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KUPKA-SMALE DIFFERENTIABLE MAPS

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Abstract. We show that if a regular map belongs to the C^1-interior of the set of all Kupka-Smale differentiable maps then it satisfis Axiom A and the strong transverality condition.

1. Introduction

Let M be a closed C^∞ manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \text{Diff}(M)$. We say that f is *structurally stable* if there is a C^1-neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, there is a homeomorphism $h: M \to M$ such that $h \circ f(x) = g \circ h(x)$ for all $x \in M$. For $f \in \text{Diff}(M)$, we say that $f \in \mathcal{F}(M)$ if there is a C^1-neighborhood $\mathcal{U}$ of f such that all periodic points of $g \in \mathcal{U}$ are hyperbolic. Then by Aoki [1] and Hayashi [3], we know that if $f \in \mathcal{F}(M)$ then f satisfies Axiom A and no-cycle condition, that is, Ω-stable. An Axiom A diffeomorphism f satisfies the *strong transversality condition* if and only if the stable manifold $W^s(x)$ and the unstable manifold $W^u(x)$ are transversal for all $x \in M$. In [8], Robinson proved that a diffeomorphism f is structural stable if and only if it satisfies Axiom A and strong transversality condition.

Denote by $\mathcal{KS}(M)$ the C^1-interior of the set of all Kupka-Smale diffeomorphsms. It is clear that $\mathcal{KS}(M) \subset \mathcal{F}(M)$. By Aoki [1], he proved that

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every $f \in \mathcal{KS}(M)$ satisfies Axiom A and the strong transversality condition. For that, we study that if for a regular map(a non-invertable differetiable map whose derivative is surjective) belongs to the C^1-interior of the set of all Kupka-Smale regular maps then it satisfies Axiom A and the strong transversality condition.

2. Basic notions and main result

Let M be a closed C^∞ manifold, and let d be a metric on M induced by a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. A differentiable map $f : M \to M$ is called *regular* if the derivative $D_x f : T_x M \to T_{f(x)} M$ is surjective for any $x \in M$. Denote by $\mathfrak{R}(M)$ the set of all regular maps of M endowed with the C^1-topology. Let $f \in \mathfrak{R}(M)$. We denote by $P(f)$ the set of all periodic points of f, and $\Omega(f)$ the set of all non-wandering points of f. We say that $p \in P(f)$ is *hyperbolic* if $D_p f^{\pi(p)} : T_p M \to T_p M$ has no eigenvalues of absolute value one, then $T_p M$ admits a continuous, invariant splitting $T_p M = E^s(p) \oplus E^u(p)$. Here the dimension of $E^s(p)$ is called the *index of* p, and denote by index(p). We say that f satisfies *Axiom A* if $P(f)$ is dense in $\Omega(f)$ and $\Omega(f)$ is hyperbolic. If $D_x f : T_x M \to T_{f(x)} M$ is not injective then x is called a *singular point* for f. Mañé has proven in [5] that a differentiable map which is C^1-structurally stable has no singular points. If f satisfies Axiom A and $\Omega(f)$ is the disjoint union $\Omega_1 \cup \Omega_2$ of two closed f-invariant sets such that (i) $f|_{\Omega_1}$ is injective, and (ii) Ω_2 is contained in the closure of all source periodic points then we say that f satisfies the *strong Axiom A*. If a differentiable map f satisfies Axiom A and has no singular points in the nonwandering set then f is Ω-stable if and only if f satisfies strong Axiom A and has no cycles (see [7]). In [2], Aoki *et. al* proved that if a differentiable map belongs to the C^1-interior of the set of maps satisfying (i) periodic points are hyperbolic, and (ii) singular points in $\Omega(f)$ are sinks then it satisfies Axiom A and no-cycle condition.

Denote by $\mathbb{M}$ the topological product space $\prod_{-\infty}^{\infty} M$, and define a continuous map $\bar{f} : \mathbb{M} \to \mathbb{M}$ by $\bar{f}((x_n)) = (f(x_n))$ for $(x_n) \in \mathbb{M}$. Then we define the projection $P^0 : \mathbb{M} \to M$ by $P^0((x_n)) = x_0$ satisfies $P^0 \circ \bar{f} = f \circ P^0$. For $\Lambda \subset M$, an $\bar{f}$-invariant set Λ_f is defined by $\Lambda_f = \{(x_n) \in \mathbb{M} : x_n \in \Lambda, f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{Z}\}$. Then Λ_f is $\bar{f}$-invariant and $P^0(\Lambda_f)$ is f-invariant, that is, $f(P^0(\Lambda_f)) = P^0(\Lambda_f)$. A closed f-invariant set Λ is said to be *hyperbolic* if $D\bar{f}$-invariant continuous splitting $T\mathbb{M}|_{\Lambda_f} =$

$E^s \oplus E^u$ with constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D\bar{f}^n|_{E^s}\| \leq C\lambda^n \text{ and } \|D\bar{f}^{-n}|_{E^u}\| \leq C\lambda^n$$

for $n \geq 0$.

For any $\epsilon > 0$ and $\bar{x} = (x_n) \in M_f$, we denote $W^s_\epsilon(\bar{x}) = \{y \in M : d(x_n, f^n(y)) \leq \epsilon \text{ for } n \geq 0\}$, and $W^u_\epsilon(\bar{x}) = \{y \in M : \text{there is } \bar{y} = (y_n) \in M_f \text{ such that } y_0 = y \text{ and } d(x_{-n}, y_{-n}) \leq \epsilon \text{ for } n \geq 0\}$. Then $W^s_\epsilon(\bar{x})$ and $W^u_\epsilon(\bar{x})$ are called the *local stable set* and *local unstable set*, respectively. We defined by the *stable set* and the *unstable set* as follows:

$$W^s(\bar{x}) = \{y \in M : \lim_{n\to\infty} d(x_n, f^n(y)) = 0\} \text{ and}$$

$$W^u(\bar{x}) = \{y \in M : \text{ there is } \bar{y} = (y_n) \in M_f \text{ such that } y_0 = y \text{ and } \lim_{n\to\infty} d(x_{-n}, y_{-n}) = 0\}.$$

Let Λ be a hyperbolic set. For $\bar{x} \in \Lambda_f$, we define

$$W^s(\bar{x}) = \bigcup_{n=0}^{\infty} f^{-n}\big(W^s_\epsilon(\bar{f}^n(\bar{x}))\big) \text{ and}$$

$$W^u(\bar{x}) = \bigcup_{n=0}^{\infty} f^n(W^u_\epsilon(\bar{f}^{-n}(\bar{x}))).$$

Note that if $\bar{x} \in \Lambda_f$ and $\epsilon > 0$ is small enough then by [4, Theorem 5.1], $W^s_\epsilon(\bar{x}, f), W^u_\epsilon(\bar{x}, f)$ are disks or one point sets satisfy

$$T_{x_0} W^s_\epsilon(\bar{x}) = \bar{P}^0(E^s(\bar{x})) \text{ and } T_{x_0} W^u_\epsilon(\bar{x}) = \bar{P}^0(E^u(\bar{x})).$$

For $\bar{x}, \bar{y} \in M_f$, we say that $W^s(\bar{x})$ is *transversal* to $W^u(\bar{y})$ if $f^{n+m}|_{W^u_\epsilon(\bar{f}^{-m}(\bar{y}))}$ is transversal to $W^s_\epsilon(\bar{f}^n(\bar{x}))$ for any $\epsilon > 0$ and $n, m \geq 0$. For $f \in \mathfrak{R}(M)$, we say that f is *Kupka-Samle* if

(a) for every periodic point of f is hyperbolic, and
(b) for any $(\bar{p}, \bar{q})$ of periodic points for $\bar{f}$, $W^s(\bar{p})$ is transversal to $W^u(\bar{q})$.

By Shub [9, Theorem (β)], a Kupka-Smale differentiable map is residual. We denote by $\mathcal{KS}$ the C^1-interior of the set of all Kupka-Smale differentiable maps.

For any hyperbolic $p, q \in P(f)$, we say that index(p) = index(q) if $W^s(\bar{p}) \pitchfork W^u(\bar{q}) \neq \emptyset$ and $W^u(\bar{p}) \pitchfork W^s(\bar{q}) \neq \emptyset$, where $P^0(\bar{p}) = p$ and $P^0(\bar{q}) = q$. In [9], Shub introduced the notion of the strong transversality condition for regular maps. Then we have

Theorem 2.1. *Let $f \in \mathfrak{R}(M)$. If $f \in \mathcal{KS}$ then f satisfies Axiom A and strong transversality condition.*

To prove Theorem 2.1, we need the next lemma.

Lemma 2.2. *Let $\mathcal{U}(f)$ be a C^1-neighborhood of f and let $p, q \in P(f)$ be hyperbolic points. If for any $g \in \mathcal{U}(f) \cap \mathcal{KS}$, $W^s(\bar{p}_g, g) \cap W^u(\bar{q}_g, g) \neq \emptyset$ then*

$$\mathrm{index}(p) = \mathrm{index}(q),$$

where $p = P^0(\bar{p}), q = P^0(\bar{q})$ and $\bar{p}_g, \bar{q}_g$ are the continuations of $\bar{p}, \bar{q}$, respectively.

Proof. Let $p, q \in P(f)$ be hyperbolic points such that $P^0(\bar{p}) = p$ and $P^0(\bar{q}) = q$. For simplify, we may assume that p and q are fixed points of f. Suppose, by contradiction, that $\mathrm{index}(p) \neq \mathrm{index}(q)$. If $p, q \in P(f)$ are saddles then we have

$$\mathrm{dim} W^s(\bar{p}) + \mathrm{dim} W^u(\bar{q}) < \mathrm{dim} M \text{ or } \mathrm{dim} W^u(\bar{p}) + \mathrm{dim} W^s(\bar{q}) < \mathrm{dim} M.$$

Take a Kupka-Smale differentiable map $h \in \mathcal{U}(f)$. Then by assumption, we know

$$\mathrm{dim} W^s(\bar{p}_h, h) + \mathrm{dim} W^u(\bar{q}_h, h) < \mathrm{dim} M.$$

Thus $W^s(\bar{p}_h, h) \cap W^u(\bar{q}_h, h) = \emptyset$. Since $h \in \mathcal{U}(f) \cap \mathcal{KS}$, by the hypothesis, it should be $W^s(\bar{p}_h, h) \cap W^u(\bar{q}_h, h) \neq \emptyset$. This is a contradiction.

Note that Kupka-Smale differentiable maps contain the Mores-Samle differentiable maps, that is, $\Omega(f)$ is finite and f satisfies Axiom A and the strong traversality condition (see [6]). Then we can consider that the periodic point p is sink or source. In the proof we consider that p is a source. Then we suppose that $q (\neq p)$ is a saddle. Then there is $\epsilon > 0$ such that $W^s_\epsilon(\bar{p}) = \{p\}$ and $W^u_\epsilon(\bar{q})$ is a disc. Thus we know that

$$\mathrm{dim} W^s(\bar{p}) + \mathrm{dim} W^u(\bar{q}) < \mathrm{dim} M.$$

As in the previous arguments, we get a contradiction. □

Proof of Theorem 2.1. Let $f \in \mathfrak{R}(M)$. Since $f \in \mathcal{KS}$, we know $f \in \mathcal{F}(M)$. Thus if $f \in \mathcal{KS}$ then f is Axiom A, and so, we have $M = W^s(\Omega(f)) = W^u(\Omega(f))$. To derive a contradiction, we may assume that f does not satisfy the strong transvesersality condition. Then there exist $p, q \in P(f)$ and $x \in M \setminus \Omega(f)$ such that $\bar{P}^0(\bar{x}) = x_0$, $P^0(\bar{p}) = p, P^0(\bar{q}) = q$ and

$$T_{x_0} W^s(\bar{p}) + T_{x_0} W^u(\bar{q}) \neq T_{x_0} M.$$

Since $f \in \mathcal{KS}$ by Lemma 2.2, it is impossible. Thus if $f \in \mathcal{KS}$ then f is structurally stable. □

References

[1] N. Aoki, *The set of Axiom A diffeomorphisms with no cycles.* Bol. Soc. Bras. Math. **23** (1992), 21-65.

[2] N. Aoki, K. Moriyasu and N. Sumi, C^1*-maps having hyperbolic periodic points*, Fund. Math. **169** (2001), 1-49.

[3] S. Hayashi, *Diffeomorphisms in* $\mathcal{F}^1(M)$ *satisfy Axiom A*, Ergodic Thoery & Dynam. Syst. **12** (1992), 233-253.

[4] M. Hirsh, C. Pugh and M. Shub, *"Invariant manifolds."*, Lecture Notes in Math. **583**, Springer Verlag, Berlin, 1977.

[5] R. Mañé, *Axiom A for endoemorphisms*, Lecture Notes in Math. **597**, Springer, 1977, 397-388.

[6] J. Palis, *On Morse-Smale dynamical systems*, Topology **8** (1969), 385-404.

[7] F. Przytycki, *On* Ω*-stability and structural stability of endomorphisms satisfying Axiom A*, ibid. **60** (1977), 61-77.

[8] C. Robinson, *Structural stability of* C^1*-diffeomorphisms*, J. Diff. Equat. **22** (1976), 28-73.

[9] M. Shub, *Endormorphisms of compact differentiable manifolds*, Amer. J. Math. **91** (1969), 175-199.

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