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GENERALIZATION OF INEQUALITIES ANALOGOUS TO HERMITE–HADAMARD INEQUALITY VIA FRACTIONAL INTEGRALS

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Abstract. Some Hermite–Hadamard type inequalities for the fractional integrals are established and these results have some relationship with the obtained results of [11, 12].

1. Introduction

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. One of the most famous inequalities for convex functions is Hermite–Hadamard inequality, stated as [8]:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave.

In recent years, numerous generalizations, extensions and variants of Hermite–Hadamard inequality (1) were studied extensively by many researchers and appeared in a number of papers, see [8, 10, 11, 12, 13].

Now, some necessary definitions and mathematical preliminaries of fractional calculus theory are presented, which are used further in this paper.

Definition 1 ([9])**.** Let $f \in L^1[a,b]$. The left–sided and right–sided Riemann–Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad a < x$$

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and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b$$

respectively, where $\Gamma(\cdot)$ is Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du.$$

It is to be noted that $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Using Riemann–Liouville fractional integral, many authors have studied the fractional integral inequalities and applications. For example, we refer the reader to [1, 2, 3, 5, 6] and the references cited therein. For results connected with Hermite–Hadamard type inequalities involving fractional integrals one can see [4, 7, 14].

In [14] Sarikaya et al. proved a variant of Hermite–Hadamard's inequalities in fractional integral forms as follows:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold*:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$

Remark 1. For $\alpha = 1$, inequality (2) reduces to inequality (1).

The aim of this paper is to establish left Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral using the identity obtained for fractional integrals.

2. Main results

In order to obtain our results, we modified [11, Lemma 2.1] as following:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L^1[a, b]$, then the following identity for Riemann–Liouville fractional integrals holds*:

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \frac{b-a}{2} \sum_{k=1}^{4} I_k,$$

where

$$I_1 = \int_0^{1/2} t^\alpha f'(tb + (1-t)a)dt, \qquad I_2 = \int_0^{1/2} (-t^\alpha) f'(ta + (1-t)b)dt,$$

$$I_3 = \int_{1/2}^{1} (t^\alpha - 1) f'(tb + (1-t)a)dt, \quad I_4 = \int_{1/2}^{1} (1 - t^\alpha) f'(ta + (1-t)b)dt.$$

Proof. Integrating by parts

$$\begin{aligned} I_1 &= \int_0^{1/2} t^\alpha f'(tb+(1-t)a)dt \\ &= \frac{t^\alpha f(tb+(1-t)a)}{b-a}\bigg|_0^{1/2} - \frac{\alpha}{b-a}\int_0^{1/2} t^{\alpha-1} f(tb+(1-t)a)dt \\ &= \frac{2^{-\alpha}}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b-a}\int_0^{1/2} t^{\alpha-1} f(tb+(1-t)a)dt. \end{aligned}$$

Analogously:

$$I_2 = \frac{2^{-\alpha}}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b-a}\int_0^{1/2} t^{\alpha-1} f(ta+(1-t)b)dt$$

and

$$\begin{aligned} I_3 &= \int_{1/2}^{1} (t^\alpha - 1) f'(tb+(1-t)a)dt \\ &= \frac{(t^\alpha-1) f(tb+(1-t)a)}{b-a}\bigg|_{1/2}^{1} - \frac{\alpha}{b-a}\int_{1/2}^{1} t^{\alpha-1} f(tb+(1-t)a)dt \\ &= \frac{1-2^{-\alpha}}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b-a}\int_{1/2}^{1} t^{\alpha-1} f(tb+(1-t)a)dt. \end{aligned}$$

Analogously:

$$I_4 = \frac{1-2^{-\alpha}}{b-a} f\left(\frac{a+b}{2}\right) - \frac{\alpha}{b-a}\int_{1/2}^{1} t^{\alpha-1} f(ta+(1-t)b)dt.$$

Adding above equalities, we get

$$\begin{aligned} &\frac{2}{b-a} f\left(\frac{a+b}{2}\right) \\ &\quad - \frac{\alpha}{b-a}\left[\int_0^1 t^{\alpha-1} f(tb+(1-t)a)dt + \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt\right] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now making substitution $u = tb + (1-t)a$, we have

$$\begin{aligned} \int_0^1 t^{\alpha-1} f(tb+(1-t)a)dt &= \frac{1}{(b-a)^\alpha}\int_a^b (u-a)^{\alpha-1} f(u)du \\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a), \end{aligned}$$

likewise

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b),$$

which completes our proof. □

New upper bound for the left-hand side of (2) for convex functions is proposed in the following theorem.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for Riemann–Liouville fractional integrals holds for $0 < \alpha \leq 1$:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right]\right| \\ &\leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}(|f'(a)| + |f'(b)|). \end{aligned} \tag{3}$$

Proof. By using the properties of modulus on Lemma 1, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right]\right| \leq \frac{b-a}{2}\sum_{k=1}^{4}|I_k|.$$

Now, using convexity of $|f'|$, we have

$$\begin{aligned} |I_1| &\leq \int_0^{1/2} t^\alpha |f'(tb + (1-t)a)| dt \\ &\leq |f'(b)| \int_0^{1/2} t^{\alpha+1} dt + |f'(a)| \int_0^{1/2} t^\alpha (1-t) dt \\ &= \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)| + \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|. \end{aligned}$$

Analogously:

$$|I_2| \leq \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(a)| + \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)|.$$

By using the convexity on $|f'|$ and fact that for $\alpha \in (0, 1]$ and $\forall\ t_1, t_2 \in [0, 1]$,

$$|{t_1}^\alpha - {t_2}^\alpha| \leq |t_1 - t_2|^\alpha,$$

$$\begin{aligned} |I_3| &\leq |f'(b)| \int_{1/2}^{1} (1 - t^\alpha) t\ dt + |f'(a)| \int_{1/2}^{1} (1 - t^\alpha)(1-t)\ dt \\ &\leq |f'(b)| \int_{1/2}^{1} (1-t)^\alpha t\ dt + |f'(a)| \int_{1/2}^{1} (1-t)^{\alpha+1}\ dt \\ &= \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)| + \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(a)|, \end{aligned}$$

similarly

$$|I_4| \leq \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)| + \frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|,$$

which completes the proof. ∎

Remark 2. If we take $\alpha = 1$ in Theorem 2, then inequality (3) becomes inequality as obtained in [11, Theorem 2.2].

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L^1[a, b]$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ for some fixed $p \geq 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $0 < \alpha \leq 1$:*
(4)
$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right| \\ &\leq \frac{b-a}{2^{\alpha+1}(\alpha p+1)^{1/p}} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{1/q} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{1/q} \right]. \end{aligned}$$

Proof. From Lemma 1 and using Hölder inequality with properties of modulus, we have
$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \right| \leq \frac{b-a}{2} \sum_{k=1}^{4} |I_k|.$$
By using the convexity of $|f'|^q$, we have
$$\begin{aligned} |I_1| &\leq \left(\int_0^{1/2} t^{\alpha p} dt \right)^{1/p} \left(\int_0^{1/2} |f'(tb + (1-t)a)|^q \, dt \right)^{1/q} \\ &\leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(|f'(b)|^q \int_0^{1/2} t \, dt + |f'(a)|^q \int_0^{1/2} (1-t) dt \right)^{1/q} \\ &= \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(\frac{|f'(b)|^q + 3|f'(a)|^q}{8} \right)^{1/q}, \end{aligned}$$
similarly
$$|I_2| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(\frac{3|f'(b)|^q + |f'(a)|^q}{8} \right)^{1/q},$$
now
$$|I_3| \leq \left(\int_{1/2}^{1} (1-t^\alpha)^p dt \right)^{1/p} \left(\int_{1/2}^{1} |f'(tb + (1-t)a)|^q \, dt \right)^{1/q}.$$
Let $\alpha \in (0, 1]$ and $\forall \, t_1, t_2 \in [0, 1]$,
$$|{t_1}^\alpha - {t_2}^\alpha| \leq |t_1 - t_2|^\alpha,$$
therefore
$$\int_{1/2}^{1} (1-t^\alpha)^p dt \leq \int_{1/2}^{1} (1-t)^{\alpha p} dt = \frac{1}{2^{\alpha p+1}(\alpha p+1)}.$$

Hence

$$|I_3| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(b)|^q+|f'(a)|^q}{8}\right)^{1/q},$$

and

$$|I_4| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{1/q},$$

which completes the proof. □

Remark 3. If we take $\alpha = 1$ in Theorem 3, then inequality (4) becomes inequality (2.1) of [11, Theorem 2.3].

Another similar result may be extended in the following theorem.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right]\right|\\ &\leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\left[\left(\frac{(\alpha+1)|f'(b)|^q+(\alpha+3)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}\right.\\ &\qquad\left.+\left(\frac{(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}\right].\end{aligned} \tag{5}$$

Proof. Using the well-known power-mean integral inequality for $q > 1$ [12], we have

$$|I_1| \leq \left(\int_0^{1/2} t^\alpha dt\right)^{1-1/q}\left(\int_0^{1/2} t^\alpha \left|f'(tb+(1-t)a)\right|^q dt\right)^{1/q}.$$

By convexity of $|f'|^q$

$$\begin{aligned}|I_1| &\leq \left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-1/q}\left(\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|^q\right.\\ &\qquad\left.+\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|^q\right)^{1/q}\\ &= \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+1)|f'(b)|^q+(\alpha+3)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}.\end{aligned}$$

Analogously:

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}$$

$$|I_3| \leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left(\frac{(\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q}{2(\alpha+2)} \right)^{1/q}$$

and

$$|I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left(\frac{(\alpha+1)|f'(b)|^q + (\alpha+3)|f'(a)|^q}{2(\alpha+2)} \right)^{1/q}.$$

Combining all the obtained inequalities, we get desired inequality. Which completes the proof. □

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If the function $|f'|^q$ with $q > 1$ is convex on $[a,b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left(\frac{1+2^{1/q}}{3^{1/p}} \right) [|f'(a)| + |f'(b)|]. \tag{6}$$

Proof. If we take $\alpha = 1$ in Theorem 4, then inequality (5) becomes as:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{b-a}{8} \left[\left(\frac{2|f'(a)|^q + |f'(b)|^q}{3} \right)^{1/q} + \left(\frac{|f'(a)|^q + 2|f'(b)|^q}{3} \right)^{1/q} \right], \end{aligned}$$

which can be made equivalent to (6) by using the fact:

$$\sum_{i=1}^{n} (a_i + b_i)^r \leq \sum_{i=1}^{n} a_i{}^r + \sum_{i=1}^{n} b_i{}^r$$

for $0 \leq r < 1$, $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b_2, \ldots, b_n \geq 0$. □

Remark 4. Inequality (6) is an improvement of obtained inequality as in [12, Theorem 2.1].

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $p \geq 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \\ &\leq \frac{b-a}{2^{\alpha+1}} \left(\frac{1}{\alpha p+1} \right)^{1/p} \left[\left| f'\left(\frac{a+3b}{4}\right) \right| + \left| f'\left(\frac{3a+b}{4}\right) \right| \right]. \end{aligned} \tag{7}$$

Proof. From Lemma 1 and using Hölder inequality, we have

$$|I_1| \leq \left(\int_0^{1/2} t^{\alpha p} dt \right)^{1/p} \left(\int_0^{1/2} |f'(tb + (1-t)a)|^q \, dt \right)^{1/q}.$$

Since $|f'|^q$ is concave on $[a,b]$; we can use the integral Jensen's inequality to obtain

$$\begin{aligned}\int_0^{1/2} |f'(tb+(1-t)a)|^q\,dt &= \int_0^{1/2} t^0\,|f'(tb+(1-t)a)|^q\,dt \\ &\leq \left(\int_0^{1/2} t^0 dt\right)\left|f'\left(\frac{\int_0^{1/2} t^0(tb+(1-t)a)dt}{\int_0^{1/2} t^0 dt}\right)\right|^q \\ &= \frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q.\end{aligned}$$

Analogously:

$$\begin{aligned}\int_0^{1/2} |f'(ta+(1-t)b)|^q\,dt &\leq \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \\ \int_{1/2}^{1} |f'(tb+(1-t)a)|^q\,dt &\leq \frac{1}{2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \\ \int_0^{1/2} |f'(ta+(1-t)b)|^q\,dt &\leq \frac{1}{2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q.\end{aligned}$$

□

Remark 5. If we take $\alpha = 1$ in Theorem 5, then inequality (7) becomes inequality (2.5) of [10, Theorem 5].

In the following, we obtain estimate of Hermite–Hadamard inequality (2) for concave functions.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If $|f'|^q$ is concave on $[a,b]$ for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

(8)
$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \\ &\leq \frac{b-a}{2^{\alpha+1}(\alpha+1)}\left[\left|f'\left(\frac{(\alpha+1)a+(\alpha+3)b}{2(\alpha+2)}\right)\right| + \left|f'\left(\frac{(\alpha+3)a+(\alpha+1)b}{2(\alpha+2)}\right)\right|\right].\end{aligned}$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}|f'(tx+(1-t)y)|^q &> t|f'(x)|^q + (1-t)|f'(y)|^q \\ &\geq (t|f'(x)| + (1-t)|f'(y)|)^q.\end{aligned}$$

Hence

$$|f'(tx+(1-t)y)| \geq t|f'(x)| + (1-t)|f'(y)|,$$

so, $|f'|$ is also concave. By the Jensen integral inequality, we have

$$|I_1| \leq \left(\int_0^{1/2} t^\alpha dt\right)\left|f'\left(\frac{\int_0^{1/2} t^\alpha(tb+(1-t)a)dt}{\int_0^{1/2} t^\alpha dt}\right)\right|^q$$

$$= \frac{1}{2^{\alpha+1}(\alpha+1)} \left| f'\left(\frac{(\alpha+3)a+(\alpha+1)b}{2(\alpha+2)} \right) \right|^q .$$

Analogously:

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left| f'\left(\frac{(\alpha+1)a+(\alpha+3)b}{2(\alpha+2)} \right) \right|^q ,$$
$$|I_3| \leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left| f'\left(\frac{(\alpha+1)a+(\alpha+3)b}{2(\alpha+2)} \right) \right|^q ,$$
$$|I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left| f'\left(\frac{(\alpha+3)a+(\alpha+1)b}{2(\alpha+2)} \right) \right|^q ,$$

which completes the proof. □

Corollary 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L^1[a, b]$. If $p > 1$ with $q = \frac{p}{p-1}$ and $|f'|^q$ is concave on $[a, b]$, then*

$$(9) \qquad \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \leq \frac{b-a}{8} \left[\left| f'\left(\frac{a+2b}{3} \right) \right| + \left| f'\left(\frac{2a+b}{3} \right) \right| \right] .$$

Proof. If we take $\alpha = 1$ in Theorem 5, then inequality (7) becomes (9). □

Remark 6. Inequality (9) can be made equivalent to inequality as obtained in [12, Theorem 2.2] by assuming the linearity of $|f'|$.

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