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BOUNDEDNESS IN PERTURBED NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate bounds for solutions of the perturbed nonlinear functional differential systems with a t_∞-similarity condition using the notion of h-stability.

1. Introduction and preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed nonlinear functional differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then, we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{1.3}$$

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and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [15].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(hS)*h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

The notion of h-stability (hS) was introduced by Pinto [15,16] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. He obtained a general variational h-stability and some properties about asymptotic behavior of solutions of differential systems called h-systems. Choi, Ryu [3] and Choi, Koo, and Ryu [4] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [7,8,9] and Goo et al. [11] investigated boundedness of solutions for nonlinear perturbed systems.

In this paper, we investigate bounds for solutions of the perturbed nonlinear functional differential systems using the notion of t_∞-similarity.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an absolutely integrable $n \times n$ matrix $F(t)$ over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [6, 12].

We give some related properties that we need in the sequal.

Lemma 1.4. [16] *The linear system*

$$x' = A(t)x,\ x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.6. [3] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

Theorem 1.7. [4] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

Lemma 1.8. ($Bihari - type\ inequality$) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ t \geq t_0 \geq 0.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s)ds\Big],\ t_0 \le t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.9. [5] *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau\Big)ds,\ 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau))ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.10. [9] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau)ds\Big],\ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main results

In this section, we investigate boundedness for solutions of the nonlinear perturbed differential systems via t_∞-similarity.

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

THEOREM 2.1. *Let $a, b, c, q, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)|y(t)|, \tag{2.1}$$
$$|h(t, y(t), Ty(t))| \le b(t)w(|y(t)|) + c(t)|Ty(t)|,$$

and

$$|Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds, \tag{2.2}$$

where $a, b, c, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c)+c_2\int_{t_0}^{t}(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau+c(s)\int_{t_0}^{s}q(\tau)d\tau)ds\Big],$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s}k(\tau)d\tau$$
$$+c(s)\int_{t_0}^{s}q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t, t_0, y_0)$ of (1.2) passing through (t_0, y_0) is given by

(2.3)
$$y(t, t_0, y_0) = x(t, t_0, y_0)$$
$$+ \int_{t_0}^{t} \Phi(t, s, y(s))\Big(\int_{t_0}^{s} g(\tau, y(\tau))d\tau + h(s, y(s), Ty(s))\Big)ds.$$

By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. In view of Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.1), (2.2), and (2.3), we have

$$
\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \Big(\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s,y(s),Ty(s))| \Big) ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) h(s)^{-1} \Big(a(s)|y(s)| + b(s) w(|y(s)|) \\
&\quad + c(s) \int_{t_0}^{s} q(\tau)|y(\tau)| d\tau \Big) ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) (a(s) \frac{|y(s)|}{h(s)} \\
&\quad + b(s) w(\frac{|y(s)|}{h(s)}) + c(s) \int_{t_0}^{s} q(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau) ds.
\end{aligned}
$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Lemma 1.10 yields

$$
|y(t)| \le h(t) W^{-1} \Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + c(s) \int_{t_0}^{s} q(\tau) d\tau) ds \Big]
$$

where $c = c_1 |y_0|\, h(t_0)^{-1}$. Hence, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

REMARK 2.2. Letting $w(u) = u$ and $b(t) = c(t) = 0$ in Theorem 2.1, we obtain the same result as that of Theorem 3.3 in [10].

LEMMA 2.3. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$,*

(2.4)
$$
\begin{aligned}
u(t) \le c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) u(\tau) d\tau ds \\
+ \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\tau ds, \quad 0 \le t_0 \le t.
\end{aligned}
$$

Then

$$
\begin{aligned}
u(t) \le W^{-1} \Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau \\
+ \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\tau) ds \Big],
\end{aligned} \tag{2.5}
$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Define a function $z(t)$ by the right member of (2.4). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)u(t) + \lambda_2(t)w(u(t)) + \lambda_3(t) \int_{t_0}^{t} \lambda_4(s)u(s)ds \\ &\quad + \lambda_5(t) \int_{t_0}^{t} \lambda_6(s)w(u(s))ds \\ &\leq (\lambda_1(t) + \lambda_2(t) + \lambda_3(t) \int_{t_0}^{t} \lambda_4(s)ds\lambda_5(t) \int_{t_0}^{t} \lambda_6(s)ds)w(z(t)),\ t \geq t_0, \end{aligned}$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \leq w(u)$, and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$\begin{aligned} z(t) \leq c + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)w(z(s)) \\ + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau)w(z(s)))ds. \end{aligned} \tag{2.6}$$

It follows from Lemma 1.8 that (2.6) yields the estimate (2.5). □

Theorem 2.4. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)w(|y(t))| + b(t) \int_{t_0}^{t} k(s)|y(s)|ds,\ t \geq t_0 \geq 0 \tag{2.7}$$

and

$$|h(t, y(t), Ty(t))| \leq c(t)(|y(t)| + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^{t} q(s)w(|y(s)|)ds, \tag{2.8}$$

$t \geq t_0 \geq 0$, where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(c)+c_2 \int_{t_0}^{t} (a(s)+c(s)+b(s) \int_{t_0}^{s} k(\tau)d\tau+c(s) \int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau + c(s) \int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By Lemmma 1.4, the hS condition of $x = 0$ of (1.1), (2.3), (2.7), and (2.8), we have

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}((a(s)w(|y(s)|)\\ &\quad + b(s) \int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + c(s)(|y(s)| + \int_{t_0}^{s} q(\tau)w(|y(\tau)|)d\tau)ds\\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)(c(s)\frac{|y(s)|}{h(s)} + a(s)w(\frac{|y(s)|}{h(s)}))ds\\ &\quad + \int_{t_0}^{t} c_2 h(t)(b(s) \int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau + c(s) \int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Lemma 2.3 yields

$$\begin{aligned}|y(t)| \le h(t)W^{-1}\Big[&W(c) + c_2 \int_{t_0}^{t} (a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau)d\tau\\ &+ c(s) \int_{t_0}^{s} q(\tau)d\tau)ds\Big],\end{aligned}$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. Hence, the proof is complete. □

REMARK 2.5. Letting $c(t) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.4 in [7].

THEOREM 2.6. *Let $a, b, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|), |h(t, y(t), Ty(t))| \le b(t)w(|y(t)|), \tag{2.9}$$

where $a, b \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (b(s) + \int_{t_0}^{s} a(\tau)d\tau)ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} (b(s) + \int_{t_0}^{s} a(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.3) and (2.9), we have

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} a(\tau)w(|y(\tau)|)d\tau \\ &\qquad + b(s)w(|y(s)|)\Big)ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(b(s)w(\frac{|y(s)|}{h(s)}) + \int_{t_0}^{s} a(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.\end{aligned}$$

Defining $u(t) = |y(t)||h(t)|^{-1}$, then, by Lemma 1.9, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (b(s) + \int_{t_0}^{s} a(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. □

REMARK 2.7. Letting $b(t) = 0$ in Theorem 2.6, we obtain the similar result as that of Theorem 3.5 in [11].

LEMMA 2.8. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$\begin{aligned}u(t) \le c &+ \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big(\lambda_3(\tau)w(u(\tau)) \\ &+ \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)u(r)dr\Big)d\tau ds.\end{aligned} \tag{2.10}$$

Then

$$(2.11)\quad \begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) \\ + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau)ds\Big], \end{aligned}$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) \\ + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Define a function $z(t)$ by the right member of (2.10). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)u(t) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s)w(u(s)) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)u(\tau)d\tau)ds \\ &\le (\lambda_1(t) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)d\tau)ds)w(z(t)),\ t \ge t_0, \end{aligned}$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \le w(u)$, and $u(t) \le z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$(2.12)\quad z(t) \le c + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau)w(z(s)))ds.$$

It follows from Lemma 1.8 that (2.12) yields the estimate (2.11). □

Theorem 2.9. *Let $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies*

$$(2.13)\quad |g(t, y(t))| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t}k(s)|y(s)|ds$$

and

$$(2.14)\quad |h(t, y(t), Ty(t))| \le c(t)|y(t)|,$$

where $a, b, c, k \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau)ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is well known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.13), and (2.14), we have

$$\begin{aligned}|y(t)| &\leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(\int_{t_0}^{s} (a(\tau)w(|y(\tau)|) \\ &+b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr)d\tau + c(s)|y(s)|\Big)ds \\ &\leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(c(s)\frac{|y(s)|}{h(s)} \\ &+ \int_{t_0}^{s} (a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr)d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.8, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t} (c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 2.10. Letting $c(t) = 0$ in Theorem 2.9, we obtain the similar result as that of Theorem 3.7 in [7].

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