JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **28**, No. 2, May 2015
http://dx.doi.org/10.14403/jcms.2015.28.2.307

A NONDEGENERATE BILINEAR FORM INDUCED BY COLOR POISSON BIALGEBRA

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Abstract. Let H be a color Poisson bialgebra. Here we find a canonical nondegenerate bilinear form on $\mathfrak{g}(H)\times\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{g}(H)$ and $\mathfrak{m}/\mathfrak{m}^2$ are certain color Lie algebras induced by H.

1. Introduction

For the definitions of the color Lie algebra and the color Poisson bialgebra, refer to [1, Definition 2.1 and Definition 4.4]. Let $H = (H = \oplus_{a\in G}H_a, m, \iota, \Delta, \eta, \{\cdot,\cdot\}, \epsilon)$ be a color Poisson bialgebra. For a homogeneous element $x \in H_a$, we set $|x| = a$. Set

$$\mathfrak{m} = \ker \eta.$$

Then $(\mathfrak{m}/\mathfrak{m}^2, [\cdot,\cdot], \epsilon)$ is a color Lie algebra with

$$[x+\mathfrak{m}^2, y+\mathfrak{m}^2] = \{x, y\} + \mathfrak{m}^2$$

by [1, Theorem 4.9]. Here we construct a color Lie algebra $(\mathfrak{g}(H), [\cdot,\cdot], \epsilon^{-1})$ induced by H such that there exists a canonical nondegenerate bilinear form on $\mathfrak{g}(H)\times\mathfrak{m}/\mathfrak{m}^2$. (See Theorem 2.3.)

Give a grading on the ground field $\mathbf{k}$ of characteristic zero by

$$\mathbf{k}_e = \mathbf{k}, \quad \mathbf{k}_a = \{0\}$$

for all $e \neq a \in G$, where e is the identity element of G.

Since η is a morphism of G-graded algebras, the maximal ideal $\mathfrak{m}$ is a G-graded ideal of H containing H_a for all $e \neq a \in G$. In particular, $\mathfrak{m}^2$ is also a G-graded ideal. Note that $H = \mathfrak{m}\oplus\mathbf{k}1$ as a vector space since $x = (x-\eta(x)1)+\eta(x)1$ and $x-\eta(x)1 \in \mathfrak{m}$ for all $x \in H$. Choose a basis $\mathfrak{C}$ of $\mathfrak{m}^2$ consisting of homogeneous elements and then add a set $\mathfrak{D}$ of

Received Feburary 12, 2015; Accepted April 13, 2015.
2010 Mathematics Subject Classification: 17B62, 17B63, 17B75.
Key words and phrases: color Lie algebras, color Poisson bialgebras.
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*Supported by the CNU research fund of Chungnam National University.

homogeneous elements in $\mathfrak{m}$ such that the disjoint union $\mathfrak{B} = \{1\} \sqcup \mathfrak{C} \sqcup \mathfrak{D}$ forms a basis of H. For every $x \in \mathfrak{B}$, give a grading on the dual element $x^* \in H^*$ by

$$|x^*| = |x|.$$

Denoted by H° the subset of H^* consisting of $f \in H^*$ such that there exists a finite co-dimensional graded ideal I of H with $f(I) = 0$. That is, $f \in H^\circ$ if and only if $f(I) = 0$ for some graded ideal I such that the dimension of H/I is finite, and thus there exist finite elements $x_1, \cdots, x_n$ of $\mathfrak{B}$ such that $x_1 + I, \cdots, x_n + I$ form a basis of H/I. Hence $f \in H^\circ$ is a linear combination of dual elements $x_1^*, \cdots, x_n^*$. It follows that H° is a G-graded vector space. For homogeneous elements $f, g \in H^\circ$, define a multiplication fg by

$$(fg)(x) = \sum f(x')g(x'')$$

for any homogeneous element $x \in H$, where $\Delta(x) = \sum x' \otimes x''$. Replacing ideals with finite codimension in [3, 9.1.1, 9.1.3] by G-graded ideals with finite codimension, we have $fg \in H^\circ$. Moreover H° is a G-graded algebra since the comultiplication Δ of H is a G-graded algebra morphism. Hence H° is a color Lie algebra with bracket $[\cdot, \cdot]$ defined by

$$[f, g] = fg - \varepsilon^{-1}(|f|, |g|)gf \tag{1.1}$$

for homogeneous elements $f, g \in H^\circ$ by [1, Lemma 2.4].

2. Main theorem

Denote by $\mathfrak{g}(H)$ the subspace of H° spanned by all homogeneous elements $f \in H^\circ$ such that

$$f(1) = 0, \quad f(\mathfrak{m}^2) = 0.$$

LEMMA 2.1. *The space $\mathfrak{g}(H)$ is a color Lie subalgebra of $(H^\circ, [\cdot, \cdot], \varepsilon^{-1})$, where $[\cdot, \cdot]$ is given by (1.1).*

Proof. Let f, g be homogeneous elements of $\mathfrak{g}(H)$. Then

$$[f, g](1) = (fg)(1) - \varepsilon(|f|, |g|)^{-1}(gf)(1) = 0$$

by [1, Lemma 2.6] since $\Delta(1) = 1 \otimes 1$. Let us show that $[f, g](\mathfrak{m}^2) = 0$. (Hence $[f, g] \in \mathfrak{g}(H)$.) For a homogeneous element $x \in H$, let $\Delta(x) =$

$\sum x' \otimes x''$. Since

$$\begin{aligned}\sum(x' - \eta(x')1) \otimes (x'' - \eta(x'')1) \\ &= \sum x' \otimes x'' - \sum 1 \otimes \eta(x')x'' - \sum \eta(x'')x' \otimes 1 + \eta(x)1 \otimes 1 \\ &= \Delta(x) - 1 \otimes x - x \otimes 1 + \eta(x)(1 \otimes 1)\end{aligned}$$

and $x' - \eta(x')1, x'' - \eta(x'')1 \in \mathfrak{m}$, we have

$$\Delta(x) = -\eta(x)(1 \otimes 1) + 1 \otimes x + x \otimes 1 \mod(\mathfrak{m} \otimes \mathfrak{m}). \tag{2.1}$$

Hence, for homogeneous elements $x, y \in \mathfrak{m}$,

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y) \\ &= (1 \otimes x + x \otimes 1 + \mathfrak{m} \otimes \mathfrak{m})(1 \otimes y + y \otimes 1 + \mathfrak{m} \otimes \mathfrak{m}) \qquad \text{(by 2.1)} \\ &= x \otimes y + \varepsilon(|x|, |y|)y \otimes x \mod(\mathfrak{m}^2 \otimes H + H \otimes \mathfrak{m}^2).\end{aligned}$$

Thus

$$\begin{aligned}[f, g](xy) = f(x)g(y) + \varepsilon(|x|, |y|)f(y)g(x) \\ - \varepsilon^{-1}(|f|, |g|)[g(x)f(y) + \varepsilon(|x|, |y|)g(y)f(x)] = 0\end{aligned}$$

by [1, Lemma 2.6], as claimed. Hence $\mathfrak{g}(H)$ is a color Lie subalgebra of $(H^\circ, [\cdot, \cdot], \varepsilon^{-1})$. □

LEMMA 2.2. *Let $P_\eta(H)$ be the subspace of H° spanned by all homogeneous elements $f \in H^\circ$ such that*

$$f(xy) = \eta(x)f(y) + f(x)\eta(y) \tag{2.2}$$

for all homogeneous elements $x, y \in H$. Then $P_\eta(H) = \mathfrak{g}(H)$.

Proof. If $f \in P_\eta(H)$ then $f(1) = 0$ and $f(\mathfrak{m}^2) = 0$ by (2.2). Hence $P_\eta(H) \subseteq \mathfrak{g}(H)$. Conversely, let $f \in \mathfrak{g}(H)$. For any homogeneous elements $x, y \in H$,

$$xy = (x - \eta(x)1)(y - \eta(y)1) + \eta(x)y + \eta(y)x - \eta(x)\eta(y)1,$$

thus

$$f(xy) = \eta(x)f(y) + f(x)\eta(y)$$

since $(x - \eta(x)1)(y - \eta(y)1) \in \mathfrak{m}^2$. Hence $\mathfrak{g}(H) \subseteq P_\eta(H)$. □

Let $H = (H, m, \iota, \Delta, \eta, \{\cdot, \cdot\})$ be a color Poisson bialgebra. We may assume that $\mathfrak{g}(H) \subseteq (\mathfrak{m}/\mathfrak{m}^2)^*$ since

$$H = \mathfrak{m} \oplus \mathbf{k}1, \quad f(1) = 0, \quad f(\mathfrak{m}^2) = 0$$

for any $f \in \mathfrak{g}(H)$. Thus there exists the canonical bilinear form

$$\langle \cdot, \cdot \rangle : \mathfrak{g}(H) \times \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathbf{k}, \quad (f, z) \mapsto \langle f, z \rangle = f(z). \tag{2.3}$$

Theorem 2.3. *The canonical bilinear form (2.3) is nondegenerate.*

Proof. Let $f \in \mathfrak{g}(H)$. If $\langle f, \mathfrak{m}/\mathfrak{m}^2\rangle = 0$ then $f = 0$ since $\mathfrak{g}(H) \subseteq (\mathfrak{m}/\mathfrak{m}^2)^*$.

Let $0 \neq x+\mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$. Express x by a linear combination of elements in the basis $\mathfrak{B}$. Then there exists an element $b \in \mathfrak{B}$ such that $b \in \mathfrak{m} \setminus \mathfrak{m}^2$ and the coefficient of b in the linear combination of x is nonzero. Let b^* be the dual element of b. For any homogeneous elements $y, z \in H$, we have

$$yz = (y - \eta(y)1)(z - \eta(z)1) + \eta(y)z + \eta(z)y - \eta(y)\eta(z)1.$$

Thus

$$b^*(yz) = \eta(y)b^*(z) + b^*(y)\eta(z).$$

Replacing ideals by G-graded ideals in the proof of [2, Theorem 1.3.1], we have $b^* \in \mathfrak{g}(H)$ by Lemma 2.2. Hence (2.3) is nondegenerate since $\langle b^*, x\rangle \neq 0$. □

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