilde{u}(x)=u(-x)$ and $\tilde{\sigma}^*_{|h|^{2-\gamma},s}(\tilde{u})$ denotes $\sigma^*_{|h|^{2-\gamma},s}(\tilde{u})$ with $\varrho=1$. (2.30) follows from (2.34) and (2.35). This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We only prove Theorem 1 for the case $\varphi\in\mathfrak{F}$ with satisfying (a), the rest of Theorem 1 can be obtained similarly. Assume that $h\in\Delta_\gamma(\mathbb{R}^+)$ for some $\gamma\in(1,2]$ and $\Omega\in L^q(S^{n-1})$ for some $q \in (1,2]$ with satisfying (1.1)-(1.2). By Minkowski's inequality, we can write

$$\begin{aligned}
(3.1)\qquad & \mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)(x) \\
&= \Big(\int_0^\infty \Big|\sum_{k=-\infty}^{0} \frac{1}{t^\varrho}\int_{2^{k-1}t<\rho(y)\le 2^k t} f(x-\Phi(y))\tfrac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\varrho}}dy\Big|^2 \tfrac{1}{t}dt\Big)^{1/2} \\
&\le \sum_{k=-\infty}^{0}\Big(\int_0^\infty \Big|\frac{1}{t^\varrho}\int_{2^{k-1}t<\rho(y)\le 2^k t} f(x-\Phi(y))\tfrac{\Omega(y)h(\rho(y))}{\rho(y)^{\alpha-\varrho}}dy\Big|^2 \tfrac{1}{t}dt\Big)^{1/2} \\
&\le (1-2^{-\sigma})^{-1}\Big(\int_0^\infty |\sigma^N_{h,t}*f(x)|^2\tfrac{1}{t}dt\Big)^{1/2}.
\end{aligned}$$

Let ψ be as in (2.12). For $1 \le s \le N$, $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^n$, we define the family of measures $\{\nu_{t,s}\}$ by

$$\begin{aligned}
\widehat{\nu_{t,s}}(\xi) = \widehat{\sigma^s_{h,t}}(\xi)\prod_{s<j\le N}\psi(\varphi(t)^j|R_j\pi^n_{\lambda_j}Q_j(\xi)|) \\
(3.2)\qquad - \widehat{\sigma^{s-1}_{h,t}}(\xi)\prod_{s-1<j\le N}\psi(\varphi(t)^j|R_j\pi^n_{\lambda_j}Q_j(\xi)|).
\end{aligned}$$

It is clear that

$$(3.3)\qquad \sigma^N_{h,t} = \sum_{s=1}^{N}\nu_{t,s}.$$

Here we use the convention $\Pi_{j\in\emptyset}a_j = 1$. By Lemma 2.1 and (2.1), we get

$$\begin{aligned}
(3.4)\qquad & |\widehat{\nu_{t,s}}(\xi)| \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\min\{1,\varphi(t)^s|L_s(\xi)|,(\varphi(t)^s|L_s(\xi)|)^{-1}\}^{1/(2s\gamma'q')}
\end{aligned}$$

for $1 \le s \le N$. The constant C is independent of Ω, h, q, γ and the coefficients of P_j for all $1 \le j \le n$. This together with a straightforward calculation yields that

$$\begin{aligned}
& \Big(\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}} |\widehat{\nu_{t,s}}(\xi)|^2\tfrac{1}{t}dt\Big)^{1/2} \\
(3.5)\qquad & \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})} \\
& \quad\times \min\{1,\varphi(2^{\gamma'q'k})^s|L_s(\xi)|,(\varphi(2^{\gamma'q'k})^s|L_s(\xi)|)^{-1}\}^{1/(2s\gamma'q')},
\end{aligned}$$

where C is independent of Ω, h, q, γ and the coefficients of P_j for all $1 \le j \le n$. It follows from (3.1), (3.3) and Minkowski's inequality that

$$(3.6)\qquad \mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)(x) \le C\sum_{s=1}^{N}\Big(\int_0^\infty |\nu_{t,s}*f(x)|^2\tfrac{1}{t}dt\Big)^{1/2} := C\sum_{s=1}^{N}\mathscr{M}_s(f)(x).$$

Choose a sequence of nonnegative functions $\{\Psi_k\}_{k\in\mathbb{Z}}$ in $\mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\Psi_k)\subset[\varphi(2^{q'\gamma'(k+1)})^{-s},\varphi(2^{q'\gamma'(k-1)})^{-s}],\quad \sum_{k\in\mathbb{Z}}\Psi_k(t)=1,$$

$$\left|(d/dt)^j\Psi_k(t)\right|\le C_j|t|^{-j}\ (j=1,2,\dots)\ \text{ for all } t>0 \text{ and } j\in\mathbb{N},$$

where C_j are independent of q,γ,k. Define the Fourier multiplier operator Γ_j by

$$\widehat{\Gamma_j(f)}(\xi)=\Psi_j(|R_s\pi^n_{\lambda_s}Q_s(\xi)|)\hat{f}(\xi)\ \text{ for } j\in\mathbb{Z}. \tag{3.7}$$

Then we have

$$\mathscr{M}_s(f)(x)\le\sum_{i\in\mathbb{Z}}\mathscr{M}_{s,i}(f)(x), \tag{3.8}$$

where

$$\mathscr{M}_{s,i}(f)(x)=\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\nu_{t,s}*\Gamma_{i+k}f(x)|^2\tfrac{1}{t}dt\Big)^{1/2}.$$

Below we estimate the L^p estimates for $\mathscr{M}_{s,i}$. By Lemma 2.4 and the definition of $\nu_{t,s}$, we have for $1\le s\le N$, there exists a positive constant $C=C_{n,p,\varphi}$, which is independent of Ω, h, q, γ such that

(3.9)

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\nu_{t,s}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\ 2\le p<\infty;$$

(3.10)

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\nu_{t,s}*g_k|^2\tfrac{1}{t}dt\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)},\ 1<p<2.$$

By (3.9)-(3.10) and the Littlewood-Paley theory, we have

(3.11)

$$\begin{aligned}&\|\mathscr{M}_{s,i}(f)\|_{L^p(\mathbb{R}^n)}\\ &=\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}}|\nu_{t,s}*\Gamma_{i+k}f(x)|^2\tfrac{1}{t}dt\Big)^{1/2}\|_{L^p(\mathbb{R}^n)}\\ &\le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Gamma_{i+k}f|^2\Big)^{1/2}\|_{L^p(\mathbb{R}^n)}\\ &\le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\quad 2\le p<\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathscr{M}_{s,i}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\ 1<p<2.\end{aligned} \tag{3.12}$$

On the other hand, by (2.1), (3.5), Remark 1 and Plancherel's theorem, we have

$$\begin{aligned}&\|\mathscr{M}_{s,i}(f)\|^2_{L^2(\mathbb{R}^n)} \\ &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}|\Psi_{i+k}(|R_s\pi^n_{\lambda_s}Q_s(\xi)|)|^2|\hat{f}(\xi)|^2\Big(\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}}|\widehat{\nu_{t,s}}(\xi)|^2\tfrac{1}{t}dt\Big)d\xi \\ &\le \sum_{k\in\mathbb{Z}}\int_{E_{i+k}}\Big(\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}}|\widehat{\nu_{t,s}}(\xi)|^2\tfrac{1}{t}dt\Big)|\hat{f}(\xi)|^2d\xi \\ &\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|^2_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|^2_{L^q(S^{n-1})}D_i^2\|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.13}$$

where $D_i = B_\varphi^{(2-i)/2}\chi_{\{i\ge 2\}} + B_\varphi^{i/2}\chi_{\{i<2\}}$ and

$$E_{i+k} = \{\xi\in\mathbb{R}^n:\ \varphi(2^{\gamma' q'(i+k+1)})^{-s} \le |R_s\pi^n_{\lambda_s}Q_s(\xi)| \le \varphi(2^{\gamma' q'(i+k-1)})^{-s}\}.$$

Thus

$$\begin{aligned}\|\mathscr{M}_{s,i}(f)\|_{L^2(\mathbb{R}^n)} \\ \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}D_i\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3.14}$$

By interpolation between (3.11) and (3.14) leads to

$$\begin{aligned}\|\mathscr{M}_{s,i}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}D_i^{\alpha_p}\|f\|_{L^p(\mathbb{R}^n)},\ 2\le p<\infty.\end{aligned} \tag{3.15}$$

The constant α_p depends only on p. It follows from (3.8) and (3.15) that

$$\begin{aligned}\|\mathscr{M}_{s}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\ 2\le p<\infty.\end{aligned} \tag{3.16}$$

Similarly, by (3.8) and interpolation between (3.12) and (3.14), we have

$$\begin{aligned}\|\mathscr{M}_{s}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\ 1<p<2.\end{aligned} \tag{3.17}$$

Using (3.6), (3.16)-(3.17), we get

$$\begin{aligned}\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\ 2\le p<\infty,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\|\mathscr{M}^{\varrho}_{\Omega,h,\Phi}(f)\|_{L^p(\mathbb{R}^n)} \\ \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^n)},\ 1<p<2.\end{aligned} \tag{3.19}$$

The constant $C = C_{\varrho,n,p,N,\varphi}$ is independent of Ω, h, q, γ. Finally, Theorem 1 follows directly from (3.18)-(3.19) and an extrapolation argument as in the proof of [22, Theorem 1.2].

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