

SHARP $L^p \rightarrow L^r$ ESTIMATES OF RESTRICTED AVERAGING OPERATORS OVER CURVES ON PLANES IN FINITE FIELDS

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ABSTRACT. Let \mathbb{F}_q^d be a d -dimensional vector space over a finite field \mathbb{F}_q with q elements. We endow the space \mathbb{F}_q^d with a normalized counting measure $d\mathbf{x}$. Let σ be a normalized surface measure on an algebraic variety V contained in the space $(\mathbb{F}_q^d, d\mathbf{x})$. We define the restricted averaging operator A_V by $A_V f(\mathbf{x}) = f * \sigma(\mathbf{x})$ for $\mathbf{x} \in V$, where $f : (\mathbb{F}_q^d, d\mathbf{x}) \rightarrow \mathbb{C}$. In this paper, we initially investigate $L^p \rightarrow L^r$ estimates of the restricted averaging operator A_V . As a main result, we obtain the optimal results on this problem in the case when the varieties V are any nondegenerate algebraic curves in two dimensional vector spaces over finite fields. The Fourier restriction estimates for curves on \mathbb{F}_q^2 play a crucial role in proving our results.

1. Introduction

In the Euclidean analysis, Young's inequality for a convolution function states that

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^s(\mathbb{R}^d)}$$

if $\frac{1}{r} = \frac{1}{p} + \frac{1}{s} - 1$ and $1 \leq p, r, s \leq \infty$. When the function g is replaced by the surface measure σ on a hypersurface $S \subset \mathbb{R}^d$, we have the following averaging problem over S : for which $1 \leq p, r \leq \infty$ does the averaging estimate below hold?

$$\|f * \sigma\|_{L^r(\mathbb{R}^d)} \leq C_{S,p,r,d} \|f\|_{L^p(\mathbb{R}^d)},$$

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where the constant $C_{S,p,r,d}$ is independent of the functions $f \in L^p(\mathbb{R}^d)$. This averaging problem has been well studied (see, for example, [2, 6, 8, 9]). As a variant of the averaging problem, one may ask us to determine $1 \leq p, r \leq \infty$ such that

$$\|f * \sigma\|_{L^r(S,\sigma)} \leq C_{S,p,r,d} \|f\|_{L^p(\mathbb{R}^d)},$$

where the constant $C_{S,p,r,d}$ is independent of the functions $f \in L^p(\mathbb{R}^d)$. We shall name this problem as the restricted averaging problem over S . In the Euclidean space, this problem has not been yet studied and we hope analysts to shed insight into this question.

The purpose of this paper is to construct and settle down the restricted averaging problems over algebraic curves on two dimensional vector spaces over finite fields. We begin by reviewing the notation in finite fields. Let \mathbb{F}_q^d be a d -dimensional vector space over the finite field \mathbb{F}_q with q element. Throughout this paper, we assume that the characteristic of \mathbb{F}_q is sufficiently large. We endow \mathbb{F}_q^d with the normalized counting measure $d\mathbf{x}$. Thus, if $f : (\mathbb{F}_q^d, d\mathbf{x}) \rightarrow \mathbb{C}$, then we have

$$\int_{\mathbb{F}_q^d} f(\mathbf{x}) d\mathbf{x} = q^{-d} \sum_{\mathbf{x} \in \mathbb{F}_q^d} f(\mathbf{x}).$$

Given an algebraic variety $V \subset (\mathbb{F}_q^d, d\mathbf{x})$, we shall endow the variety V with the normalized surface measure σ . Recall that the normalized surface measure σ on V is defined by the relation

$$\int_V f(\mathbf{x}) d\sigma(\mathbf{x}) = \frac{1}{|V|} \sum_{\mathbf{x} \in V} f(\mathbf{x}).$$

Notice that the normalized surface measure σ supported on V can be simply considered as a function on $(\mathbb{F}_q^d, d\mathbf{x})$ defined by

$$\sigma(\mathbf{x}) = \frac{q^d}{|V|} V(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{F}_q^d.$$

Throughout this paper, we write $V(x)$ for the characteristic function χ_V on the set V , and we denote by $|V|$ the cardinality of the set V .

1.1. Definition of the restricted averaging operator

As an analogue of the Euclidean averaging problem, Carbery-Stones-Wright ([1]) introduced and studied the averaging problem for algebraic

varieties in the finite field setting. The averaging problem for the variety $V \subset (\mathbb{F}_q^d, d\mathbf{x})$ is to decide exponents $1 \leq p, r \leq \infty$ such that

$$(1.1) \quad \|f * \sigma\|_{L^r(\mathbb{F}_q^d, d\mathbf{x})} \leq C \|f\|_{L^p(\mathbb{F}_q^d, d\mathbf{x})},$$

where the constant $C > 0$ is independent of both the finite field size q and the function $f : \mathbb{F}_q^d \rightarrow \mathbb{C}$. In addition, recall that for $\mathbf{x} \in (\mathbb{F}_q^d, d\mathbf{x})$,

$$f * \sigma(\mathbf{x}) = \int_V f(\mathbf{x} - \mathbf{y}) \, d\sigma(\mathbf{y}) := \frac{1}{|V|} \sum_{\mathbf{y} \in V} f(\mathbf{x} - \mathbf{y}).$$

In the finite field case, the averaging problems for several varieties have been well studied. In particular, the averaging problems for cones, spheres, and paraboloids have been completely solved (see [3, 5]). However, what happens to the inequality (1.1) if we replace $\|f * \sigma\|_{L^r(\mathbb{F}_q^d, d\mathbf{x})}$, the quantity of the left-hand side in (1.1), by the norm $\|f * \sigma\|_{L^r(V, d\sigma)}$? More precisely, we pose the following problem, which shall be named as the “restricted” averaging problem to an algebraic variety $V \subset (\mathbb{F}_q^d, d\mathbf{x})$.

PROBLEM 1.1. (Restricted averaging problem) Let σ be the normalized surface measure on the variety $V \subset (\mathbb{F}_q^d, d\mathbf{x})$. The restricted averaging problem to V is to determine $1 \leq p, r \leq \infty$ such that for some constant $C > 0$ independent of the field size q , the following inequality holds:

$$(1.2) \quad \|f * \sigma\|_{L^r(V, \sigma)} \leq C \|f\|_{L^p(\mathbb{F}_q^d, d\mathbf{x})} \text{ for all } f : \mathbb{F}_q^d \rightarrow \mathbb{C}.$$

In this paper, we shall provide complete solutions of Problem 1.1 in the case when the variety V is a curve on two dimensional vector spaces over finite fields and when it does not contain any line.

2. Statement of the main result

To precisely state and prove our main result, we need to introduce notations and definitions. For positive numbers A and B , we shall use $A \lesssim B$ if there is a constant $C > 0$ independent of the field size q such that $A \leq CB$. We also use $A \sim B$ to indicate that $A \lesssim B$ and $B \lesssim A$. We shall denote by A_V the restricted averaging operator to V . Namely, if σ is the normalized surface measure on the variety $V \subset (\mathbb{F}_q^d, d\mathbf{x})$ and $f : (\mathbb{F}_q^d, d\mathbf{x}) \rightarrow \mathbb{C}$, then we define $A_V f$ as the function $f * \sigma$ whose domain is restricted to the variety V . As usual, we denote by A_V^* the adjoint operator of the restricted averaging operator to V . Using the fact that

$$\langle A_V f, g \rangle_{L^2(V, \sigma)} = \langle f, A_V^* g \rangle_{L^2(\mathbb{F}_q^d, d\mathbf{x})},$$

it is not hard to see that the adjoint operator A_V^* is defined as

$$A_V^*g(\mathbf{y}) = \frac{q^d}{|V|^2} \sum_{\mathbf{x} \in V} V(\mathbf{x} - \mathbf{y})g(\mathbf{x})$$

where $g : (V, \sigma) \rightarrow \mathbb{C}$ and $\mathbf{y} \in (\mathbb{F}_q^d, d\mathbf{x})$. By duality, the inequality (1.2) in Problem 1.1 is same as the following:

$$(2.1) \quad \|A_V^*g\|_{L^{p'}(\mathbb{F}_q^d, d\mathbf{x})} \leq C\|g\|_{L^{r'}(V, \sigma)} \text{ for all } g : V \rightarrow \mathbb{C},$$

where $p' = p/(p - 1)$ and $r' = r/(r - 1)$.

DEFINITION 2.1. We write $A_V(p \rightarrow r) \lesssim 1$ if the inequality (1.2) in Problem 1.1 holds. We also use $A_V^*(r' \rightarrow p') \lesssim 1$ to indicate that the inequality (2.1) holds.

By duality, notice that $A_V(p \rightarrow r) \lesssim 1 \iff A_V^*(r' \rightarrow p') \lesssim 1$. Next, we define a nondegenerate curve on \mathbb{F}_q^2 on which we shall work.

DEFINITION 2.2. Given a polynomial $Q \in \mathbb{F}_q[\mathbf{x}]$ with $\mathbf{x} \in \mathbb{F}_q^2$, let $V = \{\mathbf{x} \in \mathbb{F}_q^2 : Q(\mathbf{x}) = 0\}$ be an algebraic curve. We say that the algebraic curve V is nondegenerate if $|V| \sim q$ and the polynomial $Q(\mathbf{x})$ does not have any linear factor.

Our main result is as follows:

THEOREM 2.3. *Let σ be the normalized surface measure on a nondegenerate curve $V \subset \mathbb{F}_q^2$. Then we have $A_V(p \rightarrow r) \lesssim 1$ if and only if $(1/p, 1/r)$ lies in the convex hull of points $(0, 0), (0, 1), (1/2, 1)$, and $(1/2, 1/2)$.*

In the remaining parts of this paper, we focus on proving Theorem 2.3 which provides us of the complete answer to the restricted averaging problem to any nondegenerate curve on \mathbb{F}_q^2 . As we will see, the proof of Theorem 2.3 is partially based on the well known extension estimate (or the restriction estimate) for a nondegenerate curve on \mathbb{F}_q^2 . In Section 3, we summarize useful information about the extension problems for curves in two dimensional space \mathbb{F}_q^2 . Finally, the complete proof of Theorem 2.3 will be given in Section 4.

3. Review of extension problems for curves

In the finite field setting, Mockenhaupt and Tao ([7]) recently formulated the extension problem for various varieties . In particular, they completely solved the problem for the parabola in two dimensions and

the cone in three dimensions. Koh and Shen ([4]) extended the sharp results for the parabola to general nondegenerate curves in two dimensions. Here, we introduce and use their results. Recall that we denote by $(\mathbb{F}_q^2, d\mathbf{x})$ the two dimensional space \mathbb{F}_q^2 equipped with the normalized counting measure $d\mathbf{x}$. Let (V, σ) be a nondegenerate curve on $(\mathbb{F}_q^2, d\mathbf{x})$, where σ denotes the normalized surface measure on V . Since the dual space of \mathbb{F}_q^2 is isomorphic to the space \mathbb{F}_q^2 as an abstract group, we can identify the space \mathbb{F}_q^2 with its dual space. For this reason, we may write \mathbb{F}_q^2 to indicate both the space \mathbb{F}_q^2 and its dual space. However, an important point in the discrete Fourier analysis is that the space \mathbb{F}_q^2 is endowed with the normalized counting measure $d\mathbf{x}$ but its dual space is endowed with the counting measure which shall be denoted by $d\mathbf{m}$. In summary, we shall use the following notation.

DEFINITION 3.1. $(\mathbb{F}_q^2, d\mathbf{x})$ means the space \mathbb{F}_q^2 with the normalized counting measure $d\mathbf{x}$. On the other hand, we shall use $(\mathbb{F}_q^2, d\mathbf{m})$ to indicate the dual space of $(\mathbb{F}_q^2, d\mathbf{x})$, where $d\mathbf{m}$ is defined as the counting measure. For simplicity, we use the notation $\mathbf{m} \in \mathbb{F}_q^2$ to indicate that \mathbf{m} is an element of the dual space $(\mathbb{F}_q^2, d\mathbf{m})$ and we write the notation $\mathbf{x} \in \mathbb{F}_q^2$ for an element in the space $(\mathbb{F}_q^2, d\mathbf{x})$.

Given a function $g : (\mathbb{F}_q^2, d\mathbf{m}) \rightarrow \mathbb{C}$, the Fourier transform of g , denoted by \widehat{g} , is actually defined on the space $(\mathbb{F}_q^2, d\mathbf{x})$. Namely, we have

$$\widehat{g}(\mathbf{x}) = \int_{\mathbb{F}_q^2} g(\mathbf{m})\chi(-\mathbf{m} \cdot \mathbf{x}) d\mathbf{m} = \sum_{\mathbf{m} \in \mathbb{F}_q^2} g(\mathbf{m})\chi(-\mathbf{m} \cdot \mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{F}_q^2,$$

where χ denotes a nontrivial additive character of \mathbb{F}_q . On the other hand, given a function $f : (\mathbb{F}_q^2, d\mathbf{x}) \rightarrow \mathbb{C}$, the inverse Fourier transform f^\vee is defined by

$$f^\vee(\mathbf{m}) = \int_{\mathbb{F}_q^2} f(\mathbf{x})\chi(\mathbf{m} \cdot \mathbf{x}) d\mathbf{x} = q^{-d} \sum_{\mathbf{x} \in \mathbb{F}_q^2} f(\mathbf{x})\chi(\mathbf{m} \cdot \mathbf{x}) \text{ for } \mathbf{m} \in \mathbb{F}_q^2.$$

In addition, if σ is the normalized surface measure on a curve $V \subset (\mathbb{F}_q^2, d\mathbf{x})$ and $f : (\mathbb{F}_q^2, d\mathbf{x}) \rightarrow \mathbb{C}$, then the inverse Fourier transform of the measure $f\sigma$ is defined by

$$(f\sigma)^\vee(\mathbf{m}) = \int_{\mathbf{x} \in V} f(\mathbf{x})\chi(\mathbf{m} \cdot \mathbf{x}) d\sigma(\mathbf{x}) = \frac{1}{|V|} \sum_{\mathbf{x} \in V} f(\mathbf{x})\chi(\mathbf{m} \cdot \mathbf{x}) \text{ for } \mathbf{m} \in \mathbb{F}_q^2.$$

With the notation above, the extension problem for the curve $V \subset (\mathbb{F}_q^2, d\mathbf{x})$ is defined as follows.

PROBLEM 3.2. (Extension problem) Determine $1 \leq p, r \leq \infty$ such that

$$\|(f\sigma)^\vee\|_{L^r(\mathbb{F}_q^2, d\mathbf{m})} \lesssim \|f\|_{L^p(V, \sigma)} \text{ for all } f : V \rightarrow \mathbb{C}.$$

In [4], Koh and Shen obtained the following result on the extension problem for nondegenerate curves $V \subset (\mathbb{F}_q^2, d\mathbf{x})$.

LEMMA 3.3. *Let σ be the normalized surface measure on a nondegenerate curve $V \subset (\mathbb{F}_q^2, d\mathbf{x})$. Then we have*

$$\|(f\sigma)^\vee\|_{L^4(\mathbb{F}_q^2, d\mathbf{m})} \lesssim \|f\|_{L^2(V, \sigma)} \text{ for all } f : V \rightarrow \mathbb{C}.$$

By duality, Lemma 3.3 implies the following restriction estimate for a nondegenerate curve $V \subset (\mathbb{F}_q^2, d\mathbf{x})$.

LEMMA 3.4. *Let σ be the normalized surface measure on a nondegenerate curve $V \subset (\mathbb{F}_q^2, d\mathbf{x})$. Then we have*

$$\|\widehat{g}\|_{L^2(V, \sigma)} \lesssim \|g\|_{L^{4/3}(\mathbb{F}_q^2, d\mathbf{m})} \text{ for all } g : \mathbb{F}_q^2 \rightarrow \mathbb{C}.$$

4. Proof of the main theorem (Theorem 2.3)

We first prove the necessary parts for $A_V(p \rightarrow r) \lesssim 1$. Assume that $A_V(p \rightarrow r) \lesssim 1$ for $1 \leq p, r \leq \infty$. Then we must have

$$(4.1) \quad \|f * \sigma\|_{L^r(V, \sigma)} \lesssim \|f\|_{L^p(\mathbb{F}_q^2, d\mathbf{x})} \text{ for all } f : \mathbb{F}_q^2 \rightarrow \mathbb{C}.$$

By duality, we also see that

$$(4.2) \quad \|A_V^* g\|_{L^{p'}(\mathbb{F}_q^2, d\mathbf{x})} \lesssim \|g\|_{L^{r'}(V, \sigma)} \text{ for all } g : V \rightarrow \mathbb{C},$$

where the adjoint operator A_V^* is given by

$$A_V^* g(\mathbf{y}) = \frac{q^2}{|V|^2} \sum_{\mathbf{x} \in V} V(\mathbf{x} - \mathbf{y})g(\mathbf{x}).$$

Taking $f = \delta_0$ where we define $\delta_0(\mathbf{x}) = 1$ for $\mathbf{x} = (0, 0)$ and is 0 otherwise, it follows from (4.1) that

$$\|\delta_0 * \sigma\|_{L^r(V, \sigma)} \lesssim \|\delta_0\|_{L^p(\mathbb{F}_q^2, d\mathbf{x})}.$$

Since $\|\delta_0 * \sigma\|_{L^r(V, \sigma)} = |V|^{-1} \sim q^{-1}$ and $\|\delta_0\|_{L^p(\mathbb{F}_q^2, d\mathbf{x})} = q^{-2/p}$, we must have

$$(4.3) \quad \frac{1}{p} \leq \frac{1}{2}.$$

On the other hand, for some $\mathbf{w} \in V$, define $\delta_{\mathbf{w}}(\mathbf{x}) = 1$ if $\mathbf{x} = \mathbf{w}$ and is 0 otherwise. Now, taking $g = \delta_{\mathbf{w}}$ in (4.2) yields

$$(4.4) \quad \|A_V^* \delta_{\mathbf{w}}\|_{L^{p'}(\mathbb{F}_q^2, d\mathbf{x})} \lesssim \|\delta_{\mathbf{w}}\|_{L^{r'}(V, \sigma)}.$$

Since $|V| \sim q$, we obtain by the definition of $\delta_{\mathbf{w}}$ that

$$(4.5) \quad \|\delta_{\mathbf{w}}\|_{L^{r'}(V, \sigma)} = |V|^{-1/r'} \sim q^{-1/r'}.$$

To estimate $\|A_V^* \delta_{\mathbf{w}}\|_{L^{p'}(\mathbb{F}_q^2, d\mathbf{x})}$, we first observe that if $\mathbf{x} \in (\mathbb{F}_q^2, d\mathbf{x})$, then

$$A_V^* \delta_{\mathbf{w}}(\mathbf{x}) = \frac{q^2}{|V|^2} \sum_{\mathbf{y} \in V} V(\mathbf{y} - \mathbf{x}) \delta_{\mathbf{w}}(\mathbf{y}) = \frac{q^2}{|V|^2} V(\mathbf{w} - \mathbf{x}).$$

From this observation, we see

$$\begin{aligned} \|A_V^* \delta_{\mathbf{w}}\|_{L^{p'}(\mathbb{F}_q^2, d\mathbf{x})} &= \left(q^{-2} \sum_{\mathbf{x} \in \mathbb{F}_q^2} |q^2 |V|^{-2} V(\mathbf{w} - \mathbf{x})|^{p'} \right)^{1/p'} \\ &= \left(q^{-2} \sum_{\mathbf{x} \in V} (q^2 |V|^{-2})^{p'} \right)^{1/p'} = q^{2-2/p'} |V|^{-2+1/p'}. \end{aligned}$$

Since $|V| \sim q$, we obtain

$$\|A_V^* \delta_{\mathbf{w}}\|_{L^{p'}(\mathbb{F}_q^2, d\mathbf{x})} \sim q^{-1/p'}.$$

From this estimation, (4.5), and (4.4), we must have $q^{-1/p'} \lesssim q^{-1/r'}$. This clearly implies that we must have

$$\frac{1}{p} \leq \frac{1}{r}.$$

Combining this with (4.3), we can conclude that if $A_V(p \rightarrow r) \lesssim 1$, then $(1/p, 1/r)$ is contained in the convex hull of points $(0, 0)$, $(0, 1)$, $(1/2, 1)$, and $(1/2, 1/2)$.

Next, we prove the sufficient conditions for $A_V(p \rightarrow r) \lesssim 1$. First, since σ is the normalized surface measure on V , it follows that if $1 \leq r_1 \leq r_2 \leq \infty$, then

$$\|f * \sigma\|_{L^{r_1}(V, \sigma)} \leq \|f * \sigma\|_{L^{r_2}(V, \sigma)}.$$

This clearly implies that if $A_V(p \rightarrow r_2) \lesssim 1$, then $A_V(p \rightarrow r_1) \lesssim 1$ for $1 \leq p \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. From this fact, it suffices to prove that $A_V(p \rightarrow r) \lesssim 1$ whenever $(1/p, 1/r)$ lies on the line segment joining $(0, 0)$ and $(1/2, 1/2)$. By the interpolation theorem, it will be enough to

prove that $A_V(\infty \rightarrow \infty) \lesssim 1$ and $A_V(2 \rightarrow 2) \lesssim 1$. In other words, it remains to prove the following two inequalities:

$$(4.6) \quad \|f * \sigma\|_{L^\infty(V,\sigma)} \lesssim \|f\|_{L^\infty(\mathbb{F}_q^2,d\mathbf{x})} \text{ for all } f : \mathbb{F}_q^2 \rightarrow \mathbb{C}.$$

and

$$(4.7) \quad \|f * \sigma\|_{L^2(V,\sigma)} \lesssim \|f\|_{L^2(\mathbb{F}_q^2,d\mathbf{x})} \text{ for all } f : \mathbb{F}_q^2 \rightarrow \mathbb{C}.$$

To obtain the inequality (4.6), we have to recall that

$$\|f * \sigma\|_{L^\infty(V,\sigma)} = \max_{\mathbf{x} \in V} |f * \sigma(\mathbf{x})| \text{ and } \|f\|_{L^\infty(\mathbb{F}_q^2,d\mathbf{x})} = \max_{\mathbf{x} \in \mathbb{F}_q^2} |f(\mathbf{x})|.$$

Then the inequality (4.6) follows immediately from the observation that for each $\mathbf{x} \in V$,

$$|f * \sigma(\mathbf{x})| = \left| \frac{1}{|V|} \sum_{\mathbf{y} \in V} f(\mathbf{x} - \mathbf{y}) \right| \leq \frac{1}{|V|} \sum_{\mathbf{y} \in V} |f(\mathbf{x} - \mathbf{y})| \leq \|f\|_{L^\infty(\mathbb{F}_q^2,d\mathbf{x})}.$$

Hence, to complete the proof of Theorem 2.3, we only need to justify the inequality (4.7). To do this, we write

$$\|f * \sigma\|_{L^2(V,\sigma)} = \|(\widehat{f^\vee \sigma^\vee})\|_{L^2(V,\sigma)}.$$

By Lemma 3.4 and Hölder’s inequality, it follows

$$\|f * \sigma\|_{L^2(V,\sigma)} \lesssim \|f^\vee \sigma^\vee\|_{L^{4/3}(\mathbb{F}_q^2,d\mathbf{m})} \leq \|f^\vee\|_{L^2(\mathbb{F}_q^2,d\mathbf{m})} \|\sigma^\vee\|_{L^4(\mathbb{F}_q^2,d\mathbf{m})}.$$

By the Plancherel theorem $\|f^\vee\|_{L^2(\mathbb{F}_q^2,d\mathbf{m})} = \|f\|_{L^2(\mathbb{F}_q^2,d\mathbf{x})}$ and by Lemma 3.3 we see that $\|\sigma^\vee\|_{L^4(\mathbb{F}_q^2,d\mathbf{m})} \lesssim \|1\|_{L^2(V,\sigma)} = 1$. Therefore, the inequality (4.7) follows and we complete the proof of Theorem 2.3.

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