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APPROXIMATE QUADRATIC MAPPINGS IN QUASI-β-NORMED SPACES*

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Abstract. In this article, we consider a modified quadratic functional equation and then investigate its generalized Hyers–Ulam stability theorem in quasi-β-normed spaces.

1. Introduction

In 1940, S.M. Ulam [17] raised the question concerning the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot,\ \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? The case of approximately additive mappings was solved by D.H. Hyers [8] under the assumption that X and Y are Banach spaces. A generalization of Hyers' theorem was provided by Th.M. Rassias [12] in 1978 and by P. Găvruta [6] in 1994.

We recall that the following functional equation

$$f(x + y) + f(x - y) = 2[f(x) + f(y)]$$

is called a quadratic functional equation which may be originated from the important parallelogram equality $\|x+y\|^2+\|x-y\|^2 = 2[\|x\|^2+\|y\|^2]$ in inner product spaces. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers–Ulam stability problem for the quadratic functional equation was proved by

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F. Skof [16] for a mapping $f : X \to Y$, where X is a normed space and Y is a Banach space. S. Czerwik [5] proved the Hyers–Ulam stability of the quadratic functional equation with the sum of powers of norms in the sense of Th. M. Rassias approach using direct method as follows.

THEOREM 1.1. *Let E_1 be a normed space and E_2 a Banach space and let $\varepsilon \geq 0$ and $r \neq 2$ be given real numbers. Let $f : E_1 \to E_2$ be a mapping satisfying the condition*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^r + \|y\|^r)$$

for all $x, y \in E_1$ ($x, y \in E_1 \backslash \{0\}$ if $r < 0$). Then there exists exactly one quadratic mapping $h : E_1 \to E_2$ such that

$$\|f(x) - \frac{f(0)}{3} - h(x)\| \leq \frac{2\varepsilon}{|2^r - 4|}\|x\|^r$$

for all $x \in E_1$ ($x \in E_1 \backslash \{0\}$ if $r < 0$), where $f(0) = 0$ in case $r > 0$. If in addition, f is measurable or the mapping $\mathbb{R} \ni t \to f(tx)$ is continuous on $\mathbb{R}$ for each fixed $x \in E_1$, then the mapping h satisfies the condition

$$h(tx) = t^2 h(x)$$

for all $x \in E_1$ and all $t \in \mathbb{R}$.

J.M. Rassias [13] proved the Hyers–Ulam stability of the quadratic functional equation with the product of powers of norms using direct method as the following theorem.

THEOREM 1.2. *Let X be a normed linear space, Y a Banach space, and let $f : X \to Y$ be a mapping. If there exist real numbers a, b with $0 \leq a + b < 2$, and $c \geq 0$ such that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq c\|x\|^a\|y\|^b$$

for all $x, y \in X$, then there exists a unique non-linear mapping $N : X \to Y$ such that

$$\|f(x) - N(x)\| \leq c_1\|x\|^{a+b}$$

and

$$N(x+y) + N(x-y) = 2N(x) + 2N(y)$$

for all $x, y \in X$, where $c_1 = \frac{c}{4-2^{a+b}}$.

On the other hand, C. Borelli and G.L. Forti [2] have proved the generalized Hyers–Ulam stability theorem of the quadratic functional equation and thus we can obtain the following stability theorem as a result.

Theorem 1.3. *Let G be an abelian group and E a Banach space, and let $f : G \to E$ be a mapping satisfying the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Assume that one of the series

$$\Phi(x,y) := \begin{cases} \sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}}\varphi(2^k x, 2^k y) < \infty, \\ \sum_{k=0}^{\infty} 2^{2k}\varphi(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}) < \infty, \qquad (a) \end{cases}$$

converges for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to E$ such that

$$\|f(x) - \frac{f(0)}{3} - Q(x)\| \le \Phi(x, x)$$

for all $x \in G$, where $f(0) = 0$ in case (a).

The stability problems of several functional equations and several functional inequalities have been extensively investigated by a number of authors and there are many interesting result concerning the stability of various functional equations and inequalities ([4],[7],[9],[10]). Recently, A. Zivari-Kazempour and M. Eshaghi Gordji [18] have determined the general solution of the quadratic functional equation

$$\begin{aligned} &f(x+2y) + f(y+2z) + f(z+2x) \\ &\quad = 2f(x+y+z) + 3[f(x) + f(y) + f(z)] \end{aligned}$$

and then have investigated its generalized Hyers–Ulam stability. Motivated from this quadratic functional equation, we consider a modified functional equation

$$\begin{aligned} (1.1) \quad & f(x+2y) + f(y+2z) + f(z+2x) + f(y+2x) + f(z+2y) \\ & +f(x+2z) = 4f(x+y+z) + 6[f(x) + f(y) + f(z)] \end{aligned}$$

and then we establish its generalized Hyers–Ulam stability of the equation (1.1) in quasi-β-normed spaces. As results, we generalize stability results of the equation (1.1) in normed spaces.

2. Generalized Hyers–Ulam stability of Eq. (1.1).

First of all, we remark that the above functional equation (1.1) is equivalent to the original quadratic functional equation [11].

Now, we recall some basic facts concerning the quasi-β-normed spaces [14]. Let β be a fixed real number with $0 < \beta \le 1$ and let X be a linear space over $\mathbb{K}$, where $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A quasi-β-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|^{\beta}\|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a quasi-β-normed space. A quasi-β-Banach space is a complete quasi-β-normed space. Let p be a real number with ($0 < p \leq 1$). Then, the quasi-β-norm $\|\cdot\|$ on X is called a (β, p)-norm if, moreover, $\|\cdot\|^p$ satisfies the following triangle inequality

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space. We notice that quasi-1-normed spaces are equivalent to quasi-normed spaces and that $(1, p)$-Banach spaces with $(1, p)$-norm are equivalent to p-Banach spaces with p-norm. We can refer to [1, 15] for the concept of quasi-normed spaces and p-Banach spaces. Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [15], each quasi-norm is equivalent to some p-norm [1].

Before making up the main subject, we use the following abbreviation for notational convenience :

$$\begin{aligned} &Df(x, y, z) \\ &\quad := f(x + 2y) + f(y + 2z) + f(z + 2x) + f(y + 2x) + f(z + 2y) \\ &\qquad + f(x + 2z) - 4f(x + y + z) - 6[f(x) + f(y) + f(z)] \end{aligned}$$

for all $x, y, z \in X$.

THEOREM 2.1. *Suppose X is a vector space and Y is a (β, p)-Banach space. Let $\varphi : X \times X \times X \to [0, \infty)$ be a function such that*

$$\Phi_1(x, y, z) := \sum_{k=0}^{\infty} \frac{1}{9^{k\beta p}} \varphi(3^k x, 3^k y, 3^k z)^p \tag{2.1}$$

is convergent for all $x, y, z \in X$. If a mapping $f : X \to Y$ satisfies

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F : X \to Y$ defined by $F(x) := \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$, $x \in X$, which satisfies the equation (1.1) and

$$\|f(x) - F(x)\| \leq \frac{1}{18^{\beta}} [\Phi_1(x, x, x)]^{\frac{1}{p}} \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = z := x$ in (2.2), we obtain

$$\|\frac{f(3x)}{9} - f(x)\| \le \frac{1}{18^\beta}\varphi(x,x,x), \ (x \in X). \tag{2.4}$$

By induction on n, one can prove the following functional inequality

$$\|f(x) - \frac{f(3^n x)}{9^n}\|^p \le \frac{1}{18^{\beta p}} \sum_{k=0}^{n-1} \frac{1}{9^{k\beta p}} \varphi(3^k x, 3^k x, 3^k x)^p \tag{2.5}$$

for all $x \in X$. In fact, it is true for $n = 1$. Assume that the inequality (2.5) holds true for n. If we replace x by $3^n x$ in (2.4), then we get

$$\|\frac{f(3^{n+1}x)}{9^{n+1}} - \frac{f(3^n x)}{9^n}\|^p \le \frac{1}{18^{\beta p} \cdot 9^{n\beta p}} \varphi(3^n x, 3^n x, 3^n x)^p \tag{2.6}$$

for all $x \in X$. Thus, by triangle inequality on $\| \cdot \|^p$, we deduce

$$\begin{aligned} &\|\frac{f(3^{n+1}x)}{9^{n+1}} - f(x)\|^p \\ &\le \|\frac{f(3^{n+1}x)}{9^{n+1}} - \frac{f(3^n x)}{9^n}\|^p + \|\frac{f(3^n x)}{9^n} - f(x)\|^p \\ &\le \frac{1}{18^{\beta p}} \sum_{k=0}^{n} \frac{1}{9^{k\beta p}} \varphi(3^k x, 3^k x, 3^k x)^p, \end{aligned} \tag{2.7}$$

which proves (2.5) for $n+1$. Now, replacing x by $3^m x$ in (2.5), we have

$$\|\frac{f(3^{n+m}x)}{9^{n+m}} - \frac{f(3^m x)}{9^m}\|^p \le \frac{1}{18^{\beta p}} \sum_{k=m}^{n+m-1} \frac{1}{9^{k\beta p}} \varphi(3^k x, 3^k x, 3^k x)^p \tag{2.8}$$

which converges to zero as $m \to \infty$ by the assumption (2.1). Thus the above inequality implies that the sequence $\{\frac{f(3^n x)}{9^n}\}$ is Cauchy for all $x \in X$ and so it is convergent in Y since the space Y is complete. Thus, we may define $F : X \to Y$ as

$$F(x) := \lim_{n\to\infty} \frac{f(3^n x)}{9^n}, \qquad (x \in X).$$

Then by the definition of F, we can see by taking $n \to \infty$ in (2.5) that the approximation (2.3) holds. To show that F satisfies the equation (1.1), we set $(x, y, z) := (3^n x, 3^n y, 3^n z)$ in (2.2), and divide the resulting inequality by 9^n. Then we get

$$\|\frac{Df(3^n x, 3^n y, 3^n z)}{9^n}\|^p \le \frac{1}{9^{n\beta p}} \varphi(3^n x, 3^n y, 3^n z)^p$$

for all $x, y, z \in X$. Taking the limit as $n \to \infty$, one obtains $DF(x, y, z) = 0$ for all $x, y, z \in X$. Hence F satisfies the equation (1.1) and so it is quadratic.

To show the uniqueness of F, we assume that there exists a quadratic mapping $G : X \to Y$ which satisfies the inequality

$$\|f(x) - G(x)\| \leq \frac{1}{18^{\beta}} \left[\sum_{k=0}^{\infty} \frac{\varphi(3^k x, 3^k x, 3^k x)^p}{9^{k\beta p}} \right]^{\frac{1}{p}}$$

for all $x \in X$, but suppose $F(y) \neq G(y)$ for some $y \in X$. Then there exists a positive constant $\varepsilon > 0$ such that $0 < \varepsilon < \|F(y) - G(y)\|^p$. For such given $\varepsilon > 0$, it follows from (2.1) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\frac{2}{18^{\beta p}} \sum_{k=n_0}^{\infty} \frac{\varphi(3^k y, 3^k y, 3^k y)^p}{9^{k\beta p}} < \varepsilon$. Since F and G are quadratic mappings, we see from the equality $F(3^{n_0} y) = 9^{n_0} F(y)$ and $G(3^{n_0} y) = 9^{n_0} G(y)$ that

$$\begin{aligned}
&\|F(y) - G(y)\|^p = \frac{1}{9^{n_0 \beta p}} \|F(3^{n_0} y) - G(3^{n_0} y)\|^p \\
&\leq \frac{1}{9^{n_0 \beta p}} [\|F(3^{n_0} y) - f(3^{n_0} y)\|^p + \|f(3^{n_0} y) - G(3^{n_0} y)\|^p] \\
&\leq \frac{1}{9^{n_0 \beta p}} \cdot \frac{2}{18^{\beta p}} \sum_{i=0}^{\infty} \frac{\varphi(3^{i+n_0} y, 3^{i+n_0} y, 3^{i+n_0} y)^p}{9^{i\beta p}} \\
&= \frac{2}{18^{\beta p}} \sum_{k=n_0}^{\infty} \frac{\varphi(3^k y, 3^k y, 3^k y)^p}{9^{k\beta p}} < \varepsilon,
\end{aligned}$$

which leads a contradiction. Hence the mapping F is a unique quadratic mapping satisfying (2.3). $\square$

THEOREM 2.2. *Let X be a vector space and Y a (β, p)-Banach space. If there exists a function $\varphi : X \times X \times X \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
\|Df(x, y, z)\| &\leq \varphi(x, y, z), \text{ and} \\
\Phi_2(x, y, z) &:= \sum_{k=1}^{\infty} 9^{k\beta p} \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) < \infty,
\end{aligned}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F : X \to Y$, defined as $F(x) = \lim_{n\to\infty} 9^n f(\frac{x}{3^n})$, $x \in X$, which satisfies the equation (1.1) and

$$\|f(x) - F(x)\| \leq \frac{1}{18^{\beta}} [\Phi_2(x, x, x)]^{\frac{1}{p}}$$

for all $x \in X$.

Proof. We see from (2.4) that

$$\|f(x) - 9f(\frac{x}{3})\|^p \leq \frac{1}{2^{\beta p}}\varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3})^p$$

for all $x \in X$. Then it follows by induction that

$$\|f(x) - 9^n f(\frac{x}{3^n})\|^p \leq \frac{1}{18^{\beta p}} \sum_{k=1}^{n} 9^{k\beta p}\varphi(\frac{x}{3^k}, \frac{x}{3^k}, \frac{x}{3^k})^p$$

for all $x \in X$. Applying the same argument as in the proof of Theorem 2.1, we get the desired results. □

As applications, we obtain the following stability results of the equation (1.1), which generalize stability results in normed spaces.

Corollary 2.3. *Suppose X is a vector space and Y is a (β, p)-Banach space. Let $f : X \to Y$ be a mapping satisfying*

$$\|Df(x, y, z)\| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique quadratic mapping $F : X \to Y$ which satisfies (1.1) and

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{2^\beta \sqrt[p]{9^{\beta p} - 1}}, \quad (x \in X). \tag{2.9}$$

Proof. Let $\varphi(x, y, z) := \varepsilon$ for all $x, y, z \in X$. Then by Theorem 2.1, we have

$$\|f(x) - F(x)\| \leq \frac{1}{18^\beta} \left[\sum_{k=0}^{\infty} \frac{\varepsilon^p}{9^{k\beta p}}\right]^{\frac{1}{p}} = \frac{\varepsilon}{2^\beta (9^{\beta p} - 1)^{\frac{1}{p}}}$$

for all $x \in X$, as desired. □

Corollary 2.4. *Suppose X is a quasi-α-normed space and Y is a (β, p)-Banach space. For given positive real numbers ε and r with $\alpha r \neq 2\beta$, let $f : X \to Y$ be a mapping satisfying*

$$\|Df(x, y, z)\| \leq \varepsilon(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{3\varepsilon}{2^\beta \sqrt[p]{|3^{2\beta p} - 3^{r\alpha p}|}}\|x\|^r$$

for all $x \in X$.

Proof. Considering a function $\varphi(x,y,z) := \varepsilon(\|x\|^r + \|y\|^r + \|z\|^r)$ and applying Theorem 2.1 and Theorem 2.2 to each cases $r\alpha < 2\beta$ or $r\alpha > 2\beta$, respectively, we obtain the required approximation for each cases $r\alpha < 2\beta$ or $r\alpha > 2\beta$, respectively. $\square$

COROLLARY 2.5. *Suppose X is a quasi-α-normed space and Y is a (β,p)-Banach space. For a given positive real number θ and three real numbers r_i, let $f : X \to Y$ be a mapping satisfying*

$$\|Df(x,y,z)\| \leq \theta\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3} \tag{2.10}$$

for all $x,y,z \in X$, where $r := r_1 + r_2 + r_3 > 0, r\alpha \neq 2\beta$. Then there exists a unique quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\theta}{2^\beta \sqrt[p]{|3^{2\beta p} - 3^{r\alpha p}|}}\|x\|^r$$

for all $x \in X$.

Proof. Considering a function $\varphi(x,y,z) := \theta\|x\|^{r_1}\|y\|^{r_2}\|z\|^{r_3}$ and then applying Theorem 2.1 and Theorem 2.2 to each cases $r\alpha < 2\beta$ or $r\alpha > 2\beta$, respectively, we obtain the desired result for each cases $r\alpha < 2\beta$ or $r\alpha > 2\beta$, respectively. $\square$

REMARK 2.6. In Corollary 2.5, let r_3 be a positive real number without loss of generality. If a mapping $f : X \to Y$ with regularity condition $f(0) = 0$ satisfies the assumption (2.10), then we find that $Df(x,y,0) = 0$, which yields the equation

$$f(x+2y) + f(2x+y) = 4f(x+y) + f(x) + f(y)$$

for all $x,y \in X$. Thus, it follows from Theorem 2.1 in [3] that f is itself a quadratic mapping in this case.

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