NEW CHARACTERIZATIONS OF COMPOSITION OPERATORS BETWEEN BLOCH TYPE SPACES IN THE UNIT BALL

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Abstract. In this paper, we give new characterizations of the boundedness and compactness of composition operators C_φ between Bloch type spaces in the unit ball $\mathbb{B}^n$, in terms of the power of the components of φ, where φ is a holomorphic self-map of $\mathbb{B}^n$.

1. Introduction

Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$ with boundary $\partial\mathbb{B}^n$. The class of all holomorphic functions on $\mathbb{B}^n$ will be denoted by $H(\mathbb{B}^n)$. Let $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{B}^n$. The composition operator is defined as follows:

$$C_\varphi(f)(z) = f(\varphi(z))$$

for any $f \in H(\mathbb{B}^n)$ and $z \in \mathbb{B}^n$.

For any $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$. For $f \in H(\mathbb{B}^n)$, let

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)$$

be the complex gradient of f, and

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of f.

Given $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ a holomorphic self-map of $\mathbb{B}^n$, the symbol $\varphi'(z)z$ is denoted by

$$\varphi'(z)z = (\Re\varphi_1(z), \ldots, \Re\varphi_n(z)).$$

For any $0 < p < \infty$, we define the Bloch type space $\mathcal{B}^p$ as the space of holomorphic functions such that

$$||f||_p = |f(0)| + \sup\left\{(1 - |z|^2)^p|\Re f(z)| : z \in \mathbb{B}^n\right\} < \infty.$$

It is important to provide function-theoretic descriptions of when φ induces a bounded or compact composition operator on various function spaces. For general references on the theory of composition operators, we refer the interested readers to the books [1, 14, 25]. Recently there has been a great interest in studying composition operators between Bloch type spaces. For example, see [2, 5, 11, 13, 15, 20, 21, 22, 23, 24]. So far there are still many unsolved problems that are the interest of numerous mathematicians. In [18], Wulan, Zheng and Zhu obtained a new result about the compactness of the composition operator on the Bloch space in the unit disk. Recently, interest has arisen to characterize the boundedness and compactness of (weighted) composition operators $(u)C_\varphi$ on Bloch type spaces in terms of the n-th power of the analytic self-map φ on the unit disc $\mathbb{D}$, such as [4, 7, 8, 9, 10, 12, 17, 19].

For the higher dimensional case, in 2012, the authors in [6] generalize Zhao's results in [19] to the polydisk. Unlike the case of composition operators on the unit disk, the essential norms are different for the cases $p \in (0,1)$ and $p \geq 1$. Dai in [3] gave several new necessary and sufficient conditions for the compactness of the composition operator on the Bloch space in the unit ball. Along with this line, this paper will consider the new characterizations of the boundedness and compactness of composition operators between Bloch type spaces in the unit ball.

2. Some lemmas

In this section, we present some lemmas which will be used in the proofs of our main results in the next section.

The proof of the following lemma can be found in Lemma 2.2 in [19].

Lemma 2.1. *Let $p > 0$, $m \in \mathbb{N}$ and $0 \leq x \leq 1$. And let $H_{m,p}(x) = x^{m-1}(1 - x^2)^p$, $r_m = (\frac{m-1}{m-1+2p})^{1/2}$ $(m \geq 2)$, and $r_m = 0$ for $m = 1$. Then for $m \geq 1$, $H_{m,p}$ has the following properties.*

(i) $\max\limits_{0\leq x\leq 1} H_{m,p}(x) = H_{m,p}(r_m) = \begin{cases} 1, & m=1; \\ \left(\frac{2p}{m-1+2p}\right)^p \left(\frac{m-1}{m-1+2p}\right)^{(m-1)/2}, & m\geq 2, \end{cases}$

and

$$\lim_{m\to\infty} m^p \max_{x\in[0,1]} H_{m,p}(x) = (\frac{2p}{e})^p.$$

(ii) $H_{m,p}(x)$ *is decreasing on* $[r_m, r_{m+1}]$, *and*

$$\lim_{m \to \infty} m^p \min_{x \in [r_m, r_{m+1}]} H_{m,p}(x) = (\frac{2p}{e})^p.$$

Combining Lemma 3.1 with Theorem 6.1 in [2], we obtain the lemma below.

Lemma 2.2. *Let* $p > 1/2$, $q > 0$. *Suppose that* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *is a holomorphic self-map of* $\mathbb{B}^n$. *Then* $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ *is bounded if and only if*

$$\sup_{z \in \mathbb{B}^n} \frac{(1 - |z|^2)^q \{(1 - |\varphi(z)|^2)|\varphi'(z)z|^2 + |\langle \varphi'(z)z, \varphi(z)\rangle|^2\}^{1/2}}{(1 - |\varphi(z)|^2)^p} < \infty.$$

Due to Lemma 3.1 and Theorem 7.4 in [2], the next lemma follows.

Lemma 2.3. *Let* $p, q > 0$. *Suppose that* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *is a holomorphic self-map of* $\mathbb{B}^n$, *and* $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ *is bounded. Then the following statements are true*:

(i) *If* $0 < p < 1/2$, *then* $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ *is compact if and only if*

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^q |\langle \varphi'(z)z, \varphi(z)\rangle|}{(1 - |\varphi(z)|^2)^p} = 0;$$

(ii) *If* $p > 1/2$, *then* $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ *is compact if and only if*

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^q \{(1 - |\varphi(z)|^2)|\varphi'(z)z|^2 + |\langle \varphi'(z)z, \varphi(z)\rangle|^2\}^{1/2}}{(1 - |\varphi(z)|^2)^p} = 0.$$

Lemma 2.4. *Let* $p > 0$, $m \in \mathbb{N}$, $\xi \in \partial \mathbb{B}^n$. *Then*

$$\lim_{m \to \infty} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle z, \xi \rangle^m\|_p \le (\frac{2p}{e})^p.$$

Proof. Note that for any $\xi \in \partial \mathbb{B}^n$,

$$\sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle z, \xi \rangle^m\|_p = \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^p |\Re(\langle z, \xi \rangle^m)|$$

$$= \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \sup_{z \in \mathbb{B}^n} m(1 - |z|^2)^p |\langle z, \xi \rangle^{m-1}| \cdot |\langle z, \xi \rangle|$$

$$\le \sup_{z \in \mathbb{B}^n} m^p (1 - |z|^2)^p |z|^{m-1}.$$

Using (i) of Lemma 2.1, the lemma follows immediately. $\square$

The following lemma is the crucial criterion for the compactness of C_φ, whose proof is similar as that of Proposition 3.11 in [1].

Lemma 2.5. *Assume that* φ *is a holomorphic self-map of* $\mathbb{B}^n$. *Then* $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ *is compact if and only if* C_φ *is bounded and for any bounded sequence* $\{f_m\}_{m \in \mathbb{N}}$ *in* $\mathcal{B}^p$ *which converges to zero uniformly on compact subsets of* $\mathbb{B}^n$, *we have*

$$\|C_\varphi f_m\|_q \to 0$$

as $m \to \infty$.

3. Main theorems

In this section, we will characterize the boundedness and compactness of composition operators between Bloch type spaces in terms of the power of the components of φ.

Theorem 3.1. *Let $p, q > 0$. Suppose that $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}^n$. Then $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ is bounded if and only if*

$$\sup_{m \in \mathbb{N}} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q < \infty.$$

Proof. Suppose that C_φ is bounded. And for $m \in \mathbb{N}$, $\xi \in \partial \mathbb{B}^n$, consider the test functions $g_{m,\xi}(z) = \frac{\langle z, \xi \rangle^m}{\|\langle z, \xi \rangle^m\|_p}$.

By Lemma 2.4, there is a constant $C > 0$, independent of m and ξ, such that $\|\langle z, \xi \rangle^m\|_p \leq C m^{1-p}$, therefore,

$$\infty > \|C_\varphi\| \geq \|C_\varphi g_{m,\xi}\|_q = \frac{\|\langle \varphi, \xi \rangle^m\|_q}{\|\langle z, \xi \rangle^m\|_p} \geq \frac{1}{C} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q.$$

Thus

$$\sup_{m \in \mathbb{N}} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q \leq C \|C_\varphi\| < \infty.$$

Conversely, for any integer $m \geq 1$, let

$$A_m = \{z \in \mathbb{B}^n : r_m \leq |\varphi(z)| \leq r_{m+1}\},$$

where r_m is defined in Lemma 2.1. Let k be the largest positive integer such that $A_m \neq \emptyset$ (k could be ∞). Thus $\mathbb{B}^n = \bigcup_{m=1}^{k} A_m$. From (ii) of Lemma 2.1, we know that, there exists a large enough integer N and a constant $\delta_1 > 0$, such that

$$\min_{z \in A_m} m^p |\varphi(z)|^{m-1}(1 - |\varphi(z)|^2)^p \geq \delta_1$$

whenever $m > N$. Noticing that also by Lemma 2.1,

$$\min_{z \in A_m} m^p |\varphi(z)|^{m-1}(1 - |\varphi(z)|^2)^p \geq m^p H_{m,p}(r_{m+1}),$$

set $\delta_2 = \min_{1 \leq m \leq N} \{m^p H_{m,p}(r_{m+1})\}$ and denote $\delta = \min\{\delta_1, \delta_2\}$. Therefore, there exists a constant $\delta > 0$, independent of m, such that

$$\min_{z \in A_m} m^p |\varphi(z)|^{m-1}(1 - |\varphi(z)|^2)^p \geq \delta.$$

Hence, we have

$$\begin{aligned}
(3.1) \quad &\sup_{z \in \mathbb{B}^n} \frac{(1 - |z|^2)^q |\langle \varphi'(z)z, \varphi(z) \rangle|}{(1 - |\varphi(z)|^2)^p} \\
&\leq \sup_{m \geq 1} \sup_{z \in A_m} \frac{m^p |\varphi(z)|^{m-1}(1 - |z|^2)^q |\langle \varphi'(z)z, \varphi(z) \rangle|}{m^p |\varphi(z)|^{m-1}(1 - |\varphi(z)|^2)^p} \\
&\leq \frac{1}{\delta} \sup_{m \geq 1} \sup_{z \in A_m} m^p |\varphi(z)|^{m-1}(1 - |z|^2)^q |\langle \varphi'(z)z, \varphi(z) \rangle|
\end{aligned}$$

$$\leq \frac{1}{\delta} \sup_{m\geq 1} \sup_{z\in A_m} m^p |\varphi(z)|^{m-2}(1-|z|^2)^q |\langle \varphi'(z)z, \varphi(z)\rangle|$$

$$= \frac{1}{\delta} \sup_{m\geq 1} \sup_{z\in A_m} m^p (1-|z|^2)^q \left|\langle \varphi(z), \frac{\varphi(z)}{|\varphi(z)|}\rangle^{m-1}\right| \cdot \left|\langle \varphi'(z)z, \frac{\varphi(z)}{|\varphi(z)|}\rangle\right|$$

$$\leq \frac{1}{\delta} \sup_{m\geq 1} \sup_{\xi\in\partial\mathbb{B}^n} m^{p-1} \sup_{z\in\mathbb{B}^n} m(1-|z|^2)^q |\langle \varphi(z),\xi\rangle^{m-1}| \cdot |\langle\varphi'(z)z,\xi\rangle|$$

$$\leq \frac{1}{\delta} \sup_{m\geq 1} \sup_{\xi\in\partial\mathbb{B}^n} m^{p-1} \sup_{z\in\mathbb{B}^n} (1-|z|^2)^q |\Re(\langle\varphi(z),\xi\rangle^m)|$$

$$\leq \frac{1}{\delta} \sup_{m\geq 1} \sup_{\xi\in\partial\mathbb{B}^n} m^{p-1} \|\langle\varphi,\xi\rangle^m\|_q < \infty.$$

Case 1. Whenever $|\varphi(z)| \leq 1/2$, for $i \in \{1,2,\ldots,n\}$, set $\xi = e_i = (0,\ldots,1,\ldots,0)$, where the i-th component is 1, otherwise 0. Then by the assumption, $\|\langle\varphi(z),e_i\rangle\|_q = \|\varphi_i\|_q = |\varphi(0)| + \sup_{z\in\mathbb{B}^n}(1-|z|^2)^q|\Re\varphi_i(z)| < \infty$. Therefore,

$$\frac{(1-|z|^2)^q\{(1-|\varphi(z)|^2)|\varphi'(z)z|^2\}^{1/2}}{(1-|\varphi(z)|^2)^p}$$

$$\leq (\frac{4}{3})^p(1-|z|^2)^q\{|\varphi'(z)z|^2\}^{1/2}$$

$$\leq (\frac{4}{3})^p(1-|z|^2)^q\{|\Re\varphi_1(z)|^2 + |\Re\varphi_2(z)|^2 \cdots + |\Re\varphi_n(z)|^2\}^{1/2} < \infty.$$

Case 2. If $|\varphi(z)| > 1/2$, by the projection theorem, there exists $\eta(z) \in \partial\mathbb{B}^n$, such that $\langle\varphi(z),\eta(z)\rangle = 0$ and

$$\varphi'(z)z = u\varphi(z) + v\eta(z),$$

where

$$u = \frac{\langle\varphi'(z)z, \varphi(z)\rangle}{|\varphi(z)|}, \ v = \langle\varphi'(z)z, \eta(z)\rangle.$$

Therefore,

$$\frac{(1-|z|^2)^q\{(1-|\varphi(z)|^2)|\varphi'(z)z|^2\}^{1/2}}{(1-|\varphi(z)|^2)^p}$$

$$= \frac{(1-|z|^2)^q\{(1-|\varphi(z)|^2)(|u|^2+|v|^2)\}^{1/2}}{(1-|\varphi(z)|^2)^p}$$

$$\leq \frac{2(1-|z|^2)^q|u|(1-|\varphi(z)|^2)^{1/2}}{(1-|\varphi(z)|^2)^p} + \frac{2(1-|z|^2)^q|v|(1-|\varphi(z)|^2)^{1/2}}{(1-|\varphi(z)|^2)^p}$$

$$(3.2) \qquad = I_1 + I_2.$$

Obviously, by (3.1), we have $I_1 < \infty$. Now let $\zeta = \varphi(z) + \sqrt{1-|\varphi(z)|^2}\eta(z)$, then $|\zeta| = 1$, and

$$\langle\varphi(z),\zeta\rangle = |\varphi(z)|^2, \ \langle\varphi'(z)z,\zeta\rangle = \langle\varphi'(z)z,\varphi(z)\rangle + \sqrt{1-|\varphi(z)|^2}v.$$

Then we obtain that

$$\sup_{m \geq 1} m^{p-1} \|\langle \varphi, \zeta \rangle^m\|_q$$

$$\geq \sup_{m \geq 1} m^{p-1} \sup_{z \in \mathbb{B}^n} m(1 - |z|^2)^q |\langle \varphi(z), \zeta \rangle^{m-1}| \cdot |\langle \varphi'(z)z, \zeta \rangle|$$

$$= \sup_{m \geq 1} m^{p-1} \sup_{z \in \mathbb{B}^n} m(1 - |z|^2)^q |\varphi(z)|^{2m-2} \cdot |\langle \varphi'(z)z, \zeta \rangle|.$$

Similar arguments as for (3.1), it is easy to show

$$\sup_{z \in \mathbb{B}^n} \frac{(1 - |z|^2)^q |\langle \varphi'(z)z, \zeta \rangle|}{(1 - |\varphi(z)|^2)^p} < \infty.$$

Note that

$$(3.3) \qquad |v|\sqrt{1 - |\varphi(z)|^2} \leq |\langle \varphi'(z)z, \zeta \rangle| + \langle \varphi'(z)z, \varphi(z) \rangle,$$

so $I_2 < \infty$. Combining the two cases above, we have actually show that

$$\sup_{z \in \mathbb{B}^n} \frac{(1 - |z|^2)^q \{(1 - |\varphi(z)|^2)|\varphi'(z)z|^2\}^{1/2}}{(1 - |\varphi(z)|^2)^p} < \infty.$$

The theorem follows by Lemma 2.2. □

Theorem 3.2. *Let $p, q > 0$, $p \neq 1/2$. Suppose that $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a holomorphic self-map of $\mathbb{B}^n$. Then $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{m \to \infty} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q = 0.$$

Proof. We show the necessity first. For $m \in \mathbb{N}$ and $\xi \in \partial \mathbb{B}^n$, consider the test functions $g_{m,\xi}(z) = \frac{\langle z, \xi \rangle^m}{\|\langle z, \xi \rangle^m\|_p}$, then it is easy to see that $\|g_{m,\xi}\|_p = 1$ and $g_{m,\xi}(z) \to 0$ uniformly on compact subsets of $\mathbb{B}^n$, as $m \to \infty$. If $C_\varphi : \mathcal{B}^p \to \mathcal{B}^q$ is compact, by Lemmas 2.4 and 2.5, we have

$$0 = \lim_{m \to \infty} \|C_\varphi g_{m,\xi}\|_q = \lim_{m \to \infty} \frac{\|\langle \varphi, \xi \rangle^m\|_q}{\|\langle z, \xi \rangle^m\|_p}$$

$$= \lim_{m \to \infty} \frac{m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q}{m^{p-1} \|\langle z, \xi \rangle^m\|_p}$$

$$\geq (\frac{e}{2p})^p \lim_{m \to \infty} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q.$$

Since ξ is arbitrary, the necessity follows.

Now we turn to prove the sufficiency. Assume that $m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q \to 0$ uniformly on $\partial \mathbb{B}^n$. For each integer $m \geq 1$, let A_m be as in Theorem 3.1, similar as discussed above, then we have

$$0 = \lim_{m \to \infty} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \|\langle \varphi, \xi \rangle^m\|_q$$

$$\geq \lim_{m \to \infty} \sup_{\xi \in \partial \mathbb{B}^n} m^{p-1} \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^q |\Re(\langle \varphi(z), \xi \rangle^m)|$$

$$= \lim_{m\to\infty} \sup_{\xi\in\partial\mathbb{B}^n} \sup_{z\in\mathbb{B}^n} m^p(1-|z|^2)^q |\langle\varphi(z),\xi\rangle^{m-1}| \cdot |\langle\varphi'(z)z,\xi\rangle|$$

$$\geq \lim_{m\to\infty} \sup_{z\in A_m} m^p(1-|z|^2)^q \left|\langle\varphi(z),\frac{\varphi(z)}{|\varphi(z)|}\rangle^{m-1}\right| \cdot \left|\langle\varphi'(z)z,\frac{\varphi(z)}{|\varphi(z)|}\rangle\right|$$

$$= \lim_{m\to\infty} \sup_{z\in A_m} m^p(1-|z|^2)^q |\varphi(z)|^{m-2} |\langle\varphi'(z)z,\varphi(z)\rangle|$$

$$\geq \lim_{m\to\infty} \sup_{z\in A_m} \frac{(1-|z|^2)^q m^p(1-|\varphi(z)|^2)^p |\varphi(z)|^{m-1}}{(1-|\varphi(z)|^2)^p} |\langle\varphi'(z)z,\varphi(z)\rangle|$$

$$\geq (\frac{2p}{e})^p \lim_{m\to\infty} \sup_{z\in A_m} \frac{(1-|z|^2)^q |\langle\varphi'(z)z,\varphi(z)\rangle|}{(1-|\varphi(z)|^2)^p}$$

$$(3.4) \qquad \geq (\frac{2p}{e})^p \lim_{|z|\to 1} \frac{(1-|z|^2)^q |\langle\varphi'(z)z,\varphi(z)\rangle|}{(1-|\varphi(z)|^2)^p}.$$

When $0 < p < 1/2$, the theorem follows by (i) of Lemma 2.3.

To prove the theorem for the case $p > 1/2$, by (ii) of Lemma 2.3, we just need to show that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^q \{(1-|\varphi(z)|^2)|\varphi'(z)z|^2\}^{1/2}}{(1-|\varphi(z)|^2)^p} = 0.$$

The same as proof in (3.2), we have

$$\frac{(1-|z|^2)^q \{(1-|\varphi(z)|^2)|\varphi'(z)z|^2\}^{1/2}}{(1-|\varphi(z)|^2)^p} \leq I_1 + I_2,$$

where I_1, I_2 are defined the same as in the previous theorem. Next we will show that $I_1 \to 0, I_2 \to 0$ as $|\varphi(z)| \to 1$. From (3.4), it is obvious that $I_1 \to 0$, as $|\varphi(z)| \to 1$. Now let $\zeta = \varphi(z) + \sqrt{1-|\varphi(z)|^2}\eta(z)$, where $\eta(z)$ is defined as in Theorem 3.1. Noticing that (3.1) and Lemma 2.1, similar arguments show that

$$\lim_{m\to\infty} m^{p-1} \|\langle\varphi,\zeta\rangle^m\|_q$$

$$\geq \lim_{m\to\infty} m^{p-1} \sup_{z\in\mathbb{B}^n} m(1-|z|^2)^q |\langle\varphi(z),\zeta\rangle^{m-1}| \cdot |\langle\varphi'(z)z,\zeta\rangle|$$

$$= \lim_{m\to\infty} m^{p-1} \sup_{z\in\mathbb{B}^n} m(1-|z|^2)^q |\varphi(z)|^{2m-2} \cdot |\langle\varphi'(z)z,\zeta\rangle|$$

and

$$(3.5) \qquad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^q |\langle\varphi'(z)z,\zeta\rangle|}{(1-|\varphi(z)|^2)^p} = 0.$$

Combining (3.4) with (3.5), again using (3.3), we obtain that $I_2 \to 0$ as $|\varphi(z)| \to 1$. This completes the proof of the theorem. $\qquad\square$

For technical reasons we can't deal with the case $p = 1/2$, so we pose the following question.

Question. What about the case $p = 1/2$, is it compatible with Theorem 3.2?

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