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AN ESTIMATE OF HEMPEL DISTANCE FOR BRIDGE SPHERES

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Abstract. Tomova [8] gave an upper bound for the distance of a bridge surface for a knot with two different bridge positions in a 3-manifold. In this paper, we show that the result of Tomova [8, Theorem 10.3] can be improved in the case when there are two different bridge spheres for a link in S^3.

1. Introduction

Hempel [4] introduced the concept of *distance* of a Heegaard surface, and it is shown by many authors that it well represents various complexities of 3-manifolds. For example, Hartshorn [2] showed that the Euler characteristic of an incompressible surface in a 3-manifold bounds the distance of its Heegaard splittings, and Scharlemann and Tomova [7] showed that the Euler characteristic of any Heegaard splitting of a 3-manifold similarly bounds the distance of any non-isotopic Heegaard splitting.

The above concept and results have been extended to *bridge surfaces* for knots and links in closed 3-manifolds, and have been studied by several authors. For example, Bachman and Schleimer [1] proved that Hartshorn's results can be extended to the distance of a bridge surface for a knot in a closed orientable 3-manifold, and also Tomova [8] proved that Scharlemann and Tomova's results can be extended to the distance of a bridge surface for a knot in a closed orientable 3-manifold. Moreover, Johnson and Tomova [6] proved that Tomova's result can be extended to a bridge surface for a tangle in a compact 3-manifold. Recently Jang [5] showed that for a link in a closed orientable 3-manifold, the result of Bachman and Schleimer [1] can be improved in the case when there exist essential meridional spheres.

In this paper, we find a property of essential simple closed curves disjoint from the *disk complex* of a 3-ball containing trivial arcs (for detail, see Lemma 2.1). This allows us to improve the result of Tomova [8, Theorem 10.3] in the case when there are two different bridge spheres for a link in S^3.

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Theorem 1.1. *Suppose L is a link in S^3 and P is a bridge sphere for L with $|P \cap L| \geq 6$. If Q is another bridge sphere for L such that Q is not equivalent to P, then $d(P, L) \leq |Q \cap L| - 2$, where $d(P, L)$ denotes the distance of the bridge sphere P.*

By the above theorem, we can improve a result of Tomova [8, Corollary 10.7] in the case of bridge sphere.

Corollary 1.2. *If P is a bridge sphere for a link L such that*

$$d(P, L) > |P \cap L| - 2,$$

then the minimal bridge sphere for L is unique up to isotopy transverse to $P \cap L$.

2. Definitions and notations

Let M be a closed orientable 3-manifold, γ a union of mutually disjoint arcs or simple closed curves properly embedded in M, F a surface embedded in M, which is in general position with respect to γ. A surface D in M is a γ-*disk*, if D is a disk intersecting γ in at most one transverse point. Let $\ell(\subset F)$ be a simple closed curve with $\ell \cap \gamma = \emptyset$. We say that ℓ is γ-*inessential* if ℓ bounds a γ-disk in F, and ℓ is γ-*essential* if it is not γ-inessential. We say that a surface D is a γ-*compressing disk* for F if; D is a γ-disk, and $D \cap F = \partial D$, and ∂D is a γ-essential simple closed curve in F. Let F_1, F_2 be surfaces in M which are in general position with respect to γ. We say that F_1 and F_2 are γ-*parallel* if they co-bound a 3-manifold homeomorphic to $F_1 \times [0, 1]$ intersecting γ in vertical arcs, where $F_1 = F_1 \times \{0\}$ and $F_2 = F_1 \times \{1\}$. We say that F_1 and F_2 are γ-*isotopic* if there exists an isotopy from F_1 to F_2 so that F_1 remains transverse to γ throughout the isotopy.

2.1. Handlebodies containing trivial arcs

Let H be a genus-$g(\geq 0)$ handlebody. If $g > 1$, *spine* Σ_H of H is a 1-complex contained in the interior of H, which is a strong deformation retract of H, where each vertex of Σ_H has valence three. Note that for a genus-0 handlebody (the 3-ball), we let the spine be a point in the interior of the 3-ball, and for a genus-1 handlebody (solid torus), we let the spine be a core circle of the solid torus. We say that a set of n arcs $\{t_1, \ldots, t_n\}$ properly in embedded in H is a *set of trivial n arcs* if $t_1 \cup \cdots \cup t_n$ is parallel to ∂H. Let H be a handlebody and $\tau = \{t_1, \ldots, t_n\}$ a set of trivial n arcs in H. Then τ can be isotoped in H so that the projection from $\partial H \times [0, 1)$ to $[0, 1)$ has exactly one critical point in each t_i. For the pair (H, τ), we let the spine $\Sigma_{(H,\tau)}$ be the union of Σ_H together with a collection of vertical arcs $\alpha_1, \ldots, \alpha_n$, where one endpoint of each α_i lies on the critical point of t_i, and the other endpoint lies on Σ_H.

2.2. Bridge decompositions

It is well known that every closed orientable 3-manifold M has a genus-g Heegaard splitting for some $g(\geq 0)$, i.e., $M = A \cup_P B$, where A and B are genus-g handlebodies in M such that $M = A \cup B$ and $A \cap B = \partial A = \partial B = P$. Let L be a link in M. We say that $(A, \tau_A) \cup_P (B, \tau_B)$ is a (g, n)-*bridge decomposition* (or *bridge decomposition* for short) for the pair (M, L) if P separates (M, L) into two components (A, τ_A) and (B, τ_B) where $\tau_A = L \cap A$ (resp. $\tau_B = L \cap B$) is a set of trivial n arcs in A (resp. B). Then we say that P is a (g, n)-*bridge surface* (or a *bridge surface* for short). It is known that each (M, L) has a (g, n)-bridge decomposition for some g and n. (For a detailed discussion, see [3, Lemma 2.1].)

Given a (g, n)-bridge decomposition $(A, \tau_A) \cup_P (B, \tau_B)$ for (M, L), there are three ways to create new bridge surfaces for (M, L): (1) adding dual one-handles disjoint from L (*stabilizing*), (2) adding dual one-handles where one of them has an arc of L as its core (*meridionally stabilizing*), and (3) introducing a pair of a canceling minimum and maximum for L (*perturbing*) (for details, see [8, Figure 15]). We say that another bridge surface Q is *equivalent* to P if Q is L-isotopic to a copy of P which may have been stabilized, meridionally stabilized and perturbed.

2.3. Sweep-outs

Let L be a link in a closed orientable 3-manifold M. Suppose $(A, \tau_A) \cup_P (B, \tau_B)$ is a bridge decomposition for (M, L). From the definition of a spine, one can construct a map $f : M \to [-1, 1]$ such that $f^{-1}(-1)$ is a spine of (A, τ_A), $f^{-1}(1)$ is a spine of (B, τ_B) and $f^{-1}(s)$ is a surface which is L-parallel to the bridge surface P for each $s \in (-1, 1)$. This map is called a *sweep-out* induced from $(A, \tau_A) \cup_P (B, \tau_B)$. For each $s \in (-1, 1)$, we put $P_s = f^{-1}(s)$, $A_s = f^{-1}([-1, s])$ and $B_s = f^{-1}([s, 1])$.

2.4. Curve complexes

Let S be a compact orientable surface with genus g and p punctures. The *curve complex* $\mathcal{C}(S)$ is defined as follows: the vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and a collections of $k+1$ vertices form a k-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in S. For two vertices x, y of $\mathcal{C}(S)$, we define the *distance* $d(x, y)$ between x and y as the minimal number of 1-simplexes of a simplicial path in $\mathcal{C}(S)$ joining x and y. Let X, Y be subsets of the vertices of $\mathcal{C}(S)$. Then we define $d(X, Y) = \min\{d(x, y) \mid x \in X, y \in Y\}$.

Let H be a handlebody and τ a set of trivial n arcs in H. Then $\mathcal{D}(H \setminus \tau)$ denotes the subset of $\mathcal{C}(\partial H \setminus \partial\tau)$ consisting of the vertices with representatives bounding disks in $H \setminus \tau$. Suppose that M is a closed orientable 3-manifold containing a link L, and $(A, \tau_A) \cup_P (B, \tau_B)$ is a bridge decomposition for (M, L).

Then the *distance* $d(P, L)$ of $(A, \tau_A) \cup_P (B, \tau_B)$ is defined by $d(\mathcal{D}(A \setminus \tau_A), \mathcal{D}(B \setminus \tau_B))$.

Let B^3 be a 3-ball, and τ a set of trivial n arcs in B^3 with $n \geq 3$.

Lemma 2.1. *Let D be a γ-compressing disk in B^3. Then for a τ-essential simple closed curve $\ell \subset \partial(B^3 \setminus \partial\tau)$ which is disjoint from ∂D, $d(\mathcal{D}(B^3 \setminus \tau), \ell) \leq 1$.*

Proof. If $D \cap \tau = \emptyset$, then clearly Lemma 2.1 holds. Suppose $D \cap \tau$ consists of one point. Then D separates (B^3, τ) into two components (B_1^3, τ_1) and (B_2^3, τ_2) where $\ell \subset \partial B_1^3 \setminus \partial\tau_1$. Since D is a τ-compressing disk, τ_2 consists of at least two components. Hence there is a compressing disk $D'(\subset B_2^3)$ for $\partial B_2^3 \setminus \partial\tau_2$. Further, by τ_2-isotopy of B_2^3, we may suppose that D' is disjoint from the image of D. Hence we may regard D' a compressing disk in $B^3 \setminus \tau$. Since ℓ is disjoint from D', we have $d(\partial D', \ell) \leq 1$, which implies $d(\mathcal{D}(B^3 \setminus \tau), \ell) \leq 1$ □

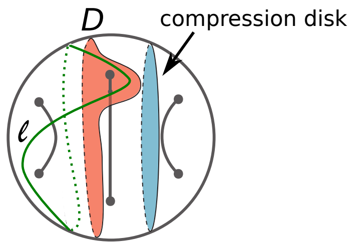

FIGURE 1

3. Proof of main results

Let L be a link in S^3. Let $(A, \tau_A) \cup_P (B, \tau_B)$ be a $(0, n_1)$-bridge decomposition ($n_1 \geq 3$) for (S^3, L), and f a sweep-out induced from $(A, \tau_A) \cup_P (B, \tau_B)$. Let π_P be the projection map from $P \times (-1, 1)$ to P.

Suppose that Q is a $(0, n_2)$-bridge surface ($n_2 \geq 3$) for (S^3, L) which is not equivalent to P. Then, by [6, Theorem 3.1 and Lemma 4.3], Q can be L-isotoped so that $f|_Q$ is Morse, and for every $s \in (-1, 1)$, $P_s \cap Q$ contains a curve that is L-essential in P_s. Moreover, as in the proof of [6, Theorem 4.2], either $d(P, L) \leq 1$ or there exists an interval $[s_-, s_+]$, where $s_- < s_+$ are critical values for $f|_Q$ such that

(1) for every $s \in (s_-, s_+)$, each component of $P_s \cap Q$ which is L-essential in P_s does not bound a disk in Q, and
(2) for a small ϵ, $P_{s_- - \epsilon} \cap Q$ contains a curve that is L-essential in $P_{s_- - \epsilon}$ and bounds a disk in $A_{s_- - \epsilon}$, and $P_{s_+ + \epsilon} \cap Q$ contains a curve that is L-essential in $P_{s_+ + \epsilon}$ and bounds a disk in $B_{s_+ + \epsilon}$.

Hence we have:

(∗) if $d(P, L) \geq 2$, there exists an interval $[s_-, s_+]$ satisfying the conditions (1) and (2).

We show that in (∗), the conclusion can be improved if $d(P, L) > 2$. Namely:

Lemma 3.1. *Let Q be as above. If $d(P, L) > 2$, there exists a subinterval $[s'_-, s'_+] \subset [s_-, s_+]$, where $s'_- < s'_+$ are critical values for $f|_Q$ such that*

(i) *for every $s' \in (s'_-, s'_+)$, each component of $P_{s'} \cap Q$ which is L-essential in $P_{s'}$ does not bound an L-disk in Q, and*

(ii) *for a small ϵ, $P_{s'_- - \epsilon} \cap Q$ contains a curve that is L-essential in $P_{s'_- -\epsilon}$ and bounds an L-disk in $A_{s'_- -\epsilon}$, and $P_{s'_+ +\epsilon} \cap Q$ contains a curve that is L-essential in $P_{s'_+ +\epsilon}$ and bounds an L-disk in $B_{s'_+ +\epsilon}$.*

Proof. Suppose, for a contradiction, that for every $s \in (s_-, s_+)$, there exists a component of $P_s \cap Q$ which is L-essential in P_s and bounds a L-disk in Q. Note that a disk in Q is an L-disk. Hence by the above condition (1), we obtain that there exists a critical value s^* (possibly $s^* = s_-$ or s_+) such that for a small ϵ, $P_{s^*-\epsilon} \cap Q$ contains a curve bounding a L-disk D_A in $A_{s^*-\epsilon}$, and $P_{s^*+\epsilon} \cap Q$ contains a curve bounding a L-disk D_B in $B_{s^*+\epsilon}$. Note that $d(\pi_P(\partial D_A), \pi_P(\partial D_B)) \leq 1$. (Recall that π_P be the projection map from $P \times (-1, 1)$ to P.) Hence by regarding D_A as D and $\pi_P(\partial D_B)$ as the ℓ in Lemma 2.1, we have $d(\mathcal{D}(A \setminus \tau_A), \pi_P(\partial D_B)) \leq 1$. Analogously we have $d(\mathcal{D}(B \setminus \tau_B), \pi(\partial D_B)) \leq 1$. These together with a triangle inequality $d_P(\mathcal{D}(A \setminus \tau_A), \mathcal{D}(B \setminus \tau_B)) \leq d(\mathcal{D}(A \setminus \tau_A), \pi_P(\partial D_B)) + d(\pi_P(\partial D_B), \mathcal{D}(B \setminus \tau_B))$ imply that $d(P, L) \leq 2$, a contradiction. Hence, there exists a component of $P_{s'} \cap Q$ which is L-essential in $P_{s'}$ and does not bound a L-disk in Q. It is easy to see that this implies Lemma 3.1(i) holds. The conclusion (ii) follows from the conclusion (i) and the above condition (1). □

Proof of Theorem 1.1. By Lemma 3.1, either (I)$d(P, L) \leq 2$ or (II)there exists an interval $[s'_-, s'_+]$, where $s'_- < s'_+$ are critical values for $f|_Q$ satisfying (i) and (ii) of Lemma 3.1. If $d(P, L) \leq 2$, then since $n_2 \leq 3$, the conclusion of Theorem 1.1 holds. Hence we consider the case (II). Let C be the union of the components of $P_{s'_- +\epsilon} \cap Q$ and $P_{s'_+ -\epsilon} \cap Q$ which are L-essential on Q. Since Q is connected, there is a component, say Q', of $Q \setminus C$ such that $\partial Q' \cap P_{s'_- +\epsilon} \neq \emptyset$ and $\partial Q' \cap P_{s'_+ -\epsilon} \neq \emptyset$. Note that each component of $\partial Q' \cap P_{s'_- +\epsilon}$ and $\partial Q' \cap P_{s'_+ -\epsilon}$ bounds an at least twice punctured disk in $(Q \setminus Q') \setminus L$ because each component of $\partial Q' \cap P_{s'_- +\epsilon}$ and $\partial Q' \cap P_{s'_+ -\epsilon}$ is L-essential on Q. Hence $\chi(Q' \setminus L) \geq \chi(Q \setminus L) + 2$. Let c_- (resp. c_+) be a component of $cl(Q') \cap P_{s'_- +\epsilon}$ (resp. $cl(Q') \cap P_{s'_+ -\epsilon}$). Hence by using arguments as in the proof of [6, Theorem 4.2], $d(\pi_P(c_-), \pi_P(c_+)) \leq -\chi(Q' \setminus L)$. By (ii) of Lemma 3.1, $\pi_P(c_-)$ (resp. $\pi_P(c_+)$) is disjoint from an L-compressing disk in A (resp. B). By Lemma 2.1,

$d(\mathcal{D}(A \setminus \tau_A), \pi_P(c_-)) \leq 1$ and $d(\pi_P(c_+), \mathcal{D}(B \setminus \tau_B)) \leq 1$. Hence, we have

$$\begin{aligned} d(P, L) &\leq d(\mathcal{D}(A \setminus \tau_A), \pi_P(c_-)) + d(\pi_P(c_-), \pi_P(c_+)) + d(\pi(c_+), \mathcal{D}(B \setminus \tau_B)) \\ &\leq 1 - \chi(Q' \setminus L) + 1 \\ &= -\chi(Q \setminus L). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let Q be a minimal bridge sphere for a link L. Suppose that Q not equivalent to P. Then, by Theorem 1.1, $d(P, L) \leq |Q \cap L| - 2 = |P \cap L| - 2$, a contradiction. □

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