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A DECOMPOSITION OF THE CURVATURE TENSOR ON $SU(3)/T(k,l)$ WITH A $SU(3)$-INVARIANT METRIC

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Abstract. In this paper, we decompose the curvature tensor (field) on the homogeneous Riemannian manifold $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant Riemannian metric into three curvature-like tensor fields, and investigate geometric properties.

1. Introduction

Let $(V, < \ , \ >)$ be an n-dimensional real inner product space. In this paper, we use the notion of a curvature-like tensor of type $(1,3)$ on $(V, < \ , \ >)$ (cf. (2.1)). We put

$$\mathfrak{L}(V) := \{L \mid L \text{ is a curvature-like tensor on } (V, < \ , \ >)\},$$

$$\mathfrak{L}_1(V) := \{L \in \mathfrak{L}(V) \mid L(u,v) = c\ u \wedge v \text{ for } u, v \in V \text{ and some } c \in \mathbb{R}\},$$

$$\mathfrak{L}_\omega(V) := \{L \in \mathfrak{L}(V) \mid \text{the Ricci tensor } Ric_L \text{ of } L \text{ is zero}\},$$

$$\mathfrak{L}_2(V) := \{L \in \mathfrak{L}_1(V)^\perp \mid \ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_\omega(V)\}.$$

Then $\mathfrak{L}(V)$ is decomposed into the orthogonal direct sum $\mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V)$. Let $L = L_1 + L_\omega + L_2$ $(L \in \mathfrak{L}(V))$ be the decomposition corresponding to $\mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V)$. The component L_ω of $L \in \mathfrak{L}(V)$ is said to be the *Weyl tensor* of L. The curvature-like tensors L_1, L_ω, L_2 of $L = L_1 + L_\omega + L_2 \in \mathfrak{L}(V)$ are given in terms of the Ricci tensor Ric_L and the scalar curvature S_L of L (cf. Lemma 2.1).

In this paper, using Lemma 2.1 we decompose the curvature tensor on the homogeneous Riemannian manifold $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ into three curvature-like tensor fields. On the manifold $SU(3)/T(k,l)$, we deal with an arbitrary $SU(3)$-invariant Riemannian metric $g = g_{(\lambda_1,\lambda_2,\lambda_3)}$.

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Geometric properties on $SU(3)/T(k,l)$ have been studied by many mathematicians (cf. [1, 6, 9, 10]).

Now, let R be the curvature tensor (field) on the homogeneous manifold $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, and $R = R^{(1)} + R^{\omega} + R^{(2)}$ the orthogonal decomposition of the curvature tensor R corresponding to

$$\mathfrak{L}(T_o(G/H)) = \mathfrak{L}_1(T_o(G/H)) \oplus \mathfrak{L}_\omega(T_o(G/H)) \oplus \mathfrak{L}_2(T_o(G/H))$$

(cf. Lemma 2.1), where $G := SU(3)$, $H := T(k,l)$ and $O := \{T(k,l)\}$.

Let $\mathfrak{m}$ be the subspace of $\mathfrak{su}(3)$ such that

$$B(\mathfrak{m}, \mathfrak{t}(k,l)) = 0 \text{ and } \mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \quad (h \in T(k,l)),$$

where $\mathfrak{su}(3)$ is the Lie algebra of $SU(3)$, B is the negative of the Killing form of $\mathfrak{su}(3)$, $\mathfrak{t}(k,l)$ is the Lie algebra of $T(k,l)$, and Ad is the adjoint representation of $SU(3)$ on $\mathfrak{su}(3)$.

In this paper, we represent the curvature-like tensors $R^{(1)}$, R^{ω} and $R^{(2)}$ in the orthogonal decomposition $R = R^{(1)} + R^{\omega} + R^{(2)}$ $(\in \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V))$ of the curvature tensor R on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ for $(k,l) \in D$, where

$$D := \mathbb{Z}^2 \setminus \{(0,t),(t,0),(t,t),(t,-t),(t,-2t),(2t,-t) \mid t \in \mathbb{R}\}$$

(cf. Theorem 4.3). And then, under the condition $(k,l) \in D \subset \mathbb{Z}^2$, we obtain the Ricci tensor $Ric^{(2)}$ of the component $R^{(2)}$ of the curvature $R = R^{(1)} + R^{\omega} + R^{(2)}$ on the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ (cf. Corollary 4.4). Furthermore, we estimate the Ricci curvature $r^{(2)}$ of the curvature-like tensor $R^{(2)}$ (cf. Proposition 4.5).

2. Preliminaries

Let $(V, < , >)$ be an n-dimensional real inner product space and $\mathfrak{gl}(V)$ the vector space of all endomorphisms of V. We denote by $\mathfrak{L}(V)$ the vector space of all tensors of type $(1,3)$ on V which satisfy the following properties:

$$L : V \times V \ \rightarrow \ \mathfrak{gl}(V)$$

is an $\mathbb{R}$-bilinear map such that, for all $v_1, v_2, v_3, v_4 \in V$,

(2.1)

$$< L(v_1,v_2)v_3, v_4 > - < L(v_2,v_1)v_3, v_4 >= - < L(v_1,v_2)v_4, v_3 >,$$

$$< L(v_1,v_2)v_3, v_4 > + < L(v_2,v_3)v_1, v_4 > + < L(v_3,v_1)v_2, v_4 >= 0.$$

A tensor $L \in \mathfrak{L}(V)$ (of type $(1,3)$ on $(V, < \ , \ >)$ which satisfies the condition (2.1)) is called a *curvature-like tensor* (cf. [3. 4]). If $L \in \mathfrak{L}(V)$, then we get from (2.1)

$$(2.2) \qquad < L(v_1, v_2)v_3, v_4 > = < L(v_3, v_4)v_1, v_2 > \quad (v_1, v_2, v_3, v_4 \in V).$$

From now on, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $(V, < \ , \ >)$. The *Ricci tensor* Ric_L of type $(0,2)$ with respect to a curvature-like tensor L on V is defined by

$$(2.3) \qquad Ric_L(v,w) := \sum_{i=1}^{n} < L(e_i, v)w, e_i > \quad (v, w \in V).$$

The *Ricci tensor* Ric_L of type $(1,1)$ with respect to $L \in \mathfrak{L}(V)$ is defined by

$$(2.4) \qquad < \mathrm{Ric}_L(v), w > = Ric_L(v,w) \quad (v, w \in V).$$

For $L \in \mathfrak{L}(V)$, we obtain from (2.1) $\sim$ (2.4)

$$Ric_L(v,w) = < \mathrm{Ric}_L(v), w > = Ric_L(w,v) = < \mathrm{Ric}_L(w), v >$$

for $v, w \in V$.

The trace of Ric_L for $L \in \mathfrak{L}(V)$

$$(2.5) \qquad S_L := \sum_{i=1}^{n} < \mathrm{Ric}_L(e_i), e_i > = \sum_{i,j=1}^{n} < L(e_j, e_i)e_i, e_j >$$

is called the *scalar curvature* with respect to $L \in \mathfrak{L}(V)$. The *sectional curvature* $K_L(\sigma)$ $(L \in \mathfrak{L}(V))$ for each plane $\sigma = \{v, w\}_{\mathbb{R}} (\subset V)$ is defined by

$$K_L(\sigma) = \frac{< L(v,w)w, v >}{< v, v >< w, w > - < v, w >^2}.$$

In general, the inner product $< \ , \ >$ on $\mathfrak{L}(V)$ is defined by

$$(2.6) \qquad < L, L' > = \sum_{i,j,k,l=1}^{n} L_{ijk}{}^{l} \cdot L'_{ijk}{}^{l},$$

where $L_{ijk}{}^{l} = < L(e_i, e_j)e_k, e_l >$.

Let $\mathfrak{L}_1(V)$ be the subspace of $\mathfrak{L}(V)$ which consists of all elements $L \in \mathfrak{L}(V)$ such that

$$L(v,w) = c \ v \wedge w \text{ for } v, w \in V \text{ and some } c \in \mathbb{R}.$$

Here $v \wedge w$ is an element of $\mathfrak{gl}(V)$ which is defined by

$$v \wedge w : V \ni z \longmapsto (v \wedge w)(z) = < w, z > v - < v, z > w \in V.$$

We put

$$\mathfrak{L}_1(V)^\perp := \{L \in \mathfrak{L}(V) \mid \ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_1(V)\}.$$

Then $\mathfrak{L}_1(V)^\perp = \{L \in \mathfrak{L}(V) \mid S_L = 0\}$. In fact, for $L \in \mathfrak{L}(V)$ and $L' \in \mathfrak{L}_1(V)$, we get from (2.5) and (2.6), and the definition of $\mathfrak{L}_1(V)$

$$< L, L' >= 2c\ S_L, \tag{2.7}$$

where $L'(v, w) = cv \wedge w$ for some $c \in \mathbb{R}$. From (2.7), we obtain the following;

$$\begin{aligned} < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_1(V) &\iff 2c\ S_L = 0 \text{ for all } c \in \mathbb{R} \\ &\iff S_L = 0. \end{aligned}$$

Putting

$$\{L \in \mathfrak{L}_1(V)^\perp \mid \mathrm{Ric}_L = 0\} =: \mathfrak{L}_\omega(V)$$

and

$$\{L \in \mathfrak{L}_1(V)^\perp \mid \ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_\omega(V)\} =: \mathfrak{L}_2(V),$$

we get the orthogonal direct sum decomposition of $\mathfrak{L}(V)$ as follows:

$$\mathfrak{L}(V) = \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V).$$

Putting together the results above, we obtain the following (cf. [5, Chapter 5])

Lemma 2.1. *Let V be an $n(\geq 3)$-dimensional real inner product space and $L \in \mathfrak{L}(V)$. Then components $L_1 \in \mathfrak{L}_1(V)$, $L_\omega \in \mathfrak{L}_\omega(V)$ and $L_2 \in \mathfrak{L}_2(V)$ of $L(= L_1 + L_\omega + L_2)$ are given as follows:*

$$\begin{aligned} L_1(u, v) =& \frac{S_L}{n(n-1)}\ u \wedge v, \\ L_2(u, v) =& \frac{1}{n-2}\left\{\mathrm{Ric}_L(u) \wedge v + u \wedge \mathrm{Ric}_L(v) - \frac{2S_L}{n}\ u \wedge v\right\}, \\ L_\omega(u, v) =& L(u.v) - \frac{1}{n-2}\{\mathrm{Ric}_L(u) \wedge v + u \wedge \mathrm{Ric}_L(v)\} \\ &+ \frac{S_L}{(n-1)(n-2)}\ u \wedge v. \end{aligned} \tag{2.8}$$

Proof. The fact that L_1, L_2, L_ω appeared in (2.8) belong to $\mathfrak{L}(V)$ is easily verified. And, $L = L_1 + L_\omega + L_2$. Moreover from straightforward computations we get

$$S_{L_2} = 0, \quad \mathrm{Ric}_{L_\omega} = 0, \quad < L_2, L_\omega >= 0.$$

Thus the proof of Lemma 2.1 is completed. □

3. Inequivalent isotropy irreducible representations in $SU(3)/T(k,l)$

3.1. Isotropy irreducible representations

Let G be a compact connected semisimple Lie group and H a closed subgroup of G. The homogeneous space G/H is *reductive*, that is, in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sun of vector subspaces) and $\mathrm{Ad}(h)\ \mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_o of H and $\mathrm{Ad}(h)$ denotes the adjoint representation of H in $\mathfrak{m}$.

Let τ_x $(x \in G)$ be the transformation of G/H which is induced by x. Taking differentials of τ_x at $p_o := \{H\}$ $(\in G/H)$, we obtain the fact that the tangent space $T_{p_o}(G/H) = \mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. The homogeneous space G/H is said to be *isotropy irreducible* if $(T_{p_o}(G/H), \mathrm{Ad}(H))$ is an irreducible representation.

3.2. Inequivalent isotropy irreducible summands in $SU(3)/T(k,l)$

Here and from now on, without further specification, we use the following notations:

$$G = SU(3),\quad \mathfrak{g} : \text{the Lie algebra of } SU(3),\quad i = \sqrt{-1},$$
$$T = T(k,l) = \{diag[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta} \mid \theta \in \mathbb{R}\} \text{ for } (k,l) \in \mathbb{Z}^2 \text{ and } |k| + |l| \neq 0,$$
$$\mathfrak{t}(k,l) : \text{the Lie algebra of } T(k,l),\quad \gamma = k^2 + kl + l^2,$$
$$(X,Y)_0 = B(X,Y) = -6\ Trace(XY),\ X, Y \in \mathfrak{g} : \text{the negative of the Killing form of } \mathfrak{g}.$$

Let E_{ij} be a real 3×3 matrix with 1 on entry (i,j) and 0 elsewhere. And we put

$$(3.1)\qquad \begin{aligned} X_1 &= \frac{1}{\sqrt{12}}(E_{12} - E_{21}), & X_2 &= \frac{i}{\sqrt{12}}(E_{12} + E_{21}),\\ X_3 &= \frac{1}{\sqrt{12}}(E_{13} - E_{31}), & X_4 &= \frac{i}{\sqrt{12}}(E_{13} + E_{31}),\\ X_5 &= \frac{1}{\sqrt{12}}(E_{23} - E_{32}), & X_6 &= \frac{i}{\sqrt{12}}(E_{23} + E_{32}),\end{aligned}$$

$$X_7 = \frac{i}{\sqrt{36\gamma}}\ diag[(k+2l), -(2k+l), (k-l)],$$
$$X_8 = \frac{i}{\sqrt{12\gamma}}\ diag[k, l, -(k+l)].$$

Then

$$\{X_1, \cdots, X_7\} \quad (\text{resp. } \{X_8\})$$

is an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{t}(k,l)$) with respect to $(\cdot\ ,\ \cdot)_0$ such that

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{t}(k,l) \text{ and } (\mathfrak{m}, \mathfrak{t}(k,l))_0 = 0.$$

If we put $\{X_1, X_2\}_{\mathbb{R}} = \mathfrak{m}_1$, $\{X_3, X_4\}_{\mathbb{R}} = \mathfrak{m}_2$, $\{X_5, X_6\}_{\mathbb{R}} = \mathfrak{m}_3$, and $\{X_7\}_{\mathbb{R}} = \mathfrak{m}_4$, then $\mathfrak{m}_i$ are irreducible $\mathrm{Ad}(T)$-representation spaces.

In general, two representations (μ_1, V_1) and (μ_2, V_2) of a Lie group G are called *equivalent* if there exists a linear isomorphism ρ of V_1 onto V_2 such that $\rho \circ \mu_1(x) = \mu_2(x) \circ \rho$ for all $x \in G$.

Park obtained the following

THEOREM 3.1. ([9]) *Assume that* $|k| + |l| \neq 0$ $(k, l \in \mathbb{Z})$. *Then a necessary and sufficient condition for* $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l)))$ $(i = 1, 2, 3, 4)$ *to be mutually inequivalent is*

$$k \neq 0, \quad l \neq 0, \quad k \neq \pm l, \quad k \neq -2l \ \textit{ and } \ l \neq -2k.$$

4. A decomposition of the curvature tensor on $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant Riemannian metric

4.1. The curvature tensor field on a homogeneous Riemannian space

Let G be a compact connected semisimple Lie group and H a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G and H, respectively. Let B be the negative of the Killing form of $\mathfrak{g}$. We consider the $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $B(\mathfrak{h}, \mathfrak{m}) = 0$. Then the set of G-invariant symmetric covariant 2-tensor fields on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant Riemannian metrics on G/H is identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ (cf. [2, 5, 8, 9]).

Let $< \ , \ >$ be an inner product which is invariant with respect to $\mathrm{Ad}(H)$ on $\mathfrak{m}$, where Ad denotes the adjoint representation of H in $\mathfrak{g}$.

This inner product $< \, , \, >$ determines a G-invariant Riemannian metric $g_{<,>}$ on G/H.

For the sake of the calculus, we take a neighborhood V of the identity element e in G and a subset N (resp. N_H) of G (resp. H) in such a way that

(i) $N = V \cap \exp(\mathfrak{m}), \;\; N_H = V \cap \exp(\mathfrak{h})$,

(ii) the map $N \times N_H \ni (c,h) \mapsto ch \in N \cdot N_H$ is a diffeomorphism,

(iii) the projection π of G onto G/H is a diffeomorphism of N onto a neighborhood $\pi(N)$ of the origin $\{H\}$ in G/H. Here, $\{\exp(tX) \mid t \in \mathbb{R}\}$ for $X \in \mathfrak{g}$ is a 1-parameter subgroup of G.

Now for an element $X \in \mathfrak{m}$, we define a vector field X^* on the neighborhood $\pi(N)$ of $\{H\}$ in G/H by

$$X^*_{\pi(c)} := (\tau_c)_* X_{\{H\}} \in T_{\pi(c)} G/H \quad (c \in N),$$

where τ_c denotes the transformation of G/H which is induced by c. Let $\{X_i\}_i$ be an orthonormal basis of the inner product space $(\mathfrak{m}, < \, , \, >)$. Then $\{X_i\}_i$ is an orthonormal frame on $\pi(N) (\subset G/H)$.

On the other hand, the connection function α (cf. [7, p.43]) on $\mathfrak{m} \times \mathfrak{m}$ corresponding to the invariant Riemannian connection of $(G/H, g_{< \, , \, >})$ is given as follows (cf. [7, p.52]):

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y) \quad (X, Y \in \mathfrak{m}) \, ,$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > \; = \; < [Z,X]_{\mathfrak{m}}, Y > + < X, [Z,Y]_{\mathfrak{m}} >$$

for $X, Y, Z \in \mathfrak{m}$, and $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let ∇ be the Levi-Civita connection on the Riemannian manifold $(G/H, g_{< \, , \, >})$. Then on $\pi(N)$ $(\nabla_{X^*} Y^*)_{\{H\}} = \alpha(X,Y) \;\; (X, Y \in \mathfrak{m})$. Moreover, the expression for the value at $p_o := \{H\} (\in G/H)$ of the curvature tensor field is as follows (cf. [7, p.47]):

$$\begin{aligned} R(X,Y)Z = \; & \alpha(X, \alpha(Y,Z)) - \alpha(Y, \alpha(X,Z)) \\ & - \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z] \quad (X,Y,Z \in \mathfrak{m}), \end{aligned} \tag{4.1}$$

where $X_{\mathfrak{m}}$ (resp. $X_{\mathfrak{h}}$) denotes the $\mathfrak{m}$ - component (resp. $\mathfrak{h}$ -component) of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

In general, the Ricci tensor field Ric of type (0,2) on a Riemannian manifold (M, g) is defined by

$$Ric(Y,Z) = Trace \, \{X \mapsto R(X,Y)Z\} \quad (X, Y, Z \in \mathfrak{X}(M)). \tag{4.2}$$

Let $\{Y_j\}_j$ be an orthonormal basis of the inner product $(\mathfrak{m}, <\ ,\ >)$. Since the group G is unimodular, we obtain the fact (cf. [2, p.184]) that

$$\sum_j U(Y_j, Y_j) = 0. \tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain the following expression (cf. [2, p.184-185]) for the value at p_o of the Ricci tensor field Ric on $(G/H, g_{<\ ,\ >})$:

$$\begin{aligned} Ric(Y,Y) = &-\frac{1}{2}\sum_j < [Y, Y_j]_{\mathfrak{m}}, [Y, Y_j]_{\mathfrak{m}} > + \frac{1}{2}B(Y,Y) \\ &+ \frac{1}{4}\sum_{i,j} < [Y_i, Y_j]_{\mathfrak{m}}, Y >^2 \end{aligned} \tag{4.4}$$

for $Y \in \mathfrak{m}$, where B is the negative of the Killing form of the Lie algebra $\mathfrak{g}$.

4.2. Ricci tensor fields on inequivalent isotropy irreducible homogeneous spaces

We retain the notation as in Section 4.1. The set of G-invariant symmetric tensor fields of type $(0, 2)$ on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant metrics on G/H is identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$.

Let $(\ ,\ )_o$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ such that $(\mathfrak{m}, \mathfrak{h})_o = 0$. For the sake of simplicity, we put $(\ ,\ )_o =: B$. Let $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_q$ be an orthogonal $\mathrm{Ad}(H)$-invariant decomposition of the space $(\mathfrak{m}, B)$ such that $\mathrm{Ad}(H)_{\mathfrak{m}_i}$ is irreducible for $i = 1, \ldots, q$, and assume that $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent irreducible representations. Then, the space of G-invariant symmetric tensor fields of type $(0, 2)$ on G/H is given by

$$\{\lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1, \ldots, \lambda_q \in \mathbb{R}\},$$

and the space of G-invariant Riemannian metrics on G/H is given by

$$\{\lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1 > 0, \ldots, \lambda_q > 0\}. \tag{4.5}$$

In fact, for an arbitrarily given $\mathrm{Ad}(H)$-invariant inner product $<\ ,\ >$ on $\mathfrak{m}$, we have $<\ ,\ >|_{\mathfrak{m}_i} = \lambda_i B|_{\mathfrak{m}_i}$ on each $\mathfrak{m}_i$ by the help of Shur's lemma ([cf. [12, 13]), and $< \mathfrak{m}_i, \mathfrak{m}_j >= 0$ for i, j $(i \neq j)$ since $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent (cf. [8, 9, 11]).

Note that the Ricci tensor field Ric of a G-invariant Riemannian metric on G/H is a G-invariant symmetric tensor field of type $(0, 2)$ on

G/H, and we identify Ric with an $\mathrm{Ad}(H)$-invariant symmetric bilinear form on $\mathfrak{m}$. Thus, if $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent irreducible representations, then Ric is written as

$$Ric = y_1 B|_{\mathfrak{m}_i} + \cdots + y_q B|_{\mathfrak{m}_q} \tag{4.6}$$

for some $y_1, \ldots, y_q \in \mathbb{R}$.

4.3. The Ricci tensor field and the scalar curvature on $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant metric

We retain the notation as in Section 4.2. In this section, we assume that the isotropy irreducible representations $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l))$ $(i = 1,2,3,4;\ k,l \in \mathbb{Z})$ are mutually inequivalent. For the sake of simplicity, we put

$$D := \mathbb{Z}^2 \setminus \{(0,t),(t,0),(t,t),(t,-t),(t,-2t),(2t,-t) \mid t \in \mathbb{Z}\}.$$

Let $(\ ,\ )_0$ be the negative of the Killing form of $\mathfrak{su}(3)$, and $<\ ,\ >$ an arbitrarily given $\mathrm{Ad}(T(k,l))$-invariant inner product on $\mathfrak{m}$. By Theorem 3.1, we obtain the fact that the isotropy irreducible representations $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l))$ $(i = 1,2,3,4;\ k,l \in \mathbb{Z})$ are mutually inequivalent if and only if (k,l) in $T(k,l)$ belongs to D. Since $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l))$ are mutually inequivalent, for the inner product $<\ ,\ >$ on $\mathfrak{m}$ there are corresponding positive numbers λ_1, λ_2, λ_3 and λ_4 such that

$$\begin{aligned} &\{X_1/\sqrt{\lambda_1} =: Y_1,\quad X_2/\sqrt{\lambda_1} =: Y_2,\quad X_3/\sqrt{\lambda_2} =: Y_3,\\ &\ X_4/\sqrt{\lambda_2} =: Y_4,\quad X_5/\sqrt{\lambda_3} =: Y_5,\quad X_6/\sqrt{\lambda_3} =: Y_6,\\ &\ X_7/\sqrt{\lambda_4} =: Y_7\} \end{aligned} \tag{4.7}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to the inner product $<\ ,\ >$, by virtue of (3.1), Theorem 3.1 and (4.5). This inner product $<\ ,\ >$ determines a $SU(3)$-invariant Riemannian metric $g_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ on $SU(3)/T(k,l)$.

From now on, we normalize $SU(3)$-invariant Riemannian metrics on $SU(3)/T(k,l)$ by putting $\lambda_4 = 1$, and denote by $g_{(\lambda_1,\lambda_2,\lambda_3)}$ the metric defined by

$$\lambda_1 B|_{\mathfrak{m}_1} + \lambda_2 B|_{\mathfrak{m}_2} + \lambda_3 B|_{\mathfrak{m}_3} + B|_{\mathfrak{m}_4}.$$

By virtue of (3.1), (4.4), (4.6) and (4.7), we obtain the following result.

LEMMA 4.1. ([9]) *Assume that $(k,l) \in D$. Then the Ricci tensor Ric on the Riemannian homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$*

is given as follows:

$$\begin{aligned}
&Ric(Y_i, Y_j) = 0 \quad (i \neq j),\\
&Ric(Y_1, Y_1) = Ric(Y_2, Y_2) = \frac{{\lambda_1}^2 - {\lambda_2}^2 - {\lambda_3}^2 + 6\lambda_2\lambda_3}{12\lambda_1\lambda_2\lambda_3} - \frac{(k+l)^2}{8\gamma{\lambda_1}^2},\\
&Ric(Y_3, Y_3) = Ric(Y_4, Y_4) = \frac{{\lambda_2}^2 - {\lambda_3}^2 - {\lambda_1}^2 + 6\lambda_3\lambda_1}{12\lambda_1\lambda_2\lambda_3} - \frac{l^2}{8\gamma{\lambda_2}^2},\\
&Ric(Y_5, Y_5) = Ric(Y_6, Y_6) = \frac{{\lambda_3}^2 - {\lambda_1}^2 - {\lambda_2}^2 + 6\lambda_1\lambda_2}{12\lambda_1\lambda_2\lambda_3} - \frac{k^2}{8\gamma{\lambda_3}^2},\\
&Ric(Y_7, Y_7) = \frac{1}{8\gamma}\left\{\frac{(k+l)^2}{{\lambda_1}^2} + \frac{l^2}{{\lambda_2}^2} + \frac{k^2}{{\lambda_3}^2}\right\},
\end{aligned}$$

where $\gamma := k^2 + kl + l^2$.

The trace of the Ricci tensor Ric of a Riemannian manifold (M, g), (i.e., $\sum_j Ric(e_j, e_j)$, where $\{e_j\}_j$ is a (locally defined) orthonormal frame on (M, g)), is called the *scalar curvature* of (M, g).

By virtue of Lemma 4.1, we get

LEMMA 4.2. ([9]) *The scalar curvature* $S_{(\lambda_1,\lambda_2,\lambda_3)}$ *of the Riemannian homogeneous space* $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, $(k,l) \in D$, *is given as follows:*

$$\begin{aligned}
S_{(\lambda_1,\lambda_2,\lambda_3)} =& \frac{-({\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2) + 6(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)}{6\lambda_1\lambda_2\lambda_3}\\
&- \frac{1}{8\gamma}\left\{\frac{(k+l)^2}{{\lambda_1}^2} + \frac{l^2}{{\lambda_2}^2} + \frac{k^2}{{\lambda_3}^2}\right\},
\end{aligned}$$

where $\gamma := k^2 + kl + l^2$.

4.4. A decomposition of the curvature tensor field on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$

We retain the notation as in Section 4.3. Let ∇ be the Levi-Civita connection on the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ and ${}^{\nabla}R$ the curvature tensor field with respect to ∇.

For the sake of convenience, we use the following notations:

$$\begin{aligned}
&V := T_{\{T(k,l)\}}(SU(3)/T(k,l)),\\
&(V, g_{(\lambda_1,\lambda_2,\lambda_3)}|_V) =: (V, <\ ,\ >), \quad {}^{\nabla}R =: R,\\
&\mathfrak{L}(V) := \{L \mid L \text{ is a curvature-like tensor on } V\},\\
&\mathfrak{L}_1(V) := \{L \in \mathfrak{L}(V) \mid L(X,Y) = c\, X \wedge Y \text{ for } X, Y \in V\\
&\qquad\qquad \text{and some } c \in \mathbb{R}\},\\
&\mathfrak{L}_\omega(V) := \{L \in \mathfrak{L}(V) \mid \text{the Ricci tensor of } L \text{ is zero}\},\\
&\mathfrak{L}_2(V) := \{L \in \mathfrak{L}_1(V)^\perp \mid\ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_\omega(V)\}.
\end{aligned}$$

Then, we get the orthogonal direct sum decomposition of $\mathfrak{L}(V)$ as follows:

$$\mathfrak{L}(V) = \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V).$$

So, the curvature tensor R at $p_o(= \{T(k,l)\})$ of the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ is uniquely decomposed as

$$\begin{gathered}
R = R^{(1)} + R^{\omega} + R^{(2)}\\
(R^{(1)} \in \mathfrak{L}_1(V),\ R^{\omega} \in \mathfrak{L}_\omega(V),\ R^{(2)} \in \mathfrak{L}_2(V)).
\end{gathered} \tag{4.8}$$

The curvature-like tensor R^ω appeared in (4.8) is said to be the *Weyl tensor (field)* of the curvature tensor field R on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$.

Then, by virtue of (2.8), Lemmas 4.1 and 4.2, we obtain

THEOREM 4.3. *Let $R^{(1)}$, R^ω and $R^{(2)}$ be the the curvature-like tensors appeared in the curvature tensor $R = R^{(1)} + R^\omega + R^{(2)}$ $(\in \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V))$ on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$. Assume that (k,l) belongs to D. Then*

$$\begin{aligned}
R^{(1)}(Y_i,Y_j) &= \frac{1}{42} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,\\
R^{(2)}(Y_i,Y_j) &= \frac{1}{5}\{\mathrm{Ric}(Y_i)\wedge Y_j + Y_i \wedge \mathrm{Ric}(Y_j)\} - \frac{2}{35} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,\\
R^{\omega}(Y_i,Y_j) &= R(Y_i,Y_j) - \frac{1}{5}\{\mathrm{Ric}(Y_i)\wedge Y_j + Y_i \wedge \mathrm{Ric}(Y_j)\}\\
&\qquad + \frac{1}{30} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,
\end{aligned}$$

where $\{Y_i\}_{i=1}^{7}$ is an orthonormal basis on $(\mathfrak{m}, <\ ,\ >)$ and $S_{(\lambda_1,\lambda_2,\lambda_3)}$ is the scalar curvature of $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$.

In general, the *Ricci curvature* r of a Riemannian manifold (M, g) with respect to a nonzero vector $v \in TM$ is defined by

$$r(v) = \frac{Ric(v, v)}{||v||_g^2}.$$

From Theorem 4.3, we get

Corollary 4.4. *Let $R^{(2)}$ be the curvature-like tensor appeared in the curvature tensor $R = R^{(1)} + R^{\omega} + R^{(2)}$ on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, where $(k,l) \in D$. Then the Ricci tensor of $R^{(2)}$ is given as follows:*

$$Ric^{(2)}(Y_i, Y_j) = -\frac{1}{7} S_{(\lambda_1,\lambda_2,\lambda_3)}\ \delta_{ij} + Ric(Y_i, Y_j).$$

By the help of Lemma 4.1 and Corollary 4.4, we obtain

Proposition 4.5. *Assume that $(k,l) \in D$, $k > l > 0$, and*

$$\lambda \leq \frac{3l^2}{10(k^2 + kl + l^2)}$$

in $(SU(3)/T(k,l), g_{(\lambda,\lambda,\lambda)})$, $\lambda > 0$. Then the Ricci curvature $r^{(2)}$ of the curvature-like tensor $R^{(2)}$ in the curvature tensor $R = R^{(1)} + R^{\omega} + R^{(2)}$ on $(SU(3)/T(k,l),\ g_{(\lambda,\lambda,\lambda)})$ is estimated as follows:

$$r^{(2)}(Y_1) = r^{(2)}(Y_2) \leq r^{(2)} \leq r^{(2)}(Y_7),$$

where $r^{(2)}(Y_i) = Ric^{(2)}(Y_i, Y_i)$ for $i = 1, 2, \ldots, 7$.

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