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EXISTENCE OF MILD SOLUTIONS OF PARTIAL NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract. We study the existence of mild solutions of partial neutral integrodifferential equations with unbounded delay by using the fixed point criterion for condensing operators.

1. Introduction

In this paper, we investigate the existence of mild solutions for the partial neutral integrodifferential equation with unbounded delay described in the form

$$\begin{cases} \frac{dD(t,u_t)}{dt} = AD(t,u_t) + \int_0^t B(t-s)D(s,u_s)ds + g(t,u_t), \ 0 \le t \le a, \\ u(0) = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X, t \ge 0$, are closed linear operators; X is a Banach space; the history $x_t : (-\infty, 0] \to X$ defined by $x_t(\theta) = x(t+\theta)$, belongs to the abstract phase space by Hale and Kato: $D(t,\varphi) = \varphi(0) + f(t,\varphi)$ and $g : [0,a] \times \mathcal{B} \to X$ are appropriate functions.

For the description of heat conduction in materials with fading memory, we use the partial neutral integrodifferential equation with unbounded delay [3]. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the

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temperature in different type of materials. But this description is unsatisfatory in materials with fading memory. The next system has been frequently used to describe this phenomena:

$$\begin{cases} \frac{d}{dt}[c_1u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds] = c_2\Delta u(t,x) + \int_{-\infty}^{t} k_2(t-s)\Delta u(s,x)ds, \\ u(t,x) = 0,\ x \in \partial\Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ is open bounded with smooth boundary: $(t,x) \in \mathbb{R}^+ \times \Omega$; $u(t,x)$ represents the temperature in x at time t; c_1, c_2 are physical constants and $k_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$, are the internal energy and the heat flux relaxation, respectively. By assuming that the solution $u(\cdot)$ is known on $\mathbb{R}^-, k_1 = k_2$ and defining $B(t) = 0$ for $t \geq 0$, we can transform this system into the neutral system (1.1) [3].

2. Existence of mild solutions

Consider the integrodifferential abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds,\ t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

where $A, B(t), t \geq 0$, are closed linear operators defined on a common domain D which is dense in X. Assume that (2.1) has an associated resolvent operator $\{R(t)\}_{t\geq 0}$ on X.

Definition 2.1. A family of bounded linear operators $\{R(t)\}_{t\geq 0}$ is a *resolvent operator* for (2.1) if

(i) $R(0) = I$(the identity operator) and $R(\cdot)x \in C(\mathbb{R}^+, X)$ for every $x \in D(A)$.
(ii) For $x \in D(A)$, $AR(\cdot)x \in C(\mathbb{R}^+, X)$ and $R(\cdot)x \in C^1(\mathbb{R}^+, X)$.
(iii) For all $x \in D(A)$ and every $t \geq 0$,

$$\begin{aligned} R'(t) &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

For the axiomatic definition of the abstract phase space $\mathcal{B}$ by Hale and Kato, see [5].

Definition 2.2. A map f from a subset A of a Banach space into X is said to be *compact* or *completely continuous* if $f(B)$ is relatively compact for all bounded subsets $B \subseteq A$.

To ensure that an appropriate convolution operator between spaces of continuous functions is completely continuous, the following assumptions are needed:

Let $(X_i, ||\cdot||_i), i = 1, 2$, be Banach spaces. Let $L : I \times X_1 \to X_2$, where $I = [0, a], a \in \mathbb{R}$.

(H1) The function $L(t, \cdot) : X_1 \to X_2$ is continuous for almost all $t \in I$ and the function $L(\cdot, x) : I \to X_2$ is strongly measurable for each $x \in X_2$.

(H2) There exist an integrable function $m_L : I \to \mathbb{R}^+$ and a continuous nondecreasing function $\Omega_L : \mathbb{R}^+ \to (0, \infty)$ such that

$$||L(t, x)||_2 \le m_L(t)\Omega_L(||x||_1),\ (t, x) \in I \times X_1.$$

LEMMA 2.3. [4, Lemma 3.1] *Let $(X_i, ||\cdot||_i), i = 1, 2, 3$, be Banach spaces, $R : I \to \mathcal{L}(X_2, X_3)$, a strongly continuous map and $L : I \times X_1 \to X_2$, a function satisfying conditions (H1) and (H2). Then, the map $\Gamma : C(I, X_1) \to C(I, X_3)$ defined by*

$$\Gamma u(t) = \int_0^t R(t - s)L(s, u(s))ds$$

is continuous. Furthermore, if one of the following conditions holds,

(a) *for every $r > 0$, the set $\{L(s, x) : s \in I, ||x||_1 \le r\}$ is relatively compact in X_2;*

(b) *the map R is continuous in the operator norm and for every $r > 0$ and $t \in I$, the set $\{R(t)L(s, x) : s \in I, ||x||_1 \le r\}$ is relatively compact in X_3;*

then Γ is completely continuous.

The following is the well-known Leray-Schauder alternative theorem [3].

LEMMA 2.4. *Let C be a closed convex subset of a Banach space X and assume that $0 \in C$. Let $G : C \to C$ be a completely continuous map. Then, G has a fixed point in C or the set $\{z \in C : z = \lambda G(z), 0 < \lambda < 1\}$ is unbounded.*

To obtain another fixed point theorem, we need the following concepts [1].

DEFINITION 2.5. Let $\mathcal{D}$ be the set of all bounded subsets of a Banach space X. The *Kuratowski measure of noncompactness* is the map $\alpha : \mathcal{D} \to \mathbb{R}^+$ defined by (here $A \in \mathcal{D}$)

$$\alpha(A) = \inf\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i \text{ and } \mathrm{diam}(A_i) \le \varepsilon, i = 1, 2, \cdots, n\}.$$

DEFINITION 2.6. A map $f : A \subseteq X \to X$ is said to be *condensing* if $\alpha(f(B)) < \alpha(B)$ for all bounded sets $B \subseteq X$ with $\alpha(B) \ne 0$.

LEMMA 2.7 (Sadovskii's fixed point theorem). *Let C be a closed, convex subset of a Banach space X. Suppose that $f : C \to C$ is a continuous, condensing map. Then f has a fixed point in C.*

LEMMA 2.8. [2] *If $P = P_1 + P_2$ with P_1 a contractive operator and P_2 a compact operator, then P is a condensing operator.*

Also, we need the mean value theorem for the Bochner integral.

Lemma 2.9. [6, Lemma 2.1.3] *Suppose that f is an integrable function from I into X. Then*

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(\tau)d\tau \in \overline{co(\{f(\tau) : \tau \in [\alpha,\beta]\})}$$

for all $\alpha, \beta \in I$ with $\alpha < \beta$, where $co(\cdot)$ denotes the convex hull.

Lemma 2.10. [6, Lemma 2.2.1] *Suppose that $f : I \to X$ is continuous. Suppose also that $\alpha, \beta \in I, \alpha < \beta$, and there is an at least countable subset Λ of $[\alpha, \beta]$ such that $f'_+(t)$ exists for all $t \in [\alpha, \beta] - \Lambda$. Then*

$$f(\beta) - f(\alpha) \in (\beta-\alpha)\overline{co(\{f'_+(t) : t \in [\alpha,\beta] - \Lambda\})}.$$

Now, we consider the partial neutral integrodifferential equation (1.1).

Definition 2.11. A function $u : (-\infty, b] \to X, 0 < b \leq a$, is a *mild solution* of (1.1) on $[0, b]$ if

(i) $u \in C([0, b], X)$.
(ii) $u_0 = \varphi$.
(iii) $u(t) = R(t)[\varphi(0) + f(0, \varphi)] - f(t, u_t) + \int_0^t R(t-s)g(s, u_s)ds, t \in [0, b]$.

To obtain the existence result the following conditions are needed [3].

(H3) $g : [0, a] \times \mathcal{B} \to X$ satisfies the Carathéodory condition, and there exist a continuous function $m_g : [0, a] \to \mathbb{R}^+$ and a continuous nondecreasing function $\Omega_g : \mathbb{R}^+ \to (0, \infty)$ such that

$$||g(t, \psi)|| \leq m_g(t)\Omega_g(||\psi||_{\mathcal{B}}), (t, \psi) \in [0, a] \times \Omega.$$

(H4) $f : [0, a] \times \mathcal{B} \to X$ is completely continuous and there exist positive constants $c_1, c_2 > 0$ such that

$$||f(t, \psi)|| \leq c_1||\psi||_{\mathcal{B}} + c_2,\ (t, \psi) \in [0, a] \times \mathcal{B}.$$

(H5) Let $0 < b \leq a$ and $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x|_{[0,b]} \in C([0, b], X)\}$ endowed with the norm of the uniform convergence topology. For every $Q \subset S(b)$ bounded, the set $\{t \mapsto f(t, x_t + y_t) : x \in Q\}$ is equicontinuous on $[0, b]$.

Theorem 2.12. [3, Theorem 3.2] *Assume that f, g are continuous and that there exist continuous functions $L_f, L_g : [0, a] \to \mathbb{R}^+$ such that*

$$||f(t, \psi_1) - f(t, \psi_2)|| \leq L_f(r)||\psi_1 - \psi_2||_{\mathcal{B}}, \tag{2.2}$$
$$||g(t, \psi_1) - g(t, \psi_2)|| \leq L_g(r)||\psi_1 - \psi_2||_{\mathcal{B}} \tag{2.3}$$

for every $(t, \psi_i) \in [0, a] \times B_r(0, \mathcal{B}), i = 1, 2$, where $B_r(0, \mathcal{B})$ denotes the open ball in $\mathcal{B}$. If $K(0)L_f(0) < 1$, then there exists a unique mild solution of (1.1) on $[0, b]$, for some $0 < b \leq a$. Here $K(t)$ satisfies the axiom

$$||x_t||_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} ||x(s)|| + M(t-\sigma)||x_\sigma||_{\mathcal{B}}.$$

Also, the second existence result in [3] is the following. This result can be obtained by using a fixed point criterion for condensing operators. We prove it in detail.

THEOREM 2.13. *Let conditions (H1), (H2), (H3) and (H5) be satisfied and assume that f verifies the conditions in Theorem 2.12. Suppose, in addition, $K_b L_f < 1$ and that the following condition holds.*

(a) There exists a constant $0 < r_\varphi$ such that for each $t \in [0, a]$ there exists a compact set $W_t \subseteq X$ such that

$$R(t)g(s,\psi) \in W_t,\ \psi \in B_{r(\varphi)}(\varphi, \mathcal{B}), s \in [0, a].$$

Then there exists a mild solution of (1.1) on $[0, b]$, for some $0 < b \leq a$.

Proof. Let r, C_f, C_g be constants such that

$$||f(t,\psi)|| \leq C_f,\ ||g(t,\psi)|| \leq C_g \tag{2.4}$$

for every $(t,\psi) \in [0, b] \times B_r(\varphi, \mathcal{B})$. We choose $\rho > 0$ such that

$$\begin{aligned} \mu &= L_f K_b < 1, && (2.5)\\ ||R(t)|| &\leq M,\ 0 \leq t \leq b, && (2.6)\\ ||[R(t) - I]f(0,\varphi)||_b &+ ||f(t, y_t) - f(0,\varphi)||_b \leq \frac{(1-\mu)\rho}{3}, && (2.7)\\ MbC_g &\leq \frac{(1-\mu)\rho}{3}, && (2.8)\\ K_b\rho &+ \sup_{0\leq t\leq b} ||y_t - \varphi||_{\mathcal{B}} < r, && (2.9) \end{aligned}$$

where $K_b = \sup_{0\leq s\leq b} K(s)$ and y_t is defined below.

Now, we define the operator $\Gamma : S(b) \to S(b)$ by

$$\Gamma x(t) = R(t)f(0,\varphi) - f(t, x_t + y_t) + \int_0^t R(t-s)g(s, x_s + y_s)ds,$$

where $y : (-\infty, a] \to X$ is defined by

$$y(\theta) = \begin{cases} R(t)\varphi(0) & \text{if } 0 \leq t \leq a \\ \varphi(\theta) & \text{if } \theta \leq 0. \end{cases}$$

We claim that $\Gamma(B_\rho(0, S(b))) \subseteq B_\rho(0, S(b))$. Let $x \in B_\rho(0, S(b))$. Then $x_t + y_t \in B_r(\varphi, \mathcal{B})$ for $0 \leq t \leq b$ since

$$\begin{aligned} ||x_t + y_t - \varphi||_{\mathcal{B}} &\leq ||x_t||_{\mathcal{B}} + ||y_t - \varphi||_{\mathcal{B}} \\ &\leq K_b\rho + \sup_{0\leq t\leq b} ||y_t - \varphi||_{\mathcal{B}} \\ &< r \end{aligned}$$

by (2.9). Furthermore, we have

$$\begin{aligned} ||\Gamma x(t)|| &\leq ||R(t)f(0,\varphi)-f(0,\varphi)||+||f(t,y_t)-f(0,\varphi)|| \\ &\quad +||f(t,x_t+y_t)-f(t,y_t)|| \\ &\quad +||\int_0^t R(t-s)g(s,x_s+y_s)ds|| \\ &\leq \frac{(1-\mu)\rho}{3}+L_f||x_t||_{\mathcal{B}}+MbC_g \end{aligned}$$

by (2.7),(2.8) and (2.4). Thus, by (2.8), we obtain

$$||\Gamma x(t)|| \leq (1-\mu)\rho+L_fK_b||x||_b.$$

It follows from (2.5) that

$$\begin{aligned} ||\Gamma x(t)|| &< (1-\mu)\rho+\mu\rho \\ &= \rho, \ \ 0\leq t\leq b. \end{aligned}$$

Now, we consider the decomposition $\Gamma=\Gamma_1+\Gamma_2$:

$$\begin{aligned} \Gamma_1 x(t) &= R(t)f(0,\varphi)-f(t,x_t+y_t), \ 0\leq t\leq b, \\ \Gamma_2 x(t) &= \int_0^t R(t-s)g(s,x_s+y_s)ds, \ 0\leq t\leq b. \end{aligned}$$

Firstly, we show that Γ_1 is a contraction on $B_\rho(0,S(b))$. Let $x,z\in B_\rho(0,S(b))$. Then

$$\begin{aligned} ||\Gamma_1 x(t)-\Gamma_1 z(t)|| &= ||f(t,x_t+y_t)-f(t,z_t+y_t)|| \\ &\leq L_f||x_t-z_t||_{\mathcal{B}} \\ &\leq L_fK_b||x-z||_b \\ &< ||x-z||_b. \end{aligned}$$

Next, we prove that Γ_2 is a compact operator. Suppose that the set $\{g(s,u): 0\leq s\leq b, ||u||\leq r\}$ is relatively compact in X. Note that the set

$$C=\{R(s)g(\theta,z): s,\theta\in[0,b], z\in B_r(\varphi,S(b))\}$$

is relatively compact in X since $R(\cdot)$ is strongly continuous and g satisfies the Carathéodory condition by (H3). In view of Lemma 2.9, we have, for any $u\in B_r(\varphi,S(b))$,

$$\Gamma_2 u(t)\in t\,\overline{co(C)}.$$

Thus the set $\{\Gamma_2 u(t): u\in B_r(0,S(b))\}$ is relatively compact in X. To show that Γ_2 is compact we show that the set $\{\Gamma_2 u: u\in B_r(\varphi,S(b))\}$ is equicontinuous on $[0,b]$. Note that $R(\cdot)$ is strongly continuous and $g([0,a]\times B_r(\varphi,S(b)))$ is compact. Then for each $\varepsilon>0$ there exists $\delta>0$ such that

$$||R(t)g(s,z)-R(t')g(s,z)||\leq\varepsilon, \ t,t',s\in[0,b], \ z\in B_r(\varphi,S(b))$$

when $|t-t'|\leq\delta$. Let $u\in B_r(\varphi,S(b)), t\in[0,b], |h|\leq\delta$, and $t+h\in[0,b]$. Then

$$
\begin{aligned}
||\Gamma_2 u(t+h) - \Gamma_2 u(t)|| &\leq \int_0^t ||[R(t+h-s) - R(t-s)]g(s,u(s))||ds \\
&\quad + \sup_{0\leq\tau\leq b} ||R(\tau)|| \int_t^{t+h} ||g(s,u(s))||ds \\
&\leq \varepsilon b + \sup_{0\leq\tau\leq b} ||R(\tau)||\Omega_g(r) \int_t^{t+h} m_g(s)ds
\end{aligned}
$$

by (H3). Therefore $\{\Gamma_2 u : u \in B_r(\varphi, S(b))\}$ is equicontinuous on $[0, b]$. Hence, the Ascoli-Arzela theorem guarantees that Γ_2 is a compact operator. Consequently, $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing operator on $B_\rho(0, S(b))$. By the fixed point theorem for condensing operator (Lemma 2.8), Γ has a fixed point $x(\cdot)$ of (1.1) on $[0, b]$. Then $u = y + x$ is a mild solution of (1.1) on $[0, b]$. This completes the proof.

□

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