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AN ALGEBRAIC SOLUTION OF EINSTEIN'S FIELD EQUATIONS IN X_4

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Abstract. The main goal in the present paper is to obtain a particular solution $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ and an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}^{\nu}_{\lambda\mu}$ by means of $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ in UFT X_4.

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([6]) gave the mathematical foundation of the Einstein's unified field theory in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time) for the first time. Since then this theory had been generalized in a generalized Riemannian manifold X_n, the so-called *Einstein's n-dimensional unified field theory*(UFT X_n), and many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to obtain a particular solution $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ of Einstein's field equation in UFT X_4. In the next, we shall obtain an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}^{\nu}_{\lambda\mu}$ by means of $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ in UFT X_4.

2. Preliminary

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^{\nu}\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow

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the summation convention. The algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$(a)\ det((g_{\lambda\mu})) < 0, \quad (b)\ det((h_{\lambda\mu})) < 0, \quad (c)\ det((k_{\lambda\mu})) \geq 0. \tag{2.2}$$

Since $det((h_{\lambda\mu})) \neq 0$, we may define a unique tensor $h^{\lambda\nu}(= h^{\nu\lambda})$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined on X_n in the usual manner. The manifold X_n is assumed to be connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ which may also be split into its symmetric part $\Lambda^\nu_{\lambda\mu}$ and skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the *torsion tensor* of $\Gamma^\nu_{\lambda\mu}$.

The *Einstein's n-dimensional unified field theory in X_n*(UFT X_n) is governed by the following set of equations :

$$\partial_\omega g_{\lambda\mu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - g_{\lambda\alpha}\Gamma^\alpha_{\omega\mu} = 0 \qquad (\partial_\nu = \frac{\partial}{\partial x^\nu}), \tag{2.4}$$

and

$$(a)\ S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \quad (b)\ R_{[\lambda\mu]} = \partial_{[\lambda}P_{\mu]}, \quad (c)\ R_{(\lambda\mu)} = 0, \tag{2.5}$$

where P_μ is an arbitrary vector, called the *Einstein's vector*, and $R_{\lambda\mu}$ is the contracted curvature tensor $R^\alpha_{\lambda\mu\alpha}$ of the curvature tensor $R^\omega_{\lambda\mu\nu}$:

$$R^\omega_{\lambda\mu\nu} = \partial_\mu\Gamma^\omega_{\lambda\nu} - \partial_\nu\Gamma^\omega_{\lambda\mu} + \Gamma^\alpha_{\lambda\nu}\Gamma^\omega_{\alpha\mu} - \Gamma^\alpha_{\lambda\mu}\Gamma^\omega_{\alpha\nu}. \tag{2.6}$$

The equation (2.4) is called the *Einstein's equation*, and the solution $\Gamma^\nu_{\lambda\mu}$ of the Einstein's equation is called an *Einstein's connection*. And the vector S_λ, defined by (2.5)(a), is the called the *torsion vector*.

The following two theorems were proved by Lee([3]).

Theorem 2.1. *In UFT X_n, if the system (2.4) admits a solution $\Gamma^\nu_{\lambda\mu}$ such that its torsion tensor is, for some nonzero vector Y_λ,*

$$S_{\lambda\mu}{}^\nu = \frac{2}{n-1}\delta^\nu_{[\lambda}k_{\mu]\alpha}Y^\alpha + k_{\lambda\mu}Y^\nu, \tag{2.7}$$

then it must be of the form

$$\Gamma^\nu_{\lambda\mu} = \{{}_\lambda{}^\nu{}_\mu\} + \frac{2(2-n)}{n-1}k_{(\lambda}{}^\nu k_{\mu)\alpha}Y^\alpha + \frac{2}{n-1}\delta^\nu_{[\lambda}k_{\mu]\alpha}Y^\alpha + k_{\lambda\mu}Y^\nu, \tag{2.8}$$

where $\{{}_\lambda{}^\nu{}_\mu\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.

THEOREM 2.2. *In UFT X_n, the connection (2.8) is an Einstein's connection if and only if the vector Y_λ defining (2.8) satisfies the following condition*

$$(2.9)\quad \nabla_\nu k_{\lambda\mu} = \frac{2}{n-1} h_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha - 2k_{\nu[\lambda} Y_{\mu]} + \frac{2(n-2)}{n-1} {}^{(2)}k_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha,$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to $\{{}_\lambda{}^\nu{}_\mu\}$.

3. A particular solution of field equations in UFT X_4

In this section we shall display a particular solution of (2.4) and (2.5) in UFT X_4. Let a tensor $g_{\lambda\mu}$ be given by the following matrix :

$$(3.1)\qquad ((g_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & -e^t & e^t \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 1 & 0 \\ -e^t & 0 & 0 & -1 \end{pmatrix},$$

where $t = x^3 - x^4$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ given by the following matrices :

$$(3.2)\qquad ((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$(3.3)\qquad ((k_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -e^t & e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(3.4)\quad (a)\ det((g_{\lambda\mu})) = -1, \quad (b)\ det((h_{\lambda\mu})) = -1, \quad (c)\ det((k_{\lambda\mu})) = 0,$$

we can choose the tensor $g_{\lambda\mu}$ given by (3.1) as a basic tensor in UFT X_4, by the assumption (2.2). On the other hand, in virtue of (3.2), all the Christoffel symbols $\{{}_\lambda{}^\nu{}_\mu\}$ vanish. Hence the components of the first covariant derivatives with respect to $\{{}_\lambda{}^\nu{}_\mu\}$ are ordinary derivatives, and $H^\omega_{\lambda\mu\nu} = 0$. Define two vectors by

$$(3.5)\qquad (a)\ A_\lambda : (0,0,1,-1), \qquad (b)\ B_\lambda : (e^t, 0, 0, 0).$$

Then the skew-symmetric part $k_{\lambda\mu}$ satisfies the following relation :

$$k_{\lambda\mu} = 2A_{[\lambda}B_{\mu]}. \tag{3.6}$$

Furthermore, making use of (3.2) and (3.5), we obtain

$$(a)\ A^{\lambda}(= h^{\lambda\nu}A_{\nu}) : (0,0,1,1), \quad (b)\ B^{\lambda}(= h^{\lambda\nu}B_{\nu}) : (e^{t},0,0,0), \tag{3.7}$$

and

$$\begin{gathered} (a)\ A_{\alpha}A^{\alpha} = 0, \quad (b)\ A_{\alpha}B^{\alpha} = 0, \quad (c)\ k_{\lambda\alpha}A^{\alpha} = 0, \\ (d)\ \nabla_{\lambda}A_{\mu} = 0, \quad (e)\ \nabla_{\omega}B_{\mu} = A_{\omega}B_{\mu}. \end{gathered} \tag{3.8}$$

THEOREM 3.1. *In UFT X_4, the vector A_λ given by (3.5)(a) is a solution of the condition (2.9). In this case, the corresponding Einstein connection may be given by*

$$\Gamma^{\nu}_{\lambda\mu} = 2A_{[\lambda}B_{\mu]}A^{\nu}. \tag{3.9}$$

And its curvature tensor $R^{\omega}_{\lambda\mu\nu}$ may be given by

$$R^{\omega}_{\lambda\mu\nu} = 2A_{\lambda}A_{[\mu}B_{\nu]}A^{\omega}, \tag{3.10}$$

Proof. Substituting the vector A_λ into the condition (2.9), the vector A_λ is a solution of the condition (2.9) iff, in virtue of (3.8)(c),

$$\nabla_{\nu}\, k_{\lambda\mu} = -2k_{\nu[\lambda}\, A_{\mu]}. \tag{3.11}$$

But, making use of (3.6), (3.8)(d) and (3.8)(e),

$$\nabla_{\nu}\, k_{\lambda\mu} = A_{\lambda}A_{\nu}B_{\mu} - A_{\mu}A_{\nu}B_{\lambda} = -2k_{\nu[\lambda}\, A_{\mu]}, \tag{3.12}$$

which implies that the vector A_λ is a solution of the condition (2.9). Substituting the vector A_λ into (2.8), making use (3.6) and (3.8)(c), and remembering $\{{}_{\lambda}{}^{\nu}{}_{\mu}\} = 0$, we obtain an Einstein connection (3.9). Substituting the Einstein connection (3.9) into (2.6), we obtain (3.10) by a straightforward computation. □

Conclusion. In virtue of Theorem 3.1, if UFT X_4 is endowed with the basic tensor $g_{\lambda\mu}$ given by (3.1), then an Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ is given by (3.9), which satisfy (2.5)(a). In the next, since the contracted curvature tensor $R_{\lambda\mu}$ with respect to the connection (3.9) is given by $R_{\lambda\mu} = 0$, in virtue of (3.10), the field equation (2.5)(c) is satisfied automatically. And since the field equation (2.5)(b) is equivalent to $\partial_{[\lambda}P_{\mu]} = 0$, the field equation (2.5)(b) is satisfied by a vector $P_\mu = \partial_\mu P$, that is, the vector $P_\mu = \partial_\mu P$ is an Einstein's vector.

4. An algebraic solution of field equations in UFT X_4

Assume that we have a particular solution $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ of (2.4) and (2.5). The question arises whether there exist a tensor $\bar{g}_{\lambda\mu}$ which together with the $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is a solution of (2.4) and (2.5). In order to answer this question we put $\bar{g}_{\lambda\mu}$ in the form

$$\bar{g}_{\lambda\mu} = g_{\lambda\mu} + X_{\lambda\mu} \tag{4.1}$$

where the tensor $X_{\lambda\mu}$ has to be founded. From now on, we shall hold to the following agreement : If T is a function of $g_{\lambda\mu}$, then we denote by $\bar{T}$ the same function of $\bar{g}_{\lambda\mu}$. If, in particular, T is a tensor, so is $\bar{T}$. From (4.1), we obtain

$$(a)\ \bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu} \qquad (b)\ \bar{k}_{\lambda\mu} = k_{\lambda\mu} + q_{\lambda\mu}, \tag{4.2}$$

where $p_{\lambda\mu}$ and $q_{\lambda\mu}$ are the symmetric part and the skew-symmetric part of the tensor $X_{\lambda\mu}$, respectively. And we assume that $det((\bar{h}_{\lambda\mu})) \neq 0$. we may define a unique tensor $\bar{h}^{\lambda\nu} (= \bar{h}^{\nu\lambda})$ by

$$\bar{h}_{\lambda\mu}\bar{h}^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{4.3}$$

THEOREM 4.1. *If we put*

$$\overline{\{\lambda{}^{\omega}{}_{\nu}\}} = \{\lambda{}^{\omega}{}_{\nu}\} + P^{\omega}_{\lambda\nu}, \tag{4.4}$$

then $P^{\omega}_{\lambda\nu}$ is a tensor symmetric in the indices λ and ν, and it is given by

$$P^{\nu}_{\lambda\mu} = \frac{1}{2}\bar{h}^{\nu\alpha}(\nabla_{\lambda}\, p_{\mu\alpha} + \nabla_{\mu}\, p_{\alpha\lambda} - \nabla_{\alpha}\, p_{\lambda\mu}). \tag{4.5}$$

Proof. By the law of transformation of the Christoffel symbols, $P^{\nu}_{\lambda\mu} = \overline{\{\lambda{}^{\omega}{}_{\nu}\}} - \{\lambda{}^{\omega}{}_{\nu}\}$ is a tensor symmetric in the indices λ and ν. Multiplying by $\bar{h}_{\omega\mu}$ and summing for ω on both sides of (4.4), and using the expression (4.2)(a) in the right-hand member of (4.4), we obtain

$$\overline{[\lambda\nu,\mu]} = [\lambda\nu,\mu] + p_{\mu\alpha}\{\lambda{}^{\alpha}{}_{\nu}\} + \bar{h}_{\mu\alpha}P^{\alpha}_{\lambda\nu}. \tag{4.6}$$

In accordance with the definition of the Christoffel symbols we have, from (4.2)(a),

$$\overline{[\lambda\nu,\mu]} = [\lambda\nu,\mu] + [\lambda\nu,\mu]_p, \tag{4.7}$$

where $[\lambda\nu,\mu]_p$ are the Christoffel symbols of the first kind formed with respect to $p_{\lambda\nu}$. Substituting (4.7) into (4.6), we obtain

$$[\lambda\nu,\mu]_p = p_{\mu\alpha}\{\lambda{}^{\alpha}{}_{\nu}\} + \bar{h}_{\mu\alpha}P^{\alpha}_{\lambda\nu}. \tag{4.8}$$

If we add to this equation the one obtained by interchanging λ and μ, then the result may be written

$$\nabla_\nu p_{\lambda\mu} = \bar{h}_{\mu\alpha} P^\alpha_{\lambda\nu} + \bar{h}_{\lambda\alpha} P^\alpha_{\mu\nu}. \tag{4.9}$$

Subtracting this equation from the sum of the two others which are obtained from it by cyclic permutation of the indices λ, μ and ν, we obtain (4.5). □

In order to obtain an algebraic solution $\bar{g}_{\lambda\mu}$, $\bar{\Gamma}^\nu_{\lambda\mu}$ by means of a particular solution $g_{\lambda\mu}$, $\Gamma^\nu_{\lambda\mu}$ of (2.4) and (2.5), where $g_{\lambda\mu}$ and $\Gamma^\nu_{\lambda\mu}$ are given by (3.1) and (3.8), let us consider a tensor $\bar{g}_{\lambda\mu}$ given by the following matrix :

$$((\bar{g}_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2e^t & 0 & 1 & 0 \\ -2e^t & 0 & 0 & -1 \end{pmatrix}, \tag{4.10}$$

where $t = x^3 - x^4$. The tensor $\bar{g}_{\lambda\mu}$ may be split into its symmetric part $\bar{h}_{\lambda\mu}$ and skew-symmetric part $\bar{k}_{\lambda\mu}$ given by the following matrices :

$$((\bar{h}_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & e^t & -e^t \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 1 & 0 \\ -e^t & 0 & 0 & -1 \end{pmatrix}. \tag{4.11}$$

$$((\bar{k}_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -e^t & e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

Hence the tensor $\bar{h}_{\lambda\mu}$ may be split into $h_{\lambda\mu}$ and $p_{\lambda\mu}$ given by the following matrices :

$$((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.13}$$

$$((p_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & e^t & -e^t \\ 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 \\ -e^t & 0 & 0 & 0 \end{pmatrix}. \tag{4.14}$$

Hence we obtain

(4.15) (a) $\bar{h}_{\lambda\mu} = h_{\lambda\mu} + p_{\lambda\mu}$ (b) $q_{\lambda\mu} = 0$, (c) $\bar{k}_{\lambda\mu} = k_{\lambda\mu}(= 2A_{[\lambda}B_{\mu]})$.

Since

(4.16) (a) $det((\bar{g}_{\lambda\mu})) = -1$, (b) $det((\bar{h}_{\lambda\mu})) = -1$, (c) $det((\bar{k}_{\lambda\mu})) = 0$,

we can choose the tensor $\bar{g}_{\lambda\mu}$ given by (4.10) as a basic tensor in UFT X_4, by the assumption (2.2).

THEOREM 4.2. *In UFT X_4, for the vectors A_λ and B_λ given by (3.5), the following relations hold.*

$$\begin{aligned}&(a)\ p_{\lambda\mu} = 2A_{(\lambda}B_{\mu)} \quad (b)\ p_{\lambda\alpha}A^\alpha = 0, \quad (c)\ p_{\lambda\alpha}B^\alpha = A_\lambda B_\alpha B^\alpha,\\ &(d)\ A_\alpha \bar{h}^{\alpha\nu} = A^\nu, \quad (e)\ B_\alpha \bar{h}^{\alpha\nu} = B^\nu - A^\nu B_\beta B^\beta\\ &(f)\ P^\nu_{\lambda\mu} = A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta)\end{aligned} \tag{4.17}$$

Proof. A simple inspection based on (3.6), (3.8) and (4.15) shows (4.17)(a)∼(d). From (4.15) and (4.17)(c), we obtain

$$\bar{h}_{\lambda\alpha}B^\alpha = B_\lambda + A_\lambda B_\alpha B^\alpha \tag{4.18}$$

Multiplying $\bar{h}^{\lambda\beta}$ on both sides of (4.18) and summing for λ, we obtain

$$B^\beta = \bar{h}^{\lambda\beta}B_\lambda + A^\beta B_\alpha B^\alpha, \tag{4.19}$$

which implies (4,17)(e). Next, substituting (4.17)(a) in (4.5), and making use of (3.8) and (4,17)(e), obtain

$$P^\nu_{\lambda\mu} = \bar{h}^{\nu\alpha}A_\mu A_\lambda B_\alpha = A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta). \tag{4.20}$$

□

THEOREM 4.3. *In UFT X_4, let $g_{\lambda\mu}$ and $\Gamma^\nu_{\lambda\mu}$ be given by (3.1) and (3.9), respectively. For the basic tensor $\bar{g}_{\lambda\mu}$ given by (4.10), let $\overline{\Gamma}^\nu_{\lambda\mu}$ be a connection with the same torsion tensor as $\Gamma^\nu_{\lambda\mu}$. Then $\overline{\Gamma}^\nu_{\lambda\mu}$ is an Einstein connection which is given by*

$$\overline{\Gamma}^\nu_{\lambda\mu} = 2A_{[\lambda}B_{\mu]}A^\nu + A_\lambda A_\mu (B^\nu - A^\nu B_\beta B^\beta). \tag{4.21}$$

And its curvature tensor $\overline{R}^\omega_{\lambda\mu\nu}$ may be given by

$$\overline{R}^\omega_{\lambda\mu\nu} = 2A_\lambda A_{[\mu}B_{\nu]}A^\omega. \tag{4.22}$$

Proof. Since $\overline{\Gamma}^\nu_{\lambda\mu}$ is a connection with the same torsion tensor as $\Gamma^\nu_{\lambda\mu}$,

$$\overline{S}^\nu_{\lambda\mu} = k_{\lambda\mu}A^\nu = 2A_{[\lambda}B_{\mu]}A^\nu. \tag{4.23}$$

Hence, in virtue of Theorem 2.1, and making use of (3.8)(c), (4.4), (4.5) and (4.20), and remembering $\{\lambda{}^{\nu}{}_{\mu}\} = 0$, the connection $\overline{\Gamma}^{\nu}_{\lambda\mu}$ may be given by

$$\overline{\Gamma}^{\nu}_{\lambda\mu} = \overline{\{\lambda{}^{\nu}{}_{\mu}\}} + k_{\lambda\mu}A^{\nu} = A_{\lambda}A_{\mu}(B^{\nu} - A^{\nu}B_{\beta}B^{\beta}) + 2A_{[\lambda}B_{\mu]}A^{\nu}. \tag{4.24}$$

And, in virtue of Theorem 2.2, this connection (4.24) is an Einstein connection if and only if, making use of (3.8)(c),

$$\overline{\nabla}_{\nu}\, k_{\lambda\mu} = -2k_{\nu[\lambda}\, A_{\mu]} \tag{4.25}$$

But since, making use of (4.17)(f), (3.6) and (3.8),

$$\begin{aligned}\overline{\nabla}_{\nu}\, k_{\lambda\mu} &= \partial_{\nu}k_{\lambda\mu} - k_{\alpha\mu}\overline{\{\lambda{}^{\alpha}{}_{\nu}\}} - k_{\lambda\alpha}\overline{\{\mu{}^{\alpha}{}_{\nu}\}} \\ &= \partial_{\nu}k_{\lambda\mu} - k_{\alpha\mu}P^{\alpha}_{\lambda\nu} - k_{\lambda\alpha}P^{\alpha}_{\mu\nu} \\ &= \partial_{\nu}\, k_{\lambda\mu} = \nabla_{\nu}\, k_{\lambda\mu} = -2k_{\nu[\lambda}\, A_{\mu]},\end{aligned} \tag{4.26}$$

the connection (4.21) is an Einstein connection which satisfies (2.4). Next Substituting (4.21) into the curvature tensor :

$$\overline{R}^{\omega}_{\lambda\mu\nu} = \partial_{\mu}\overline{\Gamma}^{\omega}_{\lambda\nu} - \partial_{\nu}\overline{\Gamma}^{\omega}_{\lambda\mu} + \overline{\Gamma}^{\alpha}_{\lambda\nu}\overline{\Gamma}^{\omega}_{\alpha\mu} - \overline{\Gamma}^{\alpha}_{\lambda\mu}\overline{\Gamma}^{\omega}_{\alpha\nu}, \tag{4.27}$$

we obtain (4.22), by a straightforward computation. □

Conclusion. In virtue of Theorem 4.3, if UFT X_4 is endowed with the basic tensor $\bar{g}_{\lambda\mu}$ given by (4.10), then an Einstein connection $\overline{\Gamma}^{\nu}_{\lambda\mu}$ is given by (4.21), which satisfy (2.5)(a). Furthermore, since from (4.22), the contracted curvature tensor $\overline{R}_{\lambda\mu}$ with respect to the connection (4.21) is given by $\overline{R}_{\lambda\mu} = 0$, the field equation (2.5)(c) is satisfied automatically. On the other hand, since the field equation (2.5)(b) is equivalent to $\partial_{[\lambda}P_{\mu]} = 0$, the field equation (2.5)(b) is satisfied by a vector $P_{\mu} = \partial_{\mu}P$, that is, the vector $P_{\mu} = \partial_{\mu}P$ is an Einstein's vector. Consequently, for a particular solution $g_{\lambda\mu}$, $\Gamma^{\nu}_{\lambda\mu}$ of (2.4) and (2.5), $\bar{g}_{\lambda\mu}$ is an algebraic solution which together with the $\overline{\Gamma}^{\nu}_{\lambda\mu}$ is a solution of (2.4) and (2.5).

References

[1] A. Einstein, *The meaning of relativity*, Princeton University Press, Princeton, New Jersey, 1950

[2] J. W. Lee, *Field equations of SE(k)-manifold X_n*, International Journal of Theoretical Physics **30** (1991), 1343-1353

[3] J. W. Lee, *An Einstein's connection with zero torsion vector in even-dimensional UFT X_n*, Jour. of the Chungcheong Math. Soc. **24** (2011), no. 4, 869-881

[4] J. W. Lee, *A solution of Einstein's field equations for the third class in X_4*, Jour. of the Chungcheong Math. Soc. **26** (2013), no. 1, 167-174

[5] J. W. Lee and K. T. Chung, *A solution of Einstein's unified field equations*, Comm. Korean Math. Soc. **11** (1996), no. 4, 1047-1053.
[6] V. Hlavatý, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd. New York, 1957.

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