JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **28**, No. 2, May 2015
http://dx.doi.org/10.14403/jcms.2015.28.2.321

A FIXED POINT APPROACH TO THE STABILITY OF ADDITIVE-QUADRATIC FUNCTIONAL EQUATIONS IN MODULAR SPACES

ChangIl Kim* and Se Won Park**

Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+4f(x)+2f(-x)$$

in modular spaces by using a fixed point theorem for modular spaces.

1. Introduction and preliminaries

The question of stability for a generic functional equation was originated in 1940 by Ulam [9]. Concerning a group homomorphism, Ulam posted the question asking how likely to an automorphism a function should behave in order to guarantee the existence of an automorphism near such functions. Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [7] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [2] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in

Received March 21, 2015; Accepted April 24, 2015.
2010 Mathematics Subject Classification: Primary 39B52, 39B72, 47H09.
Key words and phrases: fixed point theorem, Hyers-Ulam stability, additive-quadratic functional equations, modular spaces.
Correspondence should be addressed to Se Won Park, sewpark1079@naver.com.

differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [6] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. This idea was refined and generalized by Musielak and Orlicz [5] in 1959.

Recently, Sadeghi [8] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and K. Wongkum, P. Chaipunya, and P. Kumam [10] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

In this paper, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+4f(x)+2f(-x) \tag{1.1}$$

in modular spaces by using a fixed point theorem for modular spaces.

DEFINITION 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}$, or $\mathbb{N})$.

(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
  (M1) $\rho(x) = 0$ if and only if $x = 0$,
  (M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
  (M3) $\rho(z) \le \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.

(2) If (M3) is replaced by
  (M4) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$
  for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex*.

The corresponding modular space, denoted by X_ρ, is then defined

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

REMARK 1.2. If a modular ρ is convex, then one has $\rho(x) \le \delta\rho(\frac{1}{\delta}x)$ for all $x \in X_\rho$ and for all real number δ with $o < \delta \le 1$.

Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, (ii) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx for some $c \in \mathbb{K}$. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the $\triangle_2$-conditions.

A modular space X_ρ is said to *satisfy the* $\triangle_2$*-condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$. Some authors varied the notion so that only $k > 0$ is required and called it *the* $\triangle_2$*-type condition*. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\triangle_2$-conditions. In [4], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of* T *around a point* $x \in C$ is the set

$$\mathbb{O}(x) := \{x, Tx, T^2x, \cdots\}.$$

The quantity $\delta_\rho(x) := \sup\{\rho(u - v) \mid u, v \in \mathbb{O}\}$ is called *the orbital diameter of* T *at* x and if $\delta_\rho(x) < \infty$, then one says that T has *a bounded orbit at* x.

Lemma 1.3. [4] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \longrightarrow C$ is a ρ-contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \ \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

For any modular ρ on X and any linear space V, we define a set $\mathbb{M}$

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\phi(x, x), \ \forall x \in V\}.$$

K. Wongkum, P. Chaipunya, and P. Kumam proved the following lemma:

Lemma 1.4. [10] *Let V be a linear space, X_ρ a ρ-complete modular space where ρ is lower semi-continuous and convex, and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\lim_{n\longrightarrow\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0, \ \ \phi(2x, 2y) \leq 2L\phi(x, y)$$

for all $x, y \in X$. Then (i) $\mathbb{M}$ is a linear space, (ii) $\widetilde{\rho}$ is a convex modular on $\mathbb{M}$, (iii) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, (iv) $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete, and (v) $\widetilde{\rho}$ is lower semi-continuous.

2. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous and convex. In this section, we prove the generalized Hyers-Ulam stability for (1.1). We start with the following theorem.

For any $f : X \longrightarrow Y$, let

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \;\; f_e(x) = \frac{1}{2}(f(x) + f(-x)).$$

THEOREM 2.1. *A mapping $f : X \longrightarrow Y$ satisfies (1.1) if and only if f is an additive-quadratic mapping.*

Proof. Suppose that $f : X \longrightarrow Y$ satisfies (1.1). By (1.1), we have

$$f_e(2x + y) + f_e(2x - y) = f_e(x + y) + f_e(x - y) + 6f_e(x)$$

for all $x, y \in X$ and clearly, f_e is a quadratic mapping. By (1.1), we have

$$f_o(2x + y) + f_o(2x - y) = f_o(x + y) + f_o(x - y) + 2f_o(x) \tag{2.1}$$

for all $x, y \in X$. Replacing y by $x + y$ in (2.1), we get

$$f_o(3x + y) + f_o(x - y) = f_o(2x + y) - f_o(y) + 2f_o(x) \tag{2.2}$$

for all $x, y \in X$ and replacing y by $-y$ in (2.2), we get

$$f_o(3x - y) + f_o(x + y) = f_o(2x - y) + f_o(y) + 2f_o(x) \tag{2.3}$$

for all $x, y \in X$. By (2.1), (2.2), and (2.3), we obtain

$$f_o(3x + y) + f_o(3x - y) = 6f_o(x) \tag{2.4}$$

for all $x, y \in X$ and letting $y = 0$ in (2.4), we obtain

$$f_o(3x) = 3f_o(x) \tag{2.5}$$

for all $x \in X$. By (2.4) and (2.5), we have

$$f_o(x + y) + f_o(x - y) = 2f_o(x)$$

for all $x \in X$ and hence f_o is an additive mapping. Thus f is an additive-quadratic mapping. The converse is trivial. □

THEOREM 2.2. *Let V be a linear space, X_ρ a ρ-complete modular space and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\phi(2x, 2y) \leq 2L\phi(x, y) \tag{2.6}$$

for all $x, y \in V$ and some L with $0 \leq L < 1$ and

$$\rho(f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-4f(x)-2f(-x)) \leq \phi(x, y) \tag{2.7}$$

for all $x, y \in V$. Then there exists a unique additive-quadratic mapping $F : V \longrightarrow X_\rho$ such that

$$\rho(F(x) - \frac{1}{2}f(x)) \leq \frac{3-2L}{8(1-L)(2-L)}\psi(x, 0) \tag{2.8}$$

for all $x \in V$, where $\psi(x, y) = \frac{1}{2}(\phi(x, y) + \phi(-x, -y))$.

Proof. Define a map $\widetilde{\rho}$ on $\mathbb{M} = \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$ by

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\psi(x, o),\ \forall x \in V\}$$

for each $g \in \mathbb{M}$. Similar to the proof of Lemma 1.4, we can show that $\widetilde{\rho}$ satisfies (ii), (iii), (iv), and (v) in Lemma 1.4.

Define $T_o : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_o g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some non-negative real number c. Then by Remark 1.2, we have

$$\rho(T_o g(x) - T_o h(x)) \leq \frac{1}{2}\rho(g(2x) - h(2x)) \leq Lc\psi(x, 0)$$

for all $x \in V$ and so $\widetilde{\rho}(T_o g - T_o h) \leq L\widetilde{\rho}(g - h)$. Hence T_o is a $\widetilde{\rho}$-contraction.

Now, we claim that T_o has a bounded orbit at $\frac{1}{2}f_o$. Since f_o is an odd mapping and ρ is convex, (2.7) implies that

$$\rho(f_o(2x+y)+f_o(2x-y)-f_o(x+y)-f_o(x-y)-2f_o(x)) \leq \psi(x, y) \tag{2.9}$$

for all $x, y \in V$. Letting $y = 0$ in (2.9), we get

$$\rho(2f_o(2x) - 4f_o(x)) \leq \psi(x, 0)$$

for all $x \in V$ and so

$$\rho(\frac{1}{2}f_o(2x) - f_o(x)) \leq \frac{1}{2^2}\psi(x, 0)$$

for all $x \in V$. For any non-negative integer n, we obtain

$$\begin{aligned}
&\rho(\frac{1}{2^n}f_o(2^nx) - f_o(x))\\
&= \rho(\frac{1}{2}[\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)] + \frac{1}{2}[f_o(2x) - 2f_o(x)])\\
&\le \frac{1}{2}\rho(\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)) + \frac{1}{2}\rho(f_o(2x) - 2f_o(x))\\
&\le \frac{1}{2}\rho(\frac{1}{2^{n-1}}f_o(2^{n-1}(2x)) - f_o(2x)) + \frac{1}{2^2}\psi(x,0)
\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\rho(\frac{1}{2^n}f_o(2^nx) - f_o(x)) \le \sum_{i=0}^{n-1}\frac{1}{2^{i+2}}\psi(2^ix,0) \tag{2.10}$$

for all $x \in V$ and for all non-negative integer n. Hence for any $n, m \in \mathbb{N}$, by (2.10), we get

$$\begin{aligned}
&\rho(\frac{1}{2^{n+1}}f_o(2^nx) - \frac{1}{2^{m+1}}f_o(2^mx))\\
&\le \frac{1}{2}\rho(\frac{1}{2^n}f_o(2^nx) - f_o(x)) + \frac{1}{2}\rho(\frac{1}{2^m}f_o(2^mx) - f_o(x))\\
&\le \sum_{i=0}^{n-1}\frac{1}{2^{i+3}}\psi(2^ix,0) + \sum_{i=0}^{m-1}\frac{1}{2^{i+3}}\psi(2^ix,0) \le \frac{1}{4(1-L)}\psi(x,0)
\end{aligned}$$

for all $x \in V$ and thus

$$\widetilde{\rho}(T_o^n\frac{1}{2}f_o - T_o^m\frac{1}{2}f_o) \le \frac{1}{4(1-L)}$$

for all $x \in V$. Hence T_o has a bounded orbit at $\frac{1}{2}f_o$. By Lemma 1.3, there is an $A \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_o^n\frac{1}{2}f_o\}$ $\widetilde{\rho}$-converges to A. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_oA - A) \le \liminf_{n\longrightarrow\infty}\widetilde{\rho}(T_oA - T_o^{n+1}\frac{1}{2}f_o) \le \liminf_{n\longrightarrow\infty}L\widetilde{\rho}(A - T_o^n\frac{1}{2}f_o) = 0$$

and hence A is a fixed point of T_o in $\mathbb{M}_{\widetilde{\rho}}$. Replacing x and y by 2^nx and 2^ny in (2.9), respectively, we have

$$\begin{aligned}
&\rho(\frac{1}{2^{n+1}}[f_o(2^n(2x+y)) + f_o(2^n(2x-y)) - f_o(2^n(x+y))\\
&- f_o(2^n(x-y)) - 2f_o(2^nx)]) \le \frac{1}{2^{n+1}}\psi(2^nx,2^ny) \le \frac{L^n}{2}\psi(x,y)
\end{aligned} \tag{2.11}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.11), we get

$$A(2x+y) + A(2x-y) - A(x+y) - A(x-y) - 2A(x) = 0 \tag{2.12}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.10), we get

$$\rho(2A(x) - f_o(x)) \leq \frac{1}{4(1-L)}\psi(x,0)$$

for all $x \in X$ and so we have

$$\widetilde{\rho}(2A - f_o) \leq \frac{1}{4(1-L)}. \tag{2.13}$$

Define $T_e : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_e g(x) = \frac{1}{4}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \leq c$ for some non-negative real number c. Then we have

$$\rho(T_e g(x) - T_e h(x)) \leq \frac{1}{4}\rho(g(2x) - h(2x)) \leq \frac{L}{2}c\psi(x,0)$$

for all $x \in V$ and so $\widetilde{\rho}(T_e g - T_e h) \leq \frac{L}{2}\widetilde{\rho}(g-h)$. Thus T_e is a $\widetilde{\rho}$-contraction.

Now, we claim that T_e has a bounded orbit at $\frac{1}{2}f_e$. Since f_e is an even mapping and ρ is convex, (2.7) implies that

$$\rho(f_e(2x+y)+f_e(2x-y)-f_e(x+y)-f_e(x-y)-6f_e(x)) \leq \psi(x,y) \tag{2.14}$$

for all $x, y \in V$. Letting $y = 0$ in (2.14), we get

$$\rho(2f_e(2x) - 8f_e(x)) \leq \psi(x,0)$$

for all $x \in V$ and so

$$\rho(\frac{1}{4}f_e(2x) - f_e(x)) \leq \frac{1}{2\cdot 4}\psi(x,0)$$

for all $x \in V$. For any non-negative integer n, we obtain

$$\begin{aligned}
&\rho(\frac{1}{4^n}f_e(2^n x) - f_e(x)) \\
&= \rho(\frac{1}{2}[\frac{1}{2\cdot 4^{n-1}}f_e(2^{n-1}(2x)) - \frac{1}{2}f_e(2x)] + \frac{1}{2}[\frac{1}{2}f_e(2x) - 2f_e(x)]) \\
&\leq \frac{1}{4}\rho(\frac{1}{4^{n-1}}f_e(2^{n-1}(2x)) - f_e(2x)) + \frac{1}{4}\rho(f_e(2x) - 4f_e(x)) \\
&\leq \frac{1}{4}\rho(\frac{1}{4^{n-1}}f_e(2^{n-1}(2x)) - f_e(2x)) + \frac{1}{2\cdot 4}\psi(x,0)
\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\rho(\frac{1}{4^n}f_e(2^n x) - f_e(x)) \leq \sum_{i=0}^{n-1}\frac{1}{2\cdot 4^{i+1}}\psi(2^i x, 0) \tag{2.15}$$

for all $x \in V$ and for all non-negative integer n. Hence for any $n, m \in \mathbb{N}$, by (2.15), we get

$$\begin{aligned}&\rho(\frac{1}{4^n}\cdot\frac{1}{2}f_e(2^nx) - \frac{1}{4^m}\cdot\frac{1}{2}f_e(2^mx))\\ &\leq \frac{1}{2}\rho(\frac{1}{4^n}f_e(2^nx) - f_e(x)) + \frac{1}{2}\rho(\frac{1}{4^m}f_e(2^mx) - f_e(x))\\ &\leq \sum_{i=0}^{n-1}\frac{1}{4^{i+2}}\psi(2^ix,0) + \sum_{i=0}^{m-1}\frac{1}{4^{i+2}}\psi(2^ix,0) \leq \frac{1}{4(2-L)}\psi(x,0)\end{aligned}$$

for all $x \in V$ and thus

$$\widetilde{\rho}(T_e^n\frac{1}{2}f_e - T_e^m\frac{1}{2}f_e) \leq \frac{1}{4(2-L)}$$

for all $x \in V$. Hence T_e has a bounded orbit at $\frac{1}{2}f_e$. By Lemma 1.3, there is a $Q \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_e^n\frac{1}{2}f_e\}$ $\widetilde{\rho}$-converges to Q. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_eQ - Q) \leq \liminf_{n\longrightarrow\infty}\widetilde{\rho}(T_eQ - T_e^{n+1}\frac{1}{2}f_e) \leq \liminf_{n\longrightarrow\infty}\frac{L}{2}\widetilde{\rho}(Q - T_e^n\frac{1}{2}f_e) = 0$$

and hence Q is a fixed point of T_e in $\mathbb{M}_{\widetilde{\rho}}$. Replacing x and y by 2^nx and 2^ny in (2.14), respectively, we have

(2.16)

$$\begin{aligned}&\rho(\frac{1}{2\cdot 4^n}[f_e(2^n(2x+y)) + f_e(2^n(2x-y)) - f_e(2^n(x+y))\\ &- f_e(2^n(x-y)) - 6f_e(2^nx)]) \leq \frac{1}{2\cdot 4^n}\psi(2^nx, 2^ny) \leq \frac{L^n}{2^{n+1}}\psi(x,y)\end{aligned}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.16), we get

$$Q(2x+y) + Q(2x-y) - Q(x+y) - Q(x-y) - 6Q(x) = 0 \tag{2.17}$$

for all $x, y \in V$. Since ρ is lower semi-continuous, by (2.15), we get

$$\rho(2Q(x) - f_e(x)) \leq \frac{1}{4(2-L)}\phi(x,0)$$

for all $x \in X$ and so we have

$$\widetilde{\rho}(2Q - f_e) \leq \frac{1}{4(2-L)}. \tag{2.18}$$

Let $F = A + Q$. Then clearly A is odd and Q is even and by (2.12) and (2.17), F is a solution of (1.1). By Theorem 2.1, F is an additive-quadratic mapping. Moreover, by (2.13) and (2.18), we have

$$\widetilde{\rho}(F - \frac{1}{2}f) \leq \frac{1}{2}\widetilde{\rho}(2A - f_o) + \frac{1}{2}\widetilde{\rho}(2Q - f_e)$$

and hence we have (2.8).

To prove the uniquness of F, let $G : V \longrightarrow X_\rho$ be another additive-quadratic mapping with (2.8). By (2.8), we get

$$\begin{aligned}\rho(\frac{1}{2}G(x) - \frac{1}{2}F(x)) &\le \frac{1}{2}\rho(G(x) - \frac{1}{2}f(x)) + \frac{1}{2}\rho(F(x) - \frac{1}{2}f(x)))\\ &\le \frac{3-2L}{8(1-L)(2-L)}\psi(x,0)\end{aligned}$$

for all $x \in V$ and so

$$\begin{aligned}\rho(\frac{1}{2}G_o(x) - \frac{1}{2}F_o(x)) &\le \frac{1}{2}\rho(\frac{1}{2}G(x) - \frac{1}{2}F(x)) + \frac{1}{2}\rho(\frac{1}{2}G(-x) - \frac{1}{2}F(-x)))\\ &\le \frac{3-2L}{8(1-L)(2-L)}\psi(x,0)\end{aligned}$$

for all $x \in V$. Since F_o and G_o are fixed points of T_o, we have

$$\begin{aligned}\rho(\frac{1}{2}G_o(x) - \frac{1}{2}F_o(x)) &\le \rho(\frac{1}{2}T_nG_o(x) - \frac{1}{2}T_nF_o(x))\\ &\le \frac{3-2L}{8(1-L)(2-L)}L^n\psi(x,0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence $F_o = G_o$ and similarly, we have $F_e = G_e$. Thus $F = G$. □

Using Theorem 2.2, we conclude the following classical generalized Hyers-Ulam stability in normed spaces.

Corollary 2.3. *Let V be a linear space, $(X, \|\cdot\|)$ a Banach space and $f : V \longrightarrow X$ a mapping with $f(0) = 0$. Suppose that the following inequality*

$$\begin{aligned}&\|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 4f(x) - f(-x)\|\\ &\le \|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}\end{aligned}$$

holds for all $x, y \in V$ and for some real number p with $0 < p < \frac{1}{2}$. Then there is a unique additive-quadratic mapping $F : V \longrightarrow X$ such that

$$\|F(x) - f(x)\| \le \frac{3-2^{2p}}{2(2-2^{2p})(4-2^{2p})}\|x\|^{2p}$$

for all $x \in X$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64-66.
[2] P. Găvruta, *A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431-436.
[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), 222-224.
[4] M. A. Khamsi, *Quasicontraction mappings in modular spaces without 2-condition*, Fixed Point Theory and Applications **2008** (2008), 1-6.
[5] J. Musielak and W. Orlicz, *On modular spaces*, Studia Mathematica **18** (1959), 591-597.
[6] H. Nakano, *Modular semi-ordered spaces*, Tokyo, Japan, 1959.
[7] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Sco. **72** (1978), 297-300.
[8] G. Sadeghi, *A fixed point approach to stability of functional equations in modular spaces*, Bulletin of the Malaysian Mathematical Sciences Society, Second Series, **37** (2014), 333-344.
[9] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.
[10] K. Wongkum, P. Chaipunya, and P. Kumam, *On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without* $\triangle_2$*-conditions*, **2015** (2015), 1-6.

*
Department of Mathematics Education
Dankook University
Yongin 448-701, Republic of Korea
E-mail: kci206@hanmail.net

**
Department of Liberal arts and Science
Shingyeong University
Hwaseong 445-741, Republic of Korea
E-mail: sewpark1079@naver.com